Biharmonic morphisms

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Abstract. Let $(X, \mathcal{H})$ and $(X', \mathcal{H}')$ be two strong biharmonic spaces in the sense of Smyrnelis whose associated harmonic spaces are Brelot spaces. A biharmonic morphism from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$ is a continuous map from X to X' which preserves the biharmonic structures of X and X'. In the present work we study this notion and characterize in some cases the biharmonic morphisms between X and X' in terms of harmonic morphisms between the harmonic spaces associated with $(X, \mathcal{H})$ and $(X', \mathcal{H}')$ and the coupling kernels of them.

Keywords: harmonic space, harmonic morphism, biharmonic space, biharmonic function, biharmonic morphism

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1. Introduction

The notion of a harmonic morphism (also called harmonic map) between two harmonic spaces was introduced by Constantinescu and Cornea in 1965 as a natural generalization of holomorphic mappings between Riemann surfaces (see [4]). This notion was later extended by Fuglede to the setting of Riemannian manifolds in [9] and to the theory of finely harmonic functions in [12]. Csink, Fitzsimmons and Øksendal ([5], [6]) also gave a probabilistic interpretation of this notion.

Our main purpose in this work is to extend the notion of a harmonic morphism to the axiomatic theory of biharmonic functions.

We recall that the axiomatic theory of biharmonic functions, inspired by the classical biharmonic equation $\Delta^2 u = 0$, was developed by E.P. Smyrnelis in [14] and [15] and applies more generally to equations of the type $L_1 L_2 u = 0$, where L_1 and L_2 are two elliptic or parabolic differential operators of second order on an open subset of $\mathbb{R}^n$. In this theory, a harmonic space is a given locally compact space X equipped with a sheaf $\mathcal{H}$ of linear spaces of pairs of real continuous functions on the open subsets of X and satisfying some axioms. With such a space, two Bauer harmonic spaces are associated. Many results of classical or axiomatic potential theories were extended by Smyrnelis to the setting of the biharmonic space theory.

In this work we study the notion of biharmonic morphisms, that is, mappings between biharmonic spaces which preserve the biharmonic structures. We will prove that the biharmonic morphisms between two biharmonic spaces $(X, \mathcal{H})$ and

$(X', \mathcal{H}')$ are exactly the harmonic morphisms of harmonic spaces associated with these spaces which act suitably on the coupling kernels. At the end of this work we will give a characterization of biharmonic morphisms in the classical case of an open set in $\mathbb{R}^n$ and between Riemannian manifolds.

Let us also point out, according to Bouleau [1] et [2], that with a strong biharmonic space there is associated a couplage of two diffusion processes (X_t) and (Y_t), which are themselves associated with semi-groups (P_t) and (Q_t). This fact allows us to look for a stochastic characterization of biharmonic morphisms. We will come back to this question in a subsequent work.

Throughout this work the word *function* means, unless otherwise stated, a function with values in $\overline{\mathbb{R}}$. If (f_1, g_1) and (f_2, g_2) are two pairs of functions on a set E, we adopt the following definitions concerning the product order:

$$(f_1, g_1) \geq (f_2, g_2) \iff f_1 \geq f_2,\ g_1 \geq g_2,$$
$$(f_1, g_1) > (f_2, g_2) \iff f_1 > f_2,\ g_1 > g_2,$$

and we simply write $(f, g) \geq 0$ (resp. $(f, g) > 0$) instead of $(f, g) \geq (0, 0)$ (resp. $(f, g) > (0, 0)$).

If X is a locally compact space, we denote by $\overline{A}$ and ∂A, respectively, the closure and the boundary of A in the Alexandroff compactification $\overline{X}$ of X.

The notation used in this work and concerning the biharmonic spaces will be as in the work of Smyrnelis which is quoted in the references.

The results of this work can be easily extended in a natural way to polyharmonic spaces of any order, the biharmonic case was considered for its simplicity.

2. Preliminary results

In this section we consider a strong biharmonic space $(X, \mathcal{H})$ in the sense of Smyrnelis [14] whose associated harmonic spaces (of Bauer) are denoted by $(X, \mathcal{H}_1)$ and $(X, \mathcal{H}_2)$. We recall that for every open subset U of X, a function $h \in \mathcal{H}_1(U)$ if and only if $(h, 0) \in \mathcal{H}(U)$ and that a function $k \in \mathcal{H}_2(U)$ if and only if, for every $x \in U$, there exists an open neighborhood U_x of x contained in U and a function u on U_x such that $(u, k) \in \mathcal{H}(U_x)$.

We denote by $\mathcal{U}(X)$ and $\mathcal{U}_i(X)$ (resp. $\mathcal{U}^+(X)$ and $\mathcal{U}_i^+(X)$), $i = 1, 2$, the cones of $\mathcal{H}$-hyperharmonic pairs and $\mathcal{H}_i$-hyperharmonic functions (non-negative, resp.) on X. We also denote by $\mathcal{S}^+(X)$ and $\mathcal{S}_i^+(X)$, $i = 1, 2$, the cones of non-negative $\mathcal{H}$-superharmonic pairs and non-negative $\mathcal{H}_i$-superharmonic functions on X. If f is a function defined on an open subset U of X, we denote by $\widehat{f}$ its lower semicontinuous regularization, i.e., the greatest lower semicontinuous minorant of f in U.

Proposition 2.1 ([16, lemme 11.6]). *Let $v \in \mathcal{U}_2^+(X)$. Then the function*

$$u_v = \widehat{\inf}\{u \in \mathcal{U}_1^+(X) : (u, v) \in \mathcal{U}^+(X)\}$$

is a non-negative $\mathcal{H}_1$-hyperharmonic function on X and the pair (u_v, v) is $\mathcal{H}$-hyperharmonic on X.

Definition 2.2. The function u_v in the above proposition is called the *pure hyperharmonic function of order* 2 *associated with* v.

A pair $(u, v) \in \mathcal{U}^+(X)$ is said to be *pure* if $u = u_v$.

Remarks. 1. If (h, k) is a pure pair on X and if k is $\mathcal{H}_2$-harmonic on an open subset ω of X, then (h, k) is $\mathcal{H}$-biharmonic on ω (see [8]).

2. If there exists a function $u \in \mathcal{S}_1^+(X)$ such that $(u, v) \in \mathcal{S}^+(X)$, then $u_v \in \mathcal{S}_1^+(X)$ and u_v is even an $\mathcal{H}_1$-potential. We deduce from this fact that if (h, k) is a non-negative biharmonic pair, then u_k is the potential part in the Riesz decomposition of the non-negative $\mathcal{H}_1$-superharmonic function h.

The following theorem can be found in [2]:

Theorem 2.3. *There exists a unique Borel kernel V on X with the following properties:*

(i) *For any continuous function φ on X with compact support K, the function $V\varphi$ is $\mathcal{H}_1$-harmonic in the complement of K.*
(ii) *For every function $v \in \mathcal{U}_2^+(X)$, Vv is the pure hyperharmonic function of order 2 associated with v.*

We recall that a *Borel kernel* on a topological space E is a mapping $N : E \times \mathcal{B}(E) \longrightarrow \overline{\mathbb{R}}_+$ such that:

1. For every $A \in \mathcal{B}(E)$, the function $x \mapsto N(x, A)$ is Borel measurable on E.
2. For every $x \in E$, the function $A \mapsto N(x, A)$ is a non-negative measure on $\mathcal{B}(E)$.

Here $\mathcal{B}(E)$ is the σ-algebra of Borel subsets of E. For a non-negative Borel function f we denote by Nf or $N(f)$ the function $\int f(y)N(\cdot, dy)$.

The kernel V in the above theorem will be called the *coupling kernel* of the harmonic spaces $(X, \mathcal{H}_1)$ and $(X, \mathcal{H}_2)$ (or simply the biharmonic space $(X, \mathcal{H})$).

The interest of pure $\mathcal{H}$-hyperharmonic pairs lies in the following theorem, which likewise can be found in [2], and which is essential, in particular, for the integral representation of $\mathcal{H}$-potentials and non-negative $\mathcal{H}$-harmonic functions on X.

Theorem 2.4. *Let $(s_1, s_2) \in \mathcal{S}^+(X)$. Then we have $V(s_2) \prec s_1$, i.e., there exists a function $t \in \mathcal{S}_1^+(X)$ such that*

$$s_1 = t + V(s_2).$$

Lemma 2.5. *There exists a positive, finite and continuous $\mathcal{H}_2$-potential whose associated pure hyperharmonic function of order 2 is a strict finite and continuous $\mathcal{H}_1$-potential.*

Proof: Let (p_0, q_0) be a positive, finite and continuous $\mathcal{H}$-potential in X. By Theorem 2.4 there exists a function $t \in \mathcal{S}_1^+(X)$ such that $p_0 = t + V(q_0)$, from which we deduce that t and $V(q_0)$ are finite and continuous. It is easy to verify that p_0 is a strict $\mathcal{H}_1$-potential. □

More generally, if (p_0, q_0) is a positive, finite and continuous $\mathcal{H}$-superharmonic pair, then the pure hyperharmonic function of order 2 associated with q_0 is a strict finite and continuous $\mathcal{H}_1$-potential.

Let (p_0, q_0) be a positive, finite and continuous pure $\mathcal{H}$-superharmonic pair. We know by [4, Theorem 8.1.1 and Exercise 8.2.3] that there exists a unique Borel kernel W on X such that:

(i) $W1 = p_0$,
(ii) for every non-negative continuous function f with compact support, Wf is $\mathcal{H}_1$-harmonic in the complement of the support of f.

Theorem 2.6. *For every non-negative Borel function f on X we have*

$$Vf = W\left(\frac{f}{q_0}\right).$$

Proof: Let us consider the Borel kernel W' defined on X by $W'f = V(fq_0)$ for every non-negative Borel function on X. Since the pair (p_0, q_0) is pure we have $Vq_0 = p_0$, hence $W'1 = p_0$. Hence, according to the properties of W', it follows that $W' = W$. The theorem is proved. □

Example. Consider the classical biharmonic space $(\mathbb{R}^n, \mathcal{H})$, $n \geq 1$, where the biharmonic sheaf $\mathcal{H}$ is given for every open subset ω of $\mathbb{R}^n$ by

$$\mathcal{H}(\omega) = \{(u, v) \in [\mathcal{C}^2(\omega)]^2 : \Delta u = -v, \Delta v = 0\}.$$

The associated harmonic spaces are identical to the classical Laplace harmonic space and hence satisfy the hypotheses of this section. This space is strong if and only if $n \geq 5$ (see [8]). The kernel V of Theorem 2.3 is given in this case by

$$Vf(x) = \frac{1}{\sigma_n(n-2)} \int \frac{f(y)}{\|x-y\|^{n-2}}\, dy$$

for every non-negative Borel function f on $\mathbb{R}^n$ and every $x \in \mathbb{R}^n$, where σ_n is the area of the unit sphere in $\mathbb{R}^n$.

If $X = \Omega$ is a Green domain in $\mathbb{R}^n$, and if the space X equipped with the sheaf induced on X by the sheaf $\mathcal{H}$ is strong, then the kernel V is given by

$$Vf(x) = \int G_\Omega(x,y)f(y)\,dy$$

for every non-negative Borel function f on Ω and every $x \in \Omega$, where G_Ω denotes the Green kernel of Ω normalized in the sense that, for all $y \in \Omega$, we have $\Delta G_\Omega(\cdot, y) = -\epsilon_y$ in the distributional sense, where ϵ_y is the Dirac measure at y.

According to [8], we know that if Ω is a domain in $\mathbb{R}^n$, $n \geq 1$, equipped with the sheaf induced by $\mathcal{H}$, then Ω is a strong biharmonic space if and only if for every (or for some) $y \in \Omega$, the pure hyperharmonic function of order 2 associated with $G_\Omega(\cdot, y)$ is superharmonic (because the harmonic spaces associated with Ω are symmetric).

Moreover, it is not difficult to see that a bounded domain Ω in $\mathbb{R}^n$ is strong and there exist positive pure biharmonic pairs on Ω. On the other hand, if Ω is not bounded, there may not exist any positive biharmonic pair, as it is the case if $\Omega = \mathbb{R}^n$ (see [8]).

We end this section by the following two lemmas which will be useful later on:

Lemma 2.7. *Let (u,v) be a pure $\mathcal{H}$-hyperharmonic pair on X and (U_n) be an increasing sequence of open sets covering X. For each n, let u_n be the pure hyperharmonic function of order 2 associated with v on U_n. Then we have $u = \sup_n u_n$.*

Proof: The pair (u,v) is $\mathcal{H}$-hyperharmonic in X, hence, for every integer n, we have $u \geq u_n$. Thus $u \geq \sup_n u_n$. On the other hand it is not difficult to verify that if $n \geq m$ then $u_n \geq u_m$ in U_m and therefore the pair $(\sup_n u_n, v) = \sup_n(u_n, v)$ is $\mathcal{H}$-hyperharmonic on every open U_n, hence on X. We deduce that $\sup_n u_n \geq u$. The lemma is proved. $\square$

We denote by $\mathcal{P}'_c(X)$ the set of finite and continuous $\mathcal{H}_2$-potentials whose associated pure hyperharmonic function is finite and continuous.

Lemma 2.8. *For every non-negative $\mathcal{H}_2$-hyperharmonic function, there exists an increasing sequence (q_n) of elements of $\mathcal{P}'_c(X)$ such that $v = \sup_n q_n$.*

Proof: We know that every non-negative $\mathcal{H}_2$-hyperharmonic function is the supremum of an increasing sequence of finite and continuous $\mathcal{H}_2$-potentials on X. Hence, to prove the lemma, it suffices to prove that every $\mathcal{H}_2$-potential is the supremum of an increasing sequence of elements of $\mathcal{P}'_c(X)$. Let q be a finite and continuous $\mathcal{H}_2$-potential on X and (p_0, q_0) a finite, positive and continuous $\mathcal{H}$-potential on X. We have $q = \sup_n \min(q, nq_0)$ and, according to Theorem 2.4, $V(\min(q, nq_0)) \prec np_0$ because $(np_0, \min(q, nq_0))$ is a non-negative $\mathcal{H}$-superharmonic pair, hence $\min(q, nq_0) \in \mathcal{P}'_c(X)$. $\square$

Remark. Lemma 2.8 allows us to prove easily Theorem 2.3 of Bouleau.

3. Biharmonic morphisms

From now on, both $(X, \mathcal{H})$ and $(X', \mathcal{H}')$ will be Brelot biharmonic spaces, that is, biharmonic spaces whose associated harmonic spaces are Brelot spaces.

Definition 3.1. A *biharmonic morphism* from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$ is a continuous mapping φ from X to X' such that, for every open subset U of X' and every $\mathcal{H}'$-hyperharmonic pair (u, v) on U, the pair $(u \circ \varphi, v \circ \varphi)$ is $\mathcal{H}$-hyperharmonic in $\varphi^{-1}(U)$.

We recall that if $(X_1, \mathcal{K}_1)$ and $(X_2, \mathcal{K}_2)$ are two harmonic spaces (in the sense of Constantinescu and Cornea [4]), a harmonic morphism from $(X_1, \mathcal{K}_1)$ to $(X_2, \mathcal{K}_2)$ is a continuous mapping φ from X_1 to X_2 such that, for every open subset U of X_2 and every $\mathcal{K}_2$-hyperharmonic function u on U, the function $u \circ \varphi$ is $\mathcal{K}_1$-hyperharmonic on $\varphi^{-1}(U)$, or equivalently, according to [5], for every $\mathcal{K}_2$-harmonic function u on U, the function $u \circ \varphi$ is $\mathcal{K}_1$-harmonic on $\varphi^{-1}(U)$.

It follows easily from Definition 3.1 that if the pair (h, k) is $\mathcal{H}'$-biharmonic on an open subset U of X', then the pair $(h \circ \varphi, k \circ \varphi)$ is $\mathcal{H}$-biharmonic on $\varphi^{-1}(U)$.

It is clear that if φ and ψ are two biharmonic morphisms from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$ and from $(X', \mathcal{H}')$ to $(X'', \mathcal{H}'')$ respectively, then $\psi \circ \varphi$ is a biharmonic morphism from $(X, \mathcal{H})$ to $(X'', \mathcal{H}'')$.

A biharmonic isomorphism from $(X, \mathcal{H})$ in $(X', \mathcal{H}')$ is a bijection φ from X onto X' such that φ and φ^{-1} are biharmonic morphisms.

In the same way as in [3], we can prove the following

Theorem 3.2. *Let φ be a continuous mapping from X to X'. Then φ is a biharmonic morphism from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$ if and only if, for every open subset U of X' and every $\mathcal{H}'$-harmonic pair (h, k) in U, the pair $(u \circ \varphi, v \circ \varphi)$ is $\mathcal{H}$-harmonic on $\varphi^{-1}(U)$.*

Proposition 3.3. *Let φ be a biharmonic morphism from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$. Then*

(i) *φ is a harmonic morphism from $(X, \mathcal{H}_1)$ to $(X', \mathcal{H}'_1)$,*
(ii) *φ is a harmonic morphism from $(X, \mathcal{H}_2)$ to $(X', \mathcal{H}'_2)$.*

Proof: Let U be an open subset of X' and u an $\mathcal{H}'_1$-hyperharmonic function on U. Then the pair $(u, 0)$ is $\mathcal{H}'$-hyperharmonic on U, hence $(u \circ \varphi, 0)$ is $\mathcal{H}$-hyperharmonic on $\varphi^{-1}(U)$, so that $u \circ \varphi$ is $\mathcal{H}'_1$-hyperharmonic on $\varphi^{-1}(U)$, which proves (i). Let v be an $\mathcal{H}'_2$-hyperharmonic function on U. Then for every $x \in \varphi^{-1}(U)$, there exist a neighborhood V_x of $\varphi(x)$ contained in U and a function u_x such (u_x, v) is $\mathcal{H}'$-hyperharmonic on V_x, thus the pair $(u_x \circ \varphi, v \circ \varphi)$ is $\mathcal{H}$-hyperharmonic on $\varphi^{-1}(V_x)$. We deduce from this that the function $v \circ \varphi$ is $\mathcal{H}$-hyperharmonic on $\varphi^{-1}(U)$. This proves (ii). □

The converse of the above proposition is not true in general as demonstrated by the following example:

It is well known according to [10] that the harmonic morphisms of $\mathbb{R}^2 = \mathbb{C}$, equipped with the classical harmonic structure defined by the Laplacian (see Example of Section 2), are exactly the functions $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ such that φ or $\bar{\varphi}$ is holomorphic. In particular the constant functions are harmonic morphisms from $\mathbb{C}$ to itself. However, it is easy to verify that these are not biharmonic morphisms. These morphisms seem to be trivial, we are going to give non-trivial ones.

Let B denote the unit ball in $\mathbb{C}$, and let $f : B \longrightarrow B$ be defined by $f(z) = z^2$. One can easily verify that the pair (u, v) of functions defined by

$$u(z) = \frac{3}{16} + \frac{1}{16}|z|^4 - \frac{1}{4}|z|^2$$

and

$$v(z) = 1 - |z|^2$$

is a pure superharmonic pair. The function $v \circ f$ is superharmonic because f is holomorphic (hence a harmonic morphism). On the other hand we have

$$u \circ f(z) = \frac{3}{16} - \frac{1}{4}|z|^4 + \frac{1}{16}|z|^8.$$

A straightforward calculation yields

$$\Delta(u \circ f)(z) = 4|z|^2(|z|^2 - 1),$$

but we do not have

$$\Delta(u \circ f)(z) \leq -v \circ f(z)$$

for every $z \in B$ as one can see by taking $|z|$ close to 0, hence the pair $(u \circ f, v \circ f)$ is not hyperharmonic (recall that if the pair (u, v) is hyperharmonic on a domain $\Omega \subset \mathbb{R}^n$ and if $u \not\equiv +\infty$, then $\Delta u \leq -v$ in the distributional sense).

These examples show that conditions (i) and (ii) of Proposition 3.3 do not characterize the biharmonic morphisms. We still need a supplementary condition related to the coupling kernels of biharmonic spaces $(X, \mathcal{H})$ and $(X', \mathcal{H}')$ as we will show in Section 4.

Theorem 3.4. *Let φ be a biharmonic morphism from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$. If there exists a positive pure $\mathcal{H}'$-superharmonic pair (p_0, q_0) such that $(p_0 \circ \varphi, q_0 \circ \varphi)$ is a pure $\mathcal{H}$-potential then, for each pure $\mathcal{H}'$-hyperharmonic pair (u, v) on X', the pair $(u \circ \varphi, v \circ \varphi)$ is a pure $\mathcal{H}$-hyperharmonic pair on X.*

Proof: Let us write $(p, q) = (p_0 \circ \varphi, q_0 \circ \varphi)$. Then p and p_0 are strict potentials on X and X' (with respect to the harmonic sheaves $\mathcal{H}_1$ and $\mathcal{H}'_1$, respectively).

By [4, Exercise 8.2.3], there exist two Borel kernels W and W' on X and X' respectively such that:

(i) $W1 = p$, $W'1 = p_0$;
(ii) Wf (resp. $W'f$) is $\mathcal{H}_1$-harmonic (resp. $\mathcal{H}'_1$-harmonic) on $X \setminus \mathrm{Supp}(f)$ (resp. $X' \setminus \mathrm{Supp}(f)$), for every continuous function f with compact support on X (resp. on X').

Let (u, v) be a pure $\mathcal{H}'$-hyperharmonic pair on X'. Then, according to Theorem 2.4, we have $u = W'\frac{v}{q_0}$ and thus $u \circ \varphi = (W'\frac{v}{q_0}) \circ \varphi$. Now consider the operators W_1 and W_2 defined on the set of non-negative bounded Borel functions on X' by $W_1 g = W(g \circ \varphi)$ and $W_2 g = (W'g) \circ \varphi$. Then, combining Theorem 8.1.1 and Exercise 8.2.3 of [4], we easily get $W_1 = W_2$. In particular, we have $W_1(\frac{v}{q_0}) = W_2(\frac{v}{q_0})$, that is, $W(\frac{v \circ \varphi}{q}) = (W'\frac{v}{q_0}) \circ \varphi$. This proves the result. $\square$

4. Characterization of proper biharmonic morphisms

Throughout this section $(X, \mathcal{H})$ and $(X', \mathcal{H}')$ are two strong biharmonic Brelot spaces.

Lemma 4.1. *Let k be a positive $\mathcal{H}_2$-harmonic function on a relatively compact open subset U of X', ω be an $\mathcal{H}$-regular open set, $\omega \subset \overline{\omega} \subset U$, and h'_ω be the pure hyperharmonic function of order 2 associated with k in ω. Then $\lim_{x \to \xi} h'_\omega(x) = 0$ for each $\xi \in \partial\omega$.*

Proof: It is easy to verify that the pair (h'_ω, k) is nothing but the solution of the Riquier problem in ω for the boundary data $(0, k|_{\partial\omega})$. This proves the lemma. $\square$

We say that a function $f : X \longrightarrow X'$ is *proper* if the inverse image under f of any compact is compact.

Theorem 4.2. *If φ is a surjective proper biharmonic morphism from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$, then for every $\mathcal{H}'$-regular relatively compact open subset ω of X' and every pure pair (u, v) on ω, the pair $(u \circ \varphi, v \circ \varphi)$ is pure on $\varphi^{-1}(\omega)$.*

Proof: Let ω be an $\mathcal{H}'$-regular relatively compact open subset of X' and k be a positive $\mathcal{H}'_2$-harmonic function in a neighborhood of $\overline{\omega}$. Then the pure hyperharmonic function h'_ω of order 2 associated with k on ω is a strict $\mathcal{H}'_1$-potential in ω (this follows obviously from Definition 2.2). Let us denote by h_1 the pure hyperharmonic function of order 2 associated with $k \circ \varphi$. Since the pair $(h'_\omega \circ \varphi, k \circ \varphi)$ is non-negative biharmonic on $\varphi^{-1}(\omega)$, we have $h'_\omega \circ \varphi \geq h_1$. On the other hand, the function $h'_\omega \circ \varphi - h_1$ is $\mathcal{H}_1$-harmonic on $\varphi^{-1}(\omega)$ and we have

$$\lim_{x \in \varphi^{-1}(\omega), x \to \xi} (h'_\omega \circ \varphi - h_1)(x) = 0$$

for every $\xi \in \partial\varphi^{-1}(\omega)$. Since the open set $\varphi^{-1}(\omega)$ is relatively compact, this implies, according to the minimum principle, that $h'_\omega \circ \varphi - h_1 = 0$ on $\varphi^{-1}(\omega)$. In other words the pair $(h'_\omega \circ \varphi, k \circ \varphi)$ is pure. Then, by Theorem 3.4, the pair $(u \circ \varphi, v \circ \varphi)$ is pure for every pure pair (u, v) on ω. □

Corollary 1. *Assume that there exists an increasing sequence (U_n) of $\mathcal{H}'$-regular open sets covering X'. If φ is a proper surjective biharmonic morphism from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$, then for every pure pair (u, v) on X', the pair $(u \circ \varphi, v \circ \varphi)$ is pure.*

Proof: Let (U_n) be an increasing sequence of $\mathcal{H}'$-regular open sets covering X' and let (u, v) be a pure pair. For every n, let us denote by u_n the pure hyperharmonic of order 2 associated with v in U_n. Then we have $u = \sup_n u_n$, hence $u \circ \varphi = \sup_n u_n \circ \varphi$. As the pairs $(u_n \circ \varphi, v \circ \varphi)$ are pure, it follows from Lemma 2.7 that $(u \circ \varphi, v \circ \varphi)$ is pure. □

Remark. If the topology of X' has a countable base, then it is known that there exists an increasing sequence (U_n) of $\mathcal{H}'$-regular open sets covering X'.

Corollary 2. *Assume that $(X', \mathcal{H}'_1)$ and $(X', \mathcal{H}'_2)$ are Brelot spaces having the same regular sets (this is the case if, for example, $\mathcal{H}'_1 = \mathcal{H}'_2$). If φ is a proper surjective biharmonic morphism from $(X, \mathcal{H})$ to $(X', \mathcal{H}')$, then for every pure pair (u, v) on X', the pair $(u \circ \varphi, v \circ \varphi)$ is pure.*

Proof: In fact, the assumptions of Corollary 1 are satisfied in this case because, in a Brelot harmonic space with positive potential, there exists an increasing sequence of regular sets covering the whole space. □

Remark. If we drop the hypothesis that φ is proper in Theorem 4.3, then the conclusion may fail. Indeed, let X be the unit ball in $\mathbb{R}^n$ and X' the unit ball in $\mathbb{R}^m$, where $n > m \geq 1$ and let $\varphi : X \longrightarrow X'$ be the projection defined by

$$\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_m).$$

Then it is clear that φ is a surjective biharmonic morphism, which is not proper. Let $\omega = \frac{1}{2}X'$ be the ball of radius $\frac{1}{2}$ in $\mathbb{R}^m$, and (u, v) be a pure pair in ω $(v > 0)$. Then $u = V_\omega v$, but $u \circ \varphi$ is not a potential on $\varphi^{-1}(\omega)$ because $u \circ \varphi$ has positive values on a part of $\partial\varphi^{-1}(\omega)$. Hence $(u \circ \varphi, v \circ \varphi)$ is not pure.

Proposition 4.3. *Let U be a relatively compact open subset of X' and (u', v') be an $\mathcal{H}'$-superharmonic pair in a neighborhood of $\overline{U}$. Then there exist an $\mathcal{H}'$-potential (p, q) and an $\mathcal{H}'$-superharmonic pair (u, v) in X' such that*

(i) *$(u, v) = (u', v') + (p, q)$ in U;*
(ii) *the pair (p, q) is $\mathcal{H}'$-harmonic in U;*
(iii) *if $(u', v') \geq 0$, one can choose $(u, v) \geq 0$ in X'.*

Proof: The proposition can be proved in the same manner as Theorem 3.2 of [4] for the harmonic case. □

Proposition 4.4. *Let $\varphi : X \longrightarrow X'$ be a continuous mapping. Assume that φ is a harmonic morphism from $(X, \mathcal{H}_1)$ to $(X', \mathcal{H}_1')$ and that for every pure positive $\mathcal{H}$-potential (p, q), the pair $(p \circ \varphi, q \circ \varphi)$ is pure. Then φ is a biharmonic morphism from X to X'.*

Proof: Let us remark first that under the hypothesis of the proposition, if (u, v) is a non-negative $\mathcal{H}'$-hyperharmonic pair on X', then $(u \circ \varphi, v \circ \varphi)$ is $\mathcal{H}$-hyperharmonic on X. In fact, this true for any $\mathcal{H}'$-potential because of the decomposition of Theorem 2.4 and the fact that φ is a harmonic morphism between the harmonic spaces $(X, \mathcal{H}_1)$ and $(X', \mathcal{H}_1')$. For a non-negative $\mathcal{H}'$-hyperharmonic pair (u, v) in X', it suffices to use the fact that (u, v) is the supremum of an increasing sequence of $\mathcal{H}'$-potentials. Suppose now that the assumptions of the proposition are satisfied, that U is an open subset of X' and (u, v) is an $\mathcal{H}'$-hyperharmonic pair on U. Let ω be an $\mathcal{H}'$-regular open subset of X' such that $\overline{\omega} \subset U$. Assume first that $(u, v) \geq 0$. According to Proposition 4.3, one can find an $\mathcal{H}'$-potential (p, q) on X', $\mathcal{H}'$-harmonic on ω, and an $\mathcal{H}'$-superharmonic non-negative pair (u_0, v_0) on X' such that $(u_0, v_0) = (u, v) + (p, q)$ on ω, and hence $(u_0 \circ \varphi, v_0 \circ \varphi) = (u \circ \varphi, v \circ \varphi) + (p \circ \varphi, q \circ \varphi)$ on $\varphi^{-1}(\omega)$. But the pair $(u_0 \circ \varphi, v_0 \circ \varphi)$ is $\mathcal{H}$-hyperharmonic on $\varphi^{-1}(\omega)$ and $(p \circ \varphi, q \circ \varphi)$ is $\mathcal{H}$-harmonic on $\varphi^{-1}(\omega)$, thus the pair $(u \circ \varphi, v \circ \varphi)$ is $\mathcal{H}$-hyperharmonic on $\varphi^{-1}(\omega)$. Since ω is arbitrary and $\mathcal{H}'$-regular subsets of X' form a base of X', it follows that the pair $(u \circ \varphi, v \circ \varphi)$ is $\mathcal{H}$-hyperharmonic on $\varphi^{-1}(U)$.

For an arbitrary pair (u, v), one can go back to the previous case by adding locally a suitable non-negative biharmonic pair to (u, v). □

Now we may prove the following

Theorem 4.5. *Assume that there exists an increasing sequence (U_n) of $\mathcal{H}'$-regular open sets covering X'. Let $\varphi : X \longrightarrow X'$ be a proper surjective continuous function. Assume also that φ is a harmonic morphism from $(X, \mathcal{H}_1)$ to $(X', \mathcal{H}_1')$ and from $(X, \mathcal{H}_2)$ to $(X', \mathcal{H}_2')$. Then the following conditions are equivalent:*

- (i) *φ is a biharmonic morphism;*
- (ii) *for every positive pure $\mathcal{H}$-potential (p, q), the pair $(p \circ \varphi, q \circ \varphi)$ is pure.*

Proof: The implication (i) $\Longrightarrow$ (ii) has been proved in Corollary 2 of Theorem 4.2. For the implication (iii) $\Longrightarrow$ (i), see Proposition 4.3. □

Let us denote by V and V' the coupling kernels of the biharmonic spaces $(X, \mathcal{H})$ and $(X', \mathcal{H}')$, respectively. In terms of coupling kernels, we have the following characterization of biharmonic morphisms:

Theorem 4.6. *Assume that there exists an increasing sequence (U_n) of $\mathcal{H}'$-regular sets covering X' and that for every relatively compact open subset U of X' there exists a positive $\mathcal{H}'$-harmonic function on U. Let $\varphi : X \longrightarrow X'$ be a proper surjective continuous function. Assume also that φ is a harmonic* *morphism from* $(X, \mathcal{H}_1)$ *to* $(X', \mathcal{H}'_1)$ *and from* $(X, \mathcal{H}_2)$ *to* $(X', \mathcal{H}'_2)$. *Then the following conditions are equivalent:*

(i) φ *is a biharmonic morphism;*

(ii) $(V'f) \circ \varphi = V(f \circ \varphi)$ *for every non-negative Borel function* f *on* X'.

Proof: The implication (ii) $\Longrightarrow$ (i) follows immediately by Theorems 4.5 and 3.3. In order to prove the implication (i) $\Longrightarrow$ (ii), let us denote by $\mathcal{P}'_c(X')$ the set of finite and continuous $\mathcal{H}'_2$-potentials whose associated pure hyperharmonic functions of order 2 are finite and continuous. Then, by the above theorem, we have $(V'q) \circ \varphi = V(q \circ \varphi)$ for each $q \in \mathcal{P}'_c(X')$. According to Lemma 2.8, we have $V'q = V(q \circ \varphi)$ for every $q \in \mathcal{P}_c(X')$. But the space of differences of finite and continuous potentials which vanish outside a compact K of X is dense in the space of finite and continuous functions with support contained in K, thus we have $V'f \circ \varphi = V(f \circ \varphi)$ for every finite and continuous function with compact support. Hence, by the monotone class theorem, we have $V'f \circ \varphi = V(f \circ \varphi)$ for every non-negative Borel function f on X. $\square$

We end this section by a characterization of biharmonic morphisms in the classical case. Let $n \geq 3$ be an integer. The kernel of couplage V relative to the biharmonic space $\mathbb{R}^n$ equipped with the sheaf $\mathcal{H}$ defined by

$$\mathcal{H}(\omega) = \{(u, v) \in [\mathcal{C}(\omega)]^2 : \Delta u = -v, \Delta v = 0\}$$

is given by

$$Vf(x) = \frac{1}{\sigma_n (n-2)} \int \frac{f(y)}{|x-y|^{n-2}}\, dy$$

for every non-negative Borel function f on $\mathbb{R}^n$, where σ_n is the area of the unit sphere in $\mathbb{R}^n$ (see [8]).

We also recall the following characterization of harmonic morphisms of $\mathbb{R}^n$ equipped with the classical sheaf associated with the Laplace operator (see [10]):

Theorem 4.7. *For a function* φ *from a domain* U *of* $\mathbb{R}^n$ *to* $\mathbb{R}^m$, $m, n \geq 2$, *the following conditions are equivalent.*

(i) φ *is a harmonic morphism.*

(ii) *The components* φ_j $(1 \leq j \leq m)$ *of* φ *and the functions* $\varphi_i \varphi_j$ $(i \neq j)$, $\varphi_i^2 - \varphi_j^2$ $(1 \leq i, j \leq m)$ *are harmonic on* U.

(iii) *The components* φ_j $(1 \leq j \leq m)$ *of* φ *are harmonic on* U *and* $\langle \nabla\varphi_i, \nabla\varphi_j \rangle = \delta_{ij} |\nabla\varphi_i|^2$ *on* U, *where* $\langle \nabla\varphi_i, \nabla\varphi_j \rangle$ *is the inner product of* $\nabla\varphi_i$ *and* $\nabla\varphi_j$.

Let us also recall that if φ is a non-constant harmonic morphism from a domain U of $\mathbb{R}^n$ to $\mathbb{R}^m$, then $n \geq m$ and $\varphi(U)$ is an open subset of $\mathbb{R}^m$ (cf. [10, Theorem 4]).

For biharmonic morphisms we can now, using Theorems 4.6 and 4.7, state the following

Theorem 4.8. *For a non-constant proper function φ from a domain U of $\mathbb{R}^n$, $n \geq 5$ (or only $n \geq 2$ if U is bounded), to $\mathbb{R}^m$, $m \geq 5$, the following conditions are equivalent.*

(i) *φ is a biharmonic morphism.*

(ii) *The components φ_j $(1 \leq j \leq m)$ of φ and the functions $\varphi_i\varphi_j$ $(i \neq j)$ and $\varphi_i^2 - \varphi_j^2$ are harmonic in U, and*

$$\int_{\varphi(U)} G_{\varphi(U)}(\varphi(x), y) f(y)\, dy = \int G_U(x, y) f(\varphi(y))\, dy$$

for every non-negative Borel function f on $\mathbb{R}^m$ and every $x \in U$.

(iii) *The components φ_j $(1 \leq j \leq m)$ of φ are harmonic on U and one has $\langle \nabla\varphi_i, \varphi_j \rangle = \delta_{ij} |\nabla\varphi_i|^2$ on U and*

$$\int_{\varphi(U)} G_{\varphi(U)}(\varphi(x), y) f(y)\, dy = \int G_U(x, y) f(\varphi(y))\, dy$$

for each non-negative Borel function f on $\mathbb{R}^m$ and every $x \in U$.

Remark. All results of this section hold for any non-constant open harmonic morphism $\phi : X \to X'$ provided that the inverse image by ϕ of any compact subset of $\phi(X)$ is compact. Let us recall here that a harmonic morphism $\phi : X \to X'$ is open if the points of X' (or just of $f(X)$) are strongly polar (see [10]).

5. Biharmonic morphisms between Riemannian manifolds

All Riemannian manifolds considered in the sequel are assumed to be connected, second countable and infinitely differentiable.

Let M be a Riemannian manifold and Δ_M be its Laplace-Beltrami operator. We shall say that a function u on M is *harmonic* if it is a solution of the harmonic equation

$$\Delta_M u = 0$$

on M. It follows that u is of class C^∞. The constant functions are of course harmonic. As shown by R.-M. Hervé [12, Chapter 7], the sheaf of harmonic functions in this sense turns the manifold M into a Brelot harmonic space (in the slightly extended sense adopted in [4] in order to include the case when M is compact).

Let M and N be two Riemannian manifolds. A continuous mapping $\phi : M \longrightarrow N$ is called a *harmonic morphism* if $v \circ \phi$ is a harmonic function on $\phi^{-1}(\omega)$ for every function v which is harmonic on an open set $\omega \subset N$ (such that $\phi^{-1}(V) \neq \emptyset$).

A function u on a Riemannian manifold M is called *biharmonic* if u is of class C^4 and $\Delta_M^2 u = 0$. If we identify the harmonic functions u with the biharmonic pairs $(u, -\Delta_M u)$ as in $\mathbb{R}^n$, then M endowed with the sheaf $\mathcal{H}_M$ of the biharmonic pairs is a biharmonic space whose associated harmonic spaces are identical to those defined above by the harmonic functions.

Let M and N be two Riemannian manifolds. A continuous mapping $\phi : M \longrightarrow N$ is called a *biharmonic morphism* if f is a biharmonic morphism between the biharmonic spaces defined on M and N by Δ_M and Δ_N, respectively.

The following result follows easily from [10, Lemma 4]:

Theorem 5.1. *Let M and N be two Riemannian manifolds and $\phi : M \longrightarrow N$ a non-constant harmonic morphism. Then ϕ is a biharmonic morphism if and only if one has*

$$\Delta_M(f \circ \phi) = (\Delta_N f) \circ \phi$$

for any C^2-function on N.

It follows from this theorem that a biharmonic morphism

$$\phi : M \longrightarrow N$$

between two Riemannian manifolds is not only a continuous mapping which preserves biharmonic functions, but it also satisfies

$$\Delta_M(u \circ \phi) = (\Delta_N u) \circ \phi$$

for any biharmonic function u on N.

As an immediate consequence of the above theorem, we have the following characterization of biharmonic morphisms between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$:

Theorem 5.2. *For a non-constant function φ from a domain U in $\mathbb{R}^m$, $m \geq 5$ (or only $n \geq 2$ if U is bounded), to $\mathbb{R}^n$, $n \geq m$, the following conditions are equivalent.*

(i) *φ is a biharmonic morphism.*
(ii) *The components φ_j $(1 \leq j \leq n)$ of φ, the functions $\varphi_i\varphi_j$ $(i \neq j)$, and the functions $\varphi_i^2 - \varphi_j^2$ are harmonic on U, and moreover $\Delta\varphi_j^2 = 2$ for some and hence any $j = 1, \ldots, n$.*
(iii) *The components φ_j $(1 \leq j \leq m)$ of φ, the functions $\varphi_i\varphi_j$ $(i \neq j)$, and the functions $\varphi_i^2 - \varphi_j^2$ are harmonic in U, and moreover $|\Delta\varphi_j| = 1$ for some and hence any $j = 1, \ldots, n$.*

We say that a Riemannian manifold M is *strong* if the biharmonic space $(M, \mathcal{H}_M)$ is strong. A strong Riemannian manifold is necessarily parabolic, that is, it possesses a Green kernel.

Let M be a strong Riemannian manifold and let us denote by V_M its coupling kernel, i.e. the kernel associated with the sheaf $\mathcal{H}_M$ as in Theorem 2.3. Then it is easy to see, as in the biharmonic space $\mathbb{R}^n$, $n \geq 5$, that, in the distributional sense, $\Delta_M(V_M f) = -f$ for any non-negative Borel function on M such that $V_M f$ is superharmonic.

Theorem 5.3. *Let φ be a harmonic morphism between two Riemannian manifolds M and N. Denote by V_M and V_N the coupling kernels relative to M and N, respectively. Then φ is a biharmonic morphism if and only if one has $V_N(q) \circ \varphi \geq V_M(q \circ \varphi)$ for every non-negative hyperharmonic function q on N, and*

$$\Delta_M[V_N(f) \circ \varphi - V_M(f \circ \varphi)] = 0$$

for all non-negative Borel functions f on N such that $V_N(f) \neq +\infty$.

Proof: Assume first that $\varphi : M \longrightarrow N$ is a biharmonic morphism and let q be a non-negative hyperharmonic function on N. Since $(V_N q, q)$ is non-negative $\mathcal{H}_N$-hyperharmonic, the pair $(V_N(q)\circ\varphi, q\circ\varphi)$ is non-negative $\mathcal{H}_M$-hyperharmonic on M. Hence, by Theorem 2.3 we have $V_N(q) \circ \varphi \geq V_M(q \circ \varphi)$. Now let f be a non-negative Borel function on N such that $V_M(f \circ \varphi) \neq +\infty$. Then we have

$$\Delta_M(V_M(f \circ \varphi)) = f \circ \varphi,$$

and by Theorem 5.1

$$\Delta_M((V_N f) \circ \varphi) = f \circ \varphi$$

in the distributional sense. Hence

$$\Delta_M[(V_N(f) \circ \varphi) - V_M(f \circ \varphi)] = 0.$$

Conversely, let $\varphi : M \longrightarrow N$ be a harmonic morphism satisfying the assumptions of Theorem 5.3. Let (p, q) be a finite non-negative $\mathcal{H}_N$-superharmonic pair on N. By Theorem 2.4 we have

$$(p, q) = (s, 0) + (V_N q, q)$$

for some non-negative $\mathcal{H}_{N1}$-superharmonic function on N. Therefore

$$\begin{aligned}(p \circ \varphi, q \circ \varphi) &= (s \circ \varphi, 0) + ((V_N q) \circ \varphi, q \circ \varphi) \\ &= (V_N(q) \circ \varphi - V_M(q \circ \varphi) + s \circ \varphi, 0) + (V_M(q \circ \varphi), q \circ \varphi).\end{aligned}$$

By the hypothesis, every term of the last of these equalities is $\mathcal{H}_M$-superharmonic, hence $(p\circ\varphi, q\circ\varphi)$ is $\mathcal{H}_M$-superharmonic on M. Moreover, if (p, q) is $\mathcal{H}$-biharmonic on an open subset U of N, then it follows from Remark 1 of Section 2 that $(p\circ\varphi, q\circ\varphi)$ is $\mathcal{H}_M$-biharmonic on $\varphi^{-1}(U)$. Since any non-negative $\mathcal{H}_N$-hyperharmonic pair on N is the supremum of an increasing sequence (u_n, v_n) of finite $\mathcal{H}_N$-hyperharmonic pairs, it follows that for any non-negative $\mathcal{H}_N$-hyperharmonic pair (u, v) on N we have the same conclusions for the pair $(u \circ \varphi, v \circ \varphi)$. Now let (u, v) be an $\mathcal{H}$-biharmonic pair on an open subset ω of N, $x \in \omega$ and ω' be a relatively compact open neighborhood of x such that $\overline{\omega'} \subset \omega$. Then by Proposition 4.3 there exist two non-negative $\mathcal{H}_N$-superharmonic pairs (p, q) and (s, t) on N such that

$$(s, t) = (u, v) + (p, q)$$

in ω and that the pair (p, q) is $\mathcal{H}_N$-biharmonic on ω'. It follows from above that $(u \circ \phi, v \circ \phi)$ is $\mathcal{H}_M$ biharmonic on $\varphi^{-1}(\omega')$. Since x and ω' are arbitrary, we conclude that the pair $(u \circ \varphi, v \circ \varphi)$ is $\mathcal{H}_M$-biharmonic on $\varphi^{-1}(\omega)$. The proof is complete. □

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