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Congruence schemes and their applications

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Abstract. Using congruence schemes we formulate new characterizations of congruence distributive, arithmetical and majority algebras. We prove new properties of the tolerance lattice and of the lattice of compatible reflexive relations of a majority algebra and generalize earlier results of H.-J. Bandelt, G. Czédli and the present authors. Algebras whose congruence lattices satisfy certain 0-conditions are also studied.

Keywords: congruence schemes, majority algebra, tolerance lattice, 0-conditions

Classification: Primary 08A10, 08A30; Secondary 06D15

1. Introduction

The congruence schemes are important tools in the study of congruence lattices of algebras. As examples we mention [12], [4], [5], [6] or [7]. For instance, in [4] it is proved that a congruence permutable algebra $\mathcal{A} = (A, F)$ is congruence distributive if and only if for all congruences α, β and γ of it and every elements $x, y \in A$, the following scheme is satisfied:

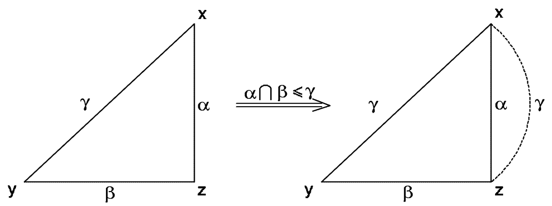

Figure 1

The results of [6] were the inspiration source of new results on the tolerance lattices of general algebras (see [8] and [10]). In this paper, by using congruence schemes, we formulate new characterizations of congruence distributive algebras, arithmetical algebras and algebras with a majority term. Using characterization of majority algebras we derive new properties of their tolerance lattices. Our results generalize the results of H.-J. Bandelt concerning the properties of the tolerance lattice of a lattice [1] and extend some results of [7] and [10]. We also prove that the compatible reflexive relations of a majority algebra form a pseudocomplemented

0-modular lattice. Finally, 0-conditions on the congruence lattice of a general algebra are investigated by using a new congruence scheme.

2. Schemes for congruence distributivity and arithmeticity

Let $\operatorname{Con}\mathcal{A}$ denote the congruence lattice of an algebra $\mathcal{A} = (A, F)$ and let $\triangle$ and ∇ stand for the identical and the total relation on A, respectively.

Proposition 2.1. *The congruence lattice of an algebra $\mathcal{A} = (A, F)$ is distributive if and only if for each $\theta, \varphi, \psi \in \operatorname{Con}\mathcal{A}$ and every $a, b \in A$ and for all sequences of elements $c_1, \dots, c_k \in A$, $k \geq 1$, the following scheme holds:*

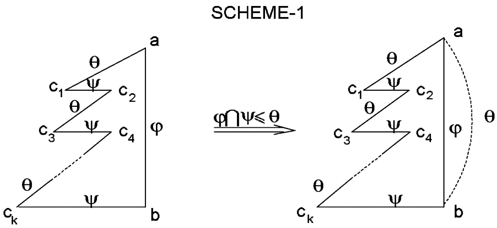

Figure 2

Proof: Suppose $\operatorname{Con}\mathcal{A}$ is distributive, $\theta, \varphi, \psi \in \operatorname{Con}\mathcal{A}$ with $\varphi \cap \psi \leq \theta$ and $(a, b) \in \varphi$, $(a, c_1) \in \theta$, $(c_1, c_2) \in \psi$, $(c_2, c_3) \in \theta, \dots, (c_k, b) \in \psi$, then

$$(a, b) \in \varphi \cap (\theta \vee \psi) = (\varphi \cap \theta) \vee (\varphi \cap \psi) \leq (\varphi \cap \theta) \vee \theta = \theta.$$

Conversely, assume that $\mathcal{A}$ satisfies SCHEME-1 and $\operatorname{Con}\mathcal{A}$ is not distributive. Then there exist $\theta, \varphi, \psi \in \operatorname{Con}\mathcal{A}$ such that $\operatorname{Con}\mathcal{A}$ contains a sublattice isomorphic to M_3 or N_5 as follows (see Figure 3):

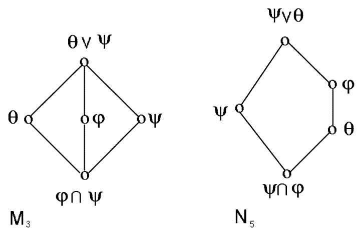

Figure 3

Clearly, in the both cases $\varphi \cap \psi \leq \theta$. Take $(a, b) \in \varphi$. Then $(a, b) \in \theta \vee \psi$, thus there exist $c_1, \dots, c_k \in A$ such that $(a, c_1) \in \theta$, $(c_1, c_2) \in \psi$, $(c_2, c_3) \in$

$\theta, \ldots, (c_k, b) \in \psi$. Applying SCHEME-1 we get $(a, b) \in \theta$, proving $\varphi \leq \theta$, a contradiction in the both cases. Thus $\operatorname{Con} \mathcal{A}$ is a distributive lattice. $\square$

Let $\rho \circ \sigma$ denote the product of two binary relations $\rho, \sigma \subseteq A \times A$. An algebra $\mathcal{A} = (A, F)$ is called *arithmetical* if $\operatorname{Con} \mathcal{A}$ is a distributive lattice and $\mathcal{A}$ is *congruence permutable*, that is $\rho \circ \sigma = \sigma \circ \rho$ holds for all $\rho, \sigma \in \operatorname{Con} \mathcal{A}$.

Theorem 2.2. *An algebra $\mathcal{A} = (A, F)$ is arithmetical, if and only if for all $\alpha, \beta, \gamma \in \operatorname{Con} \mathcal{A}$ with $\alpha \cap \beta \leq \gamma$, the following scheme is satisfied:*

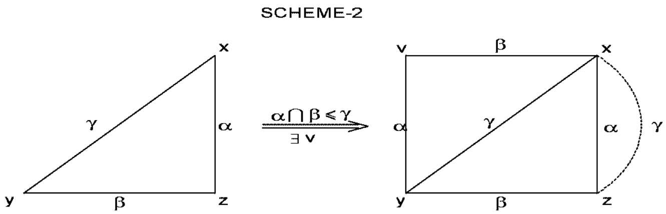

Figure 4

Proof: Suppose that $\mathcal{A} = (A, F)$ satisfies the above scheme and let $\alpha, \beta \in \operatorname{Con} \mathcal{A}$ be arbitrary. Taking $\gamma = \nabla$, we get $\alpha \cap \beta \leq \gamma$. Now, let $(x, y) \in \alpha \circ \beta$. By the scheme, there exists a $v \in A$ with $(x, v) \in \beta$ and $(v, y) \in \alpha$, thus we get $(x, y) \in \beta \circ \alpha$, proving that $\mathcal{A}$ is congruence permutable. By applying [4, Theorem 1] paraphrased in the introduction, we obtain that $\mathcal{A}$ is congruence distributive.

Conversely, let $\mathcal{A} = (A, F)$ be arithmetical and $\alpha, \beta, \gamma \in \operatorname{Con} \mathcal{A}$, $\alpha \cap \beta \leq \gamma$. Suppose $(x, z) \in \alpha$, $(z, y) \in \beta$ and $(x, y) \in \gamma$ for some $x, y, z \in A$. Due to congruence permutability of $\mathcal{A}$ there exists a $v \in A$ as it is shown in Figure 5 below:

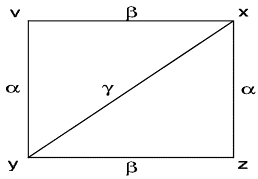

Figure 5

Using the quoted result of [4], we obtain $(x, z) \in \gamma$, hence $\mathcal{A}$ satisfies SCHEME-2. $\square$

3. A characterization of majority algebras

A term function $m(x,y,z)$ of an algebra $\mathcal{A} = (A, F)$ is called a *majority term* if $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ holds for all $x, y \in A$. For instance, any lattice $(L, \wedge, \vee)$ admits a majority term. Algebras having a majority term are called *majority algebras*. A *quasiorder* of an algebra $\mathcal{A} = (A, F)$ is a reflexive, transitive, binary relation $\rho \subseteq A \times A$ which is compatible with the operations of $\mathcal{A}$. Let $\operatorname{Quord}\mathcal{A}$ stand for the set of all quasiorders of $\mathcal{A}$. It is easy to see that $(\operatorname{Quord}\mathcal{A}, \subseteq)$ is an algebraic lattice where the "meet operation" $\wedge$ is the set intersection $\cap$ of the binary relations (see e.g. [13]).

Proposition 3.1. *Let $\mathcal{A} = (A, F)$ be an algebra and consider the following assertions.*

(i) *$\mathcal{A}$ has a majority term function m.*
(ii) *For every $a, b, c \in A$ and any compatible reflexive relations $\alpha, \beta, \gamma \subseteq A \times A$ SCHEME-3 below is satisfied.*
(iii) *Any compatible reflexive relations $\alpha, \beta, \gamma \subseteq A \times A$ satisfy*

$$\alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \beta) \circ (\alpha \cap \gamma). \tag{1}$$

(iv) *For any quasiorders α, β, γ of $\mathcal{A}$ we have:*

$$\alpha \cap (\beta \circ \gamma) = (\alpha \cap \beta) \circ (\alpha \cap \gamma). \tag{2}$$

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

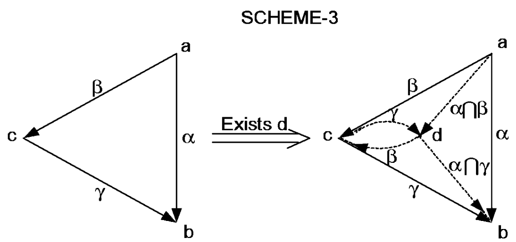

Figure 6

Proof: (i) implies (ii). Let $m : A^3 \to A$ be a majority term function of $\mathcal{A}$ and suppose $(a, b) \in \alpha$, $(a, c) \in \beta$ and $(c, b) \in \gamma$, where $a, b, c \in A$ and α, β, γ are compatible reflexive relations of $\mathcal{A}$. Take $d = m(a, c, b)$. Then we obtain:

$$(a, d) = (m(a, a, b), m(a, c, b)) \in \beta, \text{ and}$$
$$(a, d) = (m(a, c, a), m(a, c, b)) \in \alpha.$$

Thus $(a, d) \in \alpha \cap \beta$. Similarly, we get:

$$\begin{aligned}(d, b) &= (m(a, c, b), m(a, b, b)) \in \gamma, \text{ and}\\ (d, b) &= (m(a, c, b), m(b, c, b)) \in \alpha,\end{aligned}$$

whence $(d, b) \in \alpha \cap \gamma$. It is also clear that

$$\begin{aligned}(c, d) &= (m(a, c, c), m(a, c, b)) \in \gamma, \text{ and}\\ (d, c) &= (m(a, c, b), m(c, c, b)) \in \beta.\end{aligned}$$

(ii) implies (iii). Take $(a, b) \in \alpha \cap (\beta \circ \gamma)$. Then there is a $c \in A$ with $(a, c) \in \beta$ and $(c, b) \in \gamma$. As $(a, b) \in \alpha$, by applying SCHEME-3, we obtain $(a, b) \in (\alpha \cap \beta) \circ (\alpha \cap \gamma)$. Hence $\alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \beta) \circ (\alpha \cap \gamma)$.

(iii) implies (iv). If $\alpha, \beta, \gamma \in \operatorname{Quord} \mathcal{A}$, then

$$(\alpha \cap \beta) \circ (\alpha \cap \gamma) \subseteq (\alpha \circ \alpha) \cap (\beta \circ \gamma) \subseteq \alpha \cap (\beta \circ \gamma).$$

As the converse inclusion holds by assumption, we obtain relation (2). □

Remark 3.2. If $\mathcal{A}$ is an algebra with a majority term function and $\alpha, \beta_1, \dots, \beta_n$ are compatible reflexive relations of $\mathcal{A}$, then using relation (1) one can easily show by induction on $n \geq 1$ that

$$\alpha \cap (\beta_1 \circ \cdots \circ \beta_n) \subseteq (\alpha \cap \beta_1) \circ \cdots \circ (\alpha \cap \beta_n). \tag{3}$$

In the case $\beta_1 = \cdots = \beta_n = \beta$, we obtain:

$$\alpha \cap \beta^n \subseteq (\alpha \cap \beta)^n. \tag{4}$$

If $\alpha, \beta_1, \dots, \beta_n \in \operatorname{Quord} \mathcal{A}$, then (3) and $(\alpha \cap \beta_1) \circ \cdots \circ (\alpha \cap \beta_n) \subseteq \alpha^n \cap (\beta_1 \circ \cdots \circ \beta_n) \subseteq \alpha \cap (\beta_1 \circ \cdots \circ \beta_n)$ imply:

$$\alpha \cap (\beta_1 \circ \cdots \circ \beta_n) = (\alpha \cap \beta_1) \circ \cdots \circ (\alpha \cap \beta_n). \tag{5}$$

Let $\theta(a, b)$ stand for the principal congruence of an algebra $\mathcal{A} = (A, F)$ generated by the pair $(a, b) \in A^2$. If $\varphi : B \to C$ is a homomorphism of the algebra $\mathcal{B} = (B, F)$ into the algebra $\mathcal{C} = (C, F)$ then $(u, v) \in \theta(a, b)$ implies $(\varphi(u), \varphi(v)) \in \theta(\varphi(a), \varphi(b))$ (see e.g. [2, Chapter II, Section 6]).

Theorem 3.3. *Let $\mathcal{V}$ be a variety of algebras. Then the following assertions are equivalent.*

(i) *$\mathcal{V}$ has a majority term.*
(ii) *Any algebra $\mathcal{A} = (A, F) \in \mathcal{V}$ satisfies SCHEME-3.*

(iii) *For any algebra* $\mathcal{A} = (A, F) \in \mathcal{V}$ *and any compatible reflexive relations* $\alpha, \beta, \gamma \subseteq A \times A$ *we have*

$$\alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \beta) \circ (\alpha \cap \gamma).$$

(iv) *For any algebra* $\mathcal{A} \in \mathcal{V}$*, every* $\alpha, \beta, \gamma \in \operatorname{Con} \mathcal{A}$ *satisfy the equality*

$$\alpha \cap (\beta \circ \gamma) = (\alpha \cap \beta) \circ (\alpha \cap \gamma).$$

Proof: In view of Proposition 3.1, (i) implies (ii) and (ii) implies (iii). As $\operatorname{Con} \mathcal{A} \subseteq \operatorname{Quord} \mathcal{A}$, Proposition 3.1 also gets that (iii) implies (iv).

(iv) implies (i). Consider now the free algebra $F_{\mathcal{V}}(x, y, z) \in \mathcal{V}$. As $(x, z) \in \theta(x, z) \cap (\theta(x, y) \circ \theta(y, z))$, the assumption of (iv) implies $(x, z) \in (\theta(x, z) \cap \theta(x, y)) \circ (\theta(x, z) \cap \theta(y, z))$. Hence, there is a term $m(x, y, z) \in F_{\mathcal{V}}(x, y, z)$ such that

$$(x, m(x, y, z)) \in \theta(x, y) \cap \theta(x, z) \quad \text{and} \quad (m(x, y, z), z) \in \theta(x, z) \cap \theta(y, z).$$

Now, using a homomorphism $\varphi : F_{\mathcal{V}}(x, y, z) \to F_{\mathcal{V}}(x, y, z)$ with $\varphi(x) = \varphi(y) = x$ and $\varphi(z) = y$ from $(x, m(x, y, z)) \in \theta(x, y)$ we obtain

$$\begin{aligned}(x, m(x, x, y)) &= (\varphi(x), m(\varphi(x), \varphi(y), \varphi(z))) \\ &= (\varphi(x), \varphi(m(x, y, z))) \in \theta(\varphi(x), \varphi(y)) = \theta(x, x) = \triangle.\end{aligned}$$

Thus $x = m(x, x, y)$.

The identities $x = m(x, y, x) = m(y, x, x)$ can be proved in a similar way. $\square$

4. Applications

A) The quasiorder lattice $\operatorname{Quord} \mathcal{A}$ of a majority algebra $\mathcal{A}$

Let $\mathcal{A} = (A, F)$ be an algebra and $\rho_1, \rho_2 \in \operatorname{Quord} \mathcal{A}$. It is known (see e.g. [13]) that the join $\rho_1 \vee \rho_2$ in the lattice $\operatorname{Quord} \mathcal{A}$ is the transitive closure of the union $\rho_1 \cup \rho_2 \subseteq A \times A$. Hence

$$\rho_1 \vee \rho_2 = \bigcup \{\rho_{i_1} \circ \cdots \circ \rho_{i_n} \mid i_1, \ldots, i_n \in \{1;\ 2\},\ n \geq 1\}.$$

Using now relation (5) we can deduce the following result of [9] and [13]:

Corollary 4.1. *If* $\mathcal{A}$ *is an algebra with a majority term function, then* $\operatorname{Quord} \mathcal{A}$ *is a distributive lattice.*

Proof: Take any $\alpha, \beta_1, \beta_2 \in \operatorname{Quord} \mathcal{A}$. Then we can write:

$$\begin{aligned}&\alpha \wedge (\beta_1 \vee \beta_2) = \alpha \cap \left(\bigcup \{\beta_{i_1} \circ \cdots \circ \beta_{i_n} \mid i_1, \ldots, i_n \in \{1;\ 2\},\ n \geq 1\}\right) \\ &= \bigcup \{\alpha \cap (\beta_{i_1} \circ \cdots \circ \beta_{i_n}) \mid i_1, \ldots, i_n \in \{1;\ 2\},\ n \geq 1\} \\ &= \bigcup \{(\alpha \cap \beta_{i_1}) \circ \cdots \circ (\alpha \cap \beta_{i_n}) \mid i_1, \ldots, i_n \in \{1;\ 2\},\ n \geq 1\} \\ &= (\alpha \cap \beta_1) \vee (\alpha \cap \beta_2) = (\alpha \wedge \beta_1) \vee (\alpha \wedge \beta_2),\end{aligned}$$

proving that $(\operatorname{Quord} \mathcal{A}, \wedge, \vee)$ is a distributive lattice. $\square$

B) The tolerance lattice of a majority algebra

A lattice L with 0 element is called *pseudocomplemented* if for each $x \in L$ there exists an element $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0$ is equivalent to $y \leq x^*$. L is called 0-*modular*, if for any $a, b, c \in L$

$$(M_0) \qquad a \wedge c = 0 \text{ and } b \leq c \text{ imply } (a \vee b) \wedge c = b.$$

In view of the Varlet's result [16, Theorem 5] a lattice with 0 is 0-modular if and only if it does not contain an N_5 sublattice including 0. Generalizing Bandelt's result [1] on the tolerance lattice of a lattice, it was proved in [10] that the tolerance lattice of a majority algebra is a 0-modular pseudocomplemented algebraic lattice. Now we apply our former results to derive new properties of the tolerance lattice of a majority algebra.

Let $(\operatorname{Tol}\mathcal{A}, \cap, \sqcup)$ denote the tolerance lattice of an algebra $\mathcal{A} = (A, F)$. Then for all $\alpha, \beta \in \operatorname{Tol}\mathcal{A}$

$$\alpha \sqcup \beta \subseteq (\alpha \circ \beta) \cap (\beta \circ \alpha) \tag{6}$$

is true. (See e.g. [10, Lemma 2.1].) If $\mathcal{A}$ has a majority term function, then in view of [7], the inclusion

$$(\alpha \circ \beta) \cap (\beta \circ \alpha) \subseteq (\alpha \cap \beta) \circ (\alpha \sqcup \beta) \tag{7}$$

is also satisfied. For $\alpha \cap \beta = \triangle$, (6) and (7) imply

$$\alpha \sqcup \beta = (\alpha \circ \beta) \cap (\beta \circ \alpha), \tag{8}$$

a relation established in [10]. (See also [7].)

Now, using Proposition 3.1 (or equivalently Theorem 3.3) we can deduce

Theorem 4.2. *Let $\mathcal{A} = (A, F)$ be an algebra with a majority term function and $\alpha, \beta, \gamma \in \operatorname{Tol}\mathcal{A}$. Then:*

(i) *$\alpha \cap \beta \cap \gamma = \triangle$ implies $\alpha \cap (\beta \sqcup \gamma) = (\alpha \cap \beta) \sqcup (\alpha \cap \gamma)$;*
(ii) *if $\alpha \cap \gamma = \triangle$, or if $\alpha \cap \beta \cap \gamma = \triangle$ and γ is a congruence, then $(\alpha \cap \beta) \sqcup \gamma = (\alpha \sqcup \gamma) \cap (\beta \sqcup \gamma)$.*

Proof: (i) Assume $\alpha \cap \beta \cap \gamma = \triangle$. Clearly, it is enough to show only $\alpha \cap (\beta \sqcup \gamma) \subseteq (\alpha \cap \beta) \sqcup (\alpha \cap \gamma)$. By using relations (6) and (1) we get:

$$\begin{aligned} \alpha \cap (\beta \sqcup \gamma) &\subseteq \alpha \cap (\beta \circ \gamma) \cap (\gamma \circ \beta) \subseteq (\alpha \cap (\beta \circ \gamma)) \cap (\alpha \cap (\gamma \circ \beta)) \\ &\subseteq ((\alpha \cap \beta) \circ (\alpha \cap \gamma)) \cap ((\alpha \cap \gamma) \circ (\alpha \cap \beta)). \end{aligned}$$

As $(\alpha \cap \beta) \cap (\alpha \cap \gamma) = \alpha \cap \beta \cap \gamma = \triangle$, relation (8) implies

$$((\alpha \cap \beta) \circ (\alpha \cap \gamma)) \cap ((\alpha \cap \gamma) \circ (\alpha \cap \beta)) = (\alpha \cap \beta) \sqcup (\alpha \cap \gamma),$$

proving $\alpha \cap (\beta \sqcup \gamma) \subseteq (\alpha \cap \beta) \sqcup (\alpha \cap \gamma)$.

(ii) Obviously, it is sufficient to prove $(\alpha \sqcup \gamma) \cap (\beta \sqcup \gamma) \subseteq (\alpha \cap \beta) \sqcup \gamma$. First, observe that (6) implies

$$\begin{aligned}(\alpha \sqcup \gamma) \cap (\beta \sqcup \gamma) &\subseteq (\alpha \circ \gamma) \cap (\gamma \circ \alpha) \cap (\beta \circ \gamma) \cap (\gamma \circ \beta)\\ &= ((\beta \circ \gamma) \cap (\alpha \circ \gamma)) \cap ((\gamma \circ \beta) \cap (\gamma \circ \alpha)).\end{aligned}$$

Since $\beta \circ \gamma$ and $\gamma \circ \beta$ are also compatible reflexive relations of $\mathcal{A}$, by using relations (1), $\gamma \subseteq \beta \circ \gamma$ and $\gamma \subseteq \gamma \circ \beta$, we obtain

$$\begin{aligned}&(\beta \circ \gamma) \cap (\alpha \circ \gamma) \subseteq ((\beta \circ \gamma) \cap \alpha) \circ ((\beta \circ \gamma) \cap \gamma) = (\alpha \cap (\beta \circ \gamma)) \circ \gamma \quad \text{and}\\ &(\gamma \circ \beta) \cap (\gamma \circ \alpha) \subseteq ((\gamma \circ \beta) \cap \gamma) \circ ((\gamma \circ \beta) \cap \alpha) = \gamma \circ (\alpha \cap (\gamma \circ \beta)).\end{aligned}$$

Hence,

$$(\alpha \sqcup \gamma) \cap (\beta \sqcup \gamma) \subseteq [(\alpha \cap (\beta \circ \gamma)) \circ \gamma] \cap [\gamma \circ (\alpha \cap (\gamma \circ \beta))].$$

Applying formula (1) again, we get

$$[(\alpha \cap (\beta \circ \gamma)) \circ \gamma] \cap [\gamma \circ (\alpha \cap (\gamma \circ \beta))] \subseteq ((\alpha \cap \beta) \circ (\alpha \cap \gamma) \circ \gamma) \cap (\gamma \circ (\alpha \cap \gamma) \circ (\alpha \cap \beta)).$$

As the hypothesis $\alpha \cap \gamma = \triangle$ or $\gamma \in \operatorname{Con} \mathcal{A}$ gives $(\alpha \cap \gamma) \circ \gamma = \gamma \circ (\alpha \cap \gamma) = \gamma$, it follows

$$(\alpha \sqcup \gamma) \cap (\beta \sqcup \gamma) \subseteq ((\alpha \cap \beta) \circ \gamma) \cap (\gamma \circ (\alpha \cap \beta)).$$

Since $(\alpha \cap \beta) \cap \gamma = \alpha \cap \beta \cap \gamma = \triangle$, by using relation (8), we obtain

$$((\alpha \cap \beta) \circ \gamma) \cap (\gamma \circ (\alpha \cap \beta)) = (\alpha \cap \beta) \sqcup \gamma. \qquad \square$$

Remark 4.3. Notice that from Theorem 4.2(i) we can derive the known properties of the tolerance lattice of a majority algebra $\mathcal{A}$.

Indeed, in view of Theorem 4.2(i) for any $\alpha, \beta, \gamma \in \operatorname{Tol} \mathcal{A}$ the relations $\alpha \cap \beta = \alpha \cap \gamma = \triangle$ imply $\alpha \cap (\beta \sqcup \gamma) = \triangle$, and this means that $\operatorname{Tol} \mathcal{A}$ is a 0-*distributive* (see the next section). As $\operatorname{Tol} \mathcal{A}$ is an algebraic lattice this property is equivalent to the fact that $\operatorname{Tol} \mathcal{A}$ is pseudocomplemented. Observe that, in virtue of Theorem 4.2(i), $\alpha \cap \beta \cap \gamma = \triangle$ also implies $(\alpha \sqcup \beta) \cap \gamma = (\alpha \cap \gamma) \sqcup (\beta \sqcup \gamma)$. Hence for any $\alpha, \beta, \gamma \in \operatorname{Tol} \mathcal{A}$ $\alpha \cap \gamma = \triangle$ and $\beta \leq \gamma$ imply $(\alpha \sqcup \beta) \cap \gamma = \beta$, proving the 0-modularity of $\operatorname{Tol} \mathcal{A}$. Further, using Theorem 4.2(i) we obtain that $\beta \cap \gamma = \triangle$ implies $\alpha \cap (\beta \sqcup \gamma) = (\alpha \cap \beta) \sqcup (\alpha \cap \gamma)$ for any $\alpha, \beta, \gamma \in \operatorname{Tol} \mathcal{A}$, i.e. we deduce the *strong* 0-*distributive property* of $\operatorname{Tol} \mathcal{A}$ established in [7].

C) Compatible reflexive relations on a majority algebra

Let us denote by $\operatorname{Ref}\mathcal{A}$ the set of all compatible reflexive relations of an algebra $\mathcal{A}=(A,F)$ and by $\overline{\alpha}$ the transitive closure of a relation $\alpha \subseteq A\times A$.

Lemma 4.4. *If $\mathcal{A}$ is an algebra with a majority term function and $\rho,\theta \in \operatorname{Ref}\mathcal{A}$, then*

$$\overline{\rho\cap\theta}=\overline{\rho}\cap\overline{\theta}. \tag{9}$$

Proof: Since $\overline{\rho\cap\theta}\subseteq\overline{\rho}$ and $\overline{\rho\cap\theta}\subseteq\overline{\theta}$, we get $\overline{\rho\cap\theta}\subseteq\overline{\rho}\cap\overline{\theta}$. In order to show the converse inclusion, take any $(x,y)\in\overline{\rho}\cap\overline{\theta}$. Then there exist $n,m\geq 1$ such that $(x,y)\in\rho^n\cap\theta^m$. Since $\rho^n,\theta^m\in\operatorname{Ref}\mathcal{A}$, relation (4) (see Remark 3.2) implies

$$\rho^n\cap\theta^m\subseteq(\rho^n\cap\theta)^m\subseteq\overline{(\theta\cap\rho^n)}\subseteq\overline{(\theta\cap\rho)^n}\subseteq\overline{(\rho\cap\theta)}.$$

Thus we get $(x,y)\in\overline{(\rho\cap\theta)}$, proving $\overline{\rho}\cap\overline{\theta}\subseteq\overline{(\rho\cap\theta)}$. □

Remark 4.5. Clearly, for any algebra $\mathcal{A}$, $(\operatorname{Ref}\mathcal{A},\subseteq)$ is an algebraic lattice where the meet of any $\rho,\theta\in\operatorname{Ref}\mathcal{A}$ is $\rho\cap\theta$ (see e.g. [3]). Let us denote their join in $\operatorname{Ref}\mathcal{A}$ by $\rho\veebar\theta$. As we have $\rho\cup\theta\subseteq\rho\circ\theta$ and $\rho\circ\theta\in\operatorname{Ref}\mathcal{A}$, it follows

$$\rho\veebar\theta\subseteq\rho\circ\theta. \tag{10}$$

If $(L,\wedge,\vee)$ is a (bounded) pseudocomplemented lattice, then the algebra $(L,\wedge,\vee,{}^*,0,1)$ is called a *p-algebra* and $(L,\wedge,{}^*)$ a *p-semilattice*. As every algebraic distributive lattice is also a pseudocomplemented lattice, for any majority algebra $\mathcal{A}$ the lattice $(\operatorname{Quord}\mathcal{A},\cap,\vee)$ is pseudocomplemented and hence $(\operatorname{Quord}\mathcal{A},\cap,\vee,{}^*,\triangle,\nabla)$ is a p-algebra. Now we are able to prove the following

Theorem 4.6. *Let $\mathcal{A}=(A,F)$ be an algebra with a majority term function. Then the following statements hold.*

- (i) $(\operatorname{Ref}\mathcal{A},\cap,\veebar)$ *is a pseudocomplemented* 0*-modular lattice and the pseudocomplement of every $\rho\in\operatorname{Ref}\mathcal{A}$ is a quasiorder.*
- (ii) *The map $h:\operatorname{Ref}\mathcal{A}\to\operatorname{Quord}\mathcal{A}$, $h(\rho)=\overline{\rho}$ is a homomorphism of the p-algebra* $(\operatorname{Ref}\mathcal{A},\cap,\veebar,{}^*,\triangle,\nabla)$ *onto the p-algebra* $(\operatorname{Quord}\mathcal{A},\cap,\vee,{}^*,\triangle,\nabla)$.

Proof: (i) Clearly, for each $\rho\in\operatorname{Ref}\mathcal{A}$, its transitive closure $\overline{\rho}$ is a quasiorder. Let $(\overline{\rho})^*$ stand for the pseudocomplement of $\overline{\rho}$ in the lattice $(\operatorname{Quord}\mathcal{A},\cap,\vee)$. First, we prove that $(\overline{\rho})^*$ is also the pseudocomplement of ρ in the lattice $(\operatorname{Ref}\mathcal{A},\cap,\veebar)$.

Let $\varphi\in\operatorname{Ref}\mathcal{A}$. Clearly, $\varphi\leq(\overline{\rho})^*$ implies $\varphi\cap\rho\subseteq(\overline{\rho})^*\cap\overline{\rho}=\triangle$. Conversely, take a $\varphi\in\operatorname{Ref}\mathcal{A}$ with $\varphi\cap\rho=\triangle$. Then, by using Lemma 4.5, we get $\overline{\varphi}\cap\overline{\rho}=\overline{\varphi\cap\rho}=\triangle$. Hence $\varphi\leq\overline{\varphi}\leq(\overline{\rho})^*$.

Consequently, as $\varphi\cap\rho=\triangle$ is equivalent to $\varphi\leq(\overline{\rho})^*$, the lattice $(\operatorname{Ref}\mathcal{A},\cap,\veebar)$ is pseudocomplemented and the pseudocomplement ρ^* of any $\rho\in\operatorname{Ref}\mathcal{A}$ is $(\overline{\rho})^*\in$

$\operatorname{Quord}\mathcal{A}$. As $\rho = \overline{\rho}$ for all $\rho \in \operatorname{Quord}\mathcal{A}$, the pseudocomplement of a $\rho \in \operatorname{Quord}\mathcal{A}$ in the lattice $(\operatorname{Ref}\mathcal{A}, \cap, \veebar)$ is the same as in $(\operatorname{Quord}\mathcal{A}, \cap, \vee)$.

In order to prove that $(\operatorname{Ref}\mathcal{A}, \cap, \veebar)$ is 0-modular, by the way of contradiction, we assume that $\{\triangle, \alpha, \beta, \gamma, \nu\}$ is an N_5 sublattice of it with $\triangle < \alpha < \gamma < \nu$ and $\triangle < \beta < \nu$, as it is shown in Figure 7. Take $(a, b) \in \gamma$. Using formulas (10) and (1) we obtain $(a, b) \in \gamma \cap (\alpha \veebar \beta) \subseteq \gamma \cap \alpha \circ \beta \subseteq (\gamma \cap \alpha) \circ (\gamma \cap \beta) = \alpha \circ \triangle = \alpha$, whence $\gamma \leq \alpha$, a contradiction.

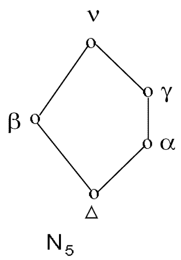

Figure 7

(ii) Since $h(\rho) = \overline{\rho} = \rho$, for each $\rho \in \operatorname{Quord}\mathcal{A}$, the map $h : \operatorname{Ref}\mathcal{A} \to \operatorname{Quord}\mathcal{A}$ is onto. Lemma 4.4 also gives

$$h(\rho \cap \theta) = \overline{\rho \cap \theta} = \overline{\rho} \cap \overline{\theta} = h(\rho) \cap h(\theta), \quad \text{for all} \quad \rho, \theta \in \operatorname{Ref}\mathcal{A}.$$

Observe, that in order to prove $h(\rho \veebar \theta) = h(\rho) \vee h(\theta)$, we have to show only $\overline{\rho \veebar \theta} = \overline{\rho} \vee \overline{\theta}$. As $\rho, \theta \leq \overline{\rho} \vee \overline{\theta} \in \operatorname{Quord}\mathcal{A}$, the inclusion $\overline{\rho \veebar \theta} \subseteq \overline{\rho} \vee \overline{\theta}$ is clear. Conversely, we have $\overline{\rho} \leq \overline{\rho \veebar \theta}$ and $\overline{\theta} \leq \overline{\rho \veebar \theta}$. Since $\overline{\rho \veebar \theta}$ is a quasiorder, we get $\overline{\rho} \vee \overline{\theta} \leq \overline{\rho \veebar \theta}$, proving $\overline{\rho \veebar \theta} = \overline{\rho} \vee \overline{\theta}$. Thus

$$h(\rho \veebar \theta) = h(\rho) \vee h(\theta), \quad \text{for all} \quad \rho, \theta \in \operatorname{Ref}\mathcal{A}.$$

It is obvious that $h(\triangle) = \triangle$ and $h(\nabla) = \nabla$. As $\rho^* \in \operatorname{Quord}\mathcal{A}$ for all $\rho \in \operatorname{Ref}\mathcal{A}$, we get $h(\rho^*) = \rho^*$. On the other hand, we have $h(\rho)^* = (\overline{\rho})^* = \rho^*$, according to the argument of the above (i). Thus we conclude

$$h(\rho^*) = h(\rho)^*.$$

All these together prove that h is a homomorphism of p-algebras. $\square$

Remark 4.7. It is an obvious consequence of Theorem 4.6 that for any majority algebra $\mathcal{A}$, $(\operatorname{Quord}\mathcal{A}, \cap, {}^*)$ is a p-subsemilattice of $(\operatorname{Ref}\mathcal{A}, \cap, {}^*)$. Theorem 4.6(i) also implies $\alpha^* \in \operatorname{Con}\mathcal{A}$, for all $\alpha \in \operatorname{Tol}\mathcal{A}$ — a result already established in [10] (and for lattices, in [14]).

5. 0-conditions

A lattice L with 0 is said to be 0-*distributive* if, for $a, b, c \in L$

$$(D_0) \qquad a \wedge c = 0 \ \text{ and } \ b \wedge c = 0 \ \text{ imply } \ (a \vee b) \wedge c = 0.$$

L is called *pseudo*-0-*distributive* if, for $a, b, c \in L$

$$(PD_0) \qquad a \wedge b = 0 \ \text{ and } \ a \wedge c = 0 \ \text{ imply } \ (a \vee b) \wedge c = b \wedge c.$$

It is known that an algebraic lattice is 0-distributive if and only if it is pseudocomplemented (see e.g. [16]). In [7] it is proved that any pseudocomplemented 0-modular lattice is pseudo-0-distributive. The lattice in Figure 8 is pseudo-0-distributive, however it is not 0-distributive.

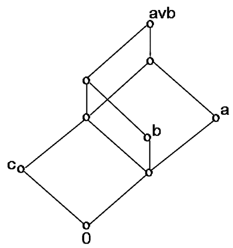

Figure 8

Lemma 5.1. *Let $\mathcal{A} = (A, F)$ be an algebra. If* $\operatorname{Con}\mathcal{A}$ *is pseudo*-0-*distributive or* 0-*distributive, then $\mathcal{A}$ satisfies the P0D-SCHEME:*

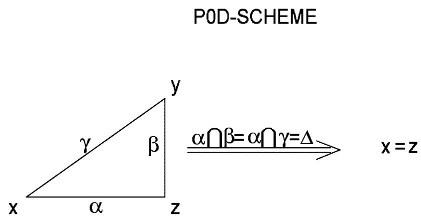

Figure 9

Proof: Suppose $\alpha \cap \beta = \alpha \cap \gamma = \triangle$ for $\alpha, \beta, \gamma \in \operatorname{Con}\mathcal{A}$ and $(x, z) \in \alpha$, $(y, z) \in \beta$, $(x, y) \in \gamma$. If $\operatorname{Con}\mathcal{A}$ is 0-distributive, then $(x, z) \in \alpha \cap (\gamma \circ \beta) \subseteq \alpha \wedge (\beta \vee \gamma) = \triangle$, and this gives $x = z$. If $\operatorname{Con}\mathcal{A}$ is pseudo-0-distributive, then using (PD_0) we get $(y, z) \in (\gamma \circ \alpha) \cap \beta \subseteq (\alpha \vee \gamma) \wedge \beta = \gamma \wedge \beta$. Hence $(y, z) \in \gamma$ and this implies $(x, z) \in \alpha \cap (\gamma \circ \gamma) = \alpha \cap \gamma = \triangle$, that is $x = z$. $\square$

In what follows, we are going to find natural conditions under which the converse of Lemma 5.1 holds.

If $\mathcal{A} = (A, F)$ is an algebra and $\alpha \in \operatorname{Con} \mathcal{A}$, then let $[a]_\alpha$ denote the α-congruence class of an element $a \in A$.

Definition 5.2. Let $\mathcal{A} = (A, F)$ be an algebra and $\alpha, \beta \in \operatorname{Con} \mathcal{A}$ be arbitrary.

(i) $\mathcal{A}$ is called *conditionally permutable* if $\alpha \cap \beta = \triangle$ implies $\alpha \circ \beta = \beta \circ \alpha$.
(ii) A subset $B \subseteq A/\alpha \times A/\beta$ is said to have a *symmetrical patchworking*, whenever $([a]_\alpha, [b]_\beta) \in B$ implies $([b]_\alpha, [a]_\beta) \in B$, for every $a, b \in A$.

Let us observe that for any algebra $\mathcal{A} = (A, F)$ and any $\alpha, \beta \in \operatorname{Con} \mathcal{A}$ with $\alpha \cap \beta = \triangle$, the map $f : A \to A/\alpha \times A/\beta$, $f(a) = ([a]_\alpha, [a]_\beta)$ is an embedding. Indeed, it is clear that $f : A \to A/\alpha \times A/\beta$ is a homomorphism of $\mathcal{A}$ into the algebra $\mathcal{A}/\alpha \times \mathcal{A}/\beta = (A/\alpha \times A/\beta, F)$. Moreover, if $f(a) = f(b)$ for $a, b \in A$, then $[a]_\alpha = [b]_\alpha$ and $[a]_\beta = [b]_\beta$ imply $b \in [a]_\alpha \cap [a]_\beta = [a]_{\alpha\cap\beta} = [a]_\triangle = \{a\}$, that is $a = b$.

Proposition 5.3. *An algebra $\mathcal{A} = (A, F)$ is conditionally permutable, whenever for each $\alpha, \beta \in \operatorname{Con} \mathcal{A}$ with $\alpha \cap \beta = \triangle$ and the corresponding embedding $f : A \to A/\alpha \times A/\beta$, $f(a) = ([a]_\alpha, [a]_\beta)$, the image $f(A)$ has a symmetrical patchworking.*

Proof: Suppose that $f(A)$ has a symmetrical patchworking and take $(a, b) \in \alpha \circ \beta$, for $\alpha, \beta \in \operatorname{Con} \mathcal{A}$ with $\alpha \cap \beta = \triangle$. Then there is a $z \in A$ with $(a, z) \in \alpha$ and $(z, b) \in \beta$, whence $[z]_\alpha = [a]_\alpha$ and $[z]_\beta = [b]_\beta$. Thus $f(z) = ([z]_\alpha, [z]_\beta) = ([a]_\alpha, [b]_\beta)$.

Now, due to the symmetrical patchworking property, there exists an $x \in A$ with $f(x) = ([x]_\alpha, [x]_\beta) = ([b]_\alpha, [a]_\beta)$, that is with $(x, b) \in \alpha$ and $(a, x) \in \beta$. Thus we get $(a, b) \in \beta \circ \alpha$, proving conditional permutability.

Conversely, suppose $\mathcal{A}$ is conditionally permutable and $([a]_\alpha, [b]_\beta) \in f(A)$, for some $a, b \in A$. Then there is a $y \in A$ with $f(y) = ([y]_\alpha, [y]_\beta) = ([a]_\alpha, [b]_\beta)$, and hence $(a, y) \in \alpha$, $(y, b) \in \beta$. Therefore, $(a, b) \in \alpha \circ \beta$. By the hypothesis, $\alpha \cap \beta = \triangle$ implies $\alpha \circ \beta = \beta \circ \alpha$. Thus $(a, b) \in \beta \circ \alpha$ and hence there is an $x \in A$ with $(a, x) \in \beta$ and $(x, b) \in \alpha$, that is, with $[x]_\beta = [a]_\beta$ and $[x]_\alpha = [b]_\alpha$. This result gives $f(x) = ([x]_\alpha, [x]_\beta) = ([b]_\alpha, [a]_\beta)$, proving $([b]_\alpha, [a]_\beta) \in f(A)$. Hence $f(A)$ has a symmetrical patchworking. □

Theorem 5.4. *Let $\mathcal{A} = (A, F)$ be a conditionally permutable algebra. Then $\operatorname{Con} \mathcal{A}$ is pseudo-0-distributive if and only if $\mathcal{A}$ satisfies the P0D-SCHEME.*

Proof: Due to Lemma 5.1, we need to show only that P0D-SCHEME implies pseudo-0-distributivity. Hence, assume that an algebra $\mathcal{A}$ satisfies the P0D-SCHEME and let $\alpha \cap \beta = \alpha \cap \gamma = \triangle$, for some $\alpha, \beta, \gamma \in \operatorname{Con} \mathcal{A}$.

Take $(x, y) \in (\alpha \vee \beta) \wedge \gamma$. As $\mathcal{A}$ is conditionally permutable, we have $\alpha \vee \beta = \alpha \circ \beta$ and hence there exists a $z \in A$ with $(x, z) \in \alpha, (z, y) \in \beta$. Since $(x, y) \in \gamma$, we can apply the P0D-SCHEME (see Figure 9) and this gives $x = z$. Thus we get

$(x,y) = (z,y) \in \beta \wedge \gamma$, proving $(\alpha \vee \beta) \wedge \gamma \leq \beta \wedge \gamma$. As the converse inequality is obvious, we obtain $(\alpha \vee \beta) \wedge \gamma = \beta \wedge \gamma$. Thus (PD_0) holds in $\operatorname{Con} \mathcal{A}$. □

In [11] there is proved that a modular lattice L with 0 is 0-distributive if and only if it has no sublattice isomorphic to one the lattices shown in Figure 10. (See also [15].)

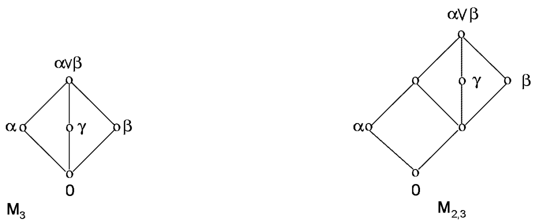

Figure 10

Lemma 5.5. *Let L be a modular lattice L with* 0*. If L pseudo*-0-*distributive, then it is* 0-*distributive, too.*

PROOF: Let L be a modular pseudo-0-distributive lattice. Choosing the elements α, β and γ in the above lattices such that $\alpha \wedge \beta = \alpha \wedge \gamma = 0$ (as shown in Figure 10), in both the cases the equality $(\alpha \vee \beta) \wedge \gamma = \beta \wedge \gamma$ is not satisfied. Hence, L contains no sublattice isomorphic to the above M_3 or $M_{2,3}$. Therefore, in view of [11], L is 0-distributive. □

Corollary 5.6. *Let $\mathcal{A} = (A, F)$ be a congruence modular conditionally permutable algebra. Then the following assertions are equivalent:*

(i) $\operatorname{Con} \mathcal{A}$ *is* 0-*distributive;*
(ii) $\operatorname{Con} \mathcal{A}$ *is pseudo*-0-*distributive;*
(iii) $\mathcal{A}$ *satisfies the P0D-SCHEME.*

PROOF: By Lemma 5.1, (i) implies (iii), and in view of Theorem 5.4, (iii) implies (ii). Finally, Lemma 5.5 gives that (ii) implies (i) and this completes the proof. □

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