Cohomology of $BO(n_1) \times \cdots \times BO(n_m)$ with local integer coefficients

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Abstract. Let $\mathcal{Z}$ be a set of all possible nonequivalent systems of local integer coefficients over the classifying space $BO(n_1) \times \cdots \times BO(n_m)$. We introduce a cohomology ring $\bigoplus_{\mathcal{G}\in\mathcal{Z}} H^*(BO(n_1)\times\cdots\times BO(n_m);\mathcal{G})$, which has a structure of a $\mathbb{Z}\oplus(\mathbb{Z}_2)^m$-graded ring, and describe it in terms of generators and relations. The cohomology ring with integer coefficients is contained as its subring. This result generalizes both the description of the cohomology with the nontrivial system of local integer coefficients of $BO(n)$ in [Č] and the description of the cohomology with integer coefficients of $BO(n_1) \times \cdots \times BO(n_m)$ in [M].

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1. Introduction

The cohomology rings of the classifying spaces for the groups $O(n)$ and $SO(n)$ with $\mathbb{Z}_2$ and $\mathbb{Z}[1/2]$ coefficients were well known very long ago, see [MS]. E. Thomas found the group structure of $H^*(BO(n))$ with integer and $\mathbb{Z}_{2^m}$ coefficients in 1960 [T]. The more complicated cohomology ring structure for integer coefficients was described in terms of generators and relations independently by E.H. Brown [B] and M. Feshbach [F] in 1982.

Since $\pi_1(BO(n)) = \mathbb{Z}_2$, there are two nonequivalent systems of local coefficients over $BO(n)$, nontrivial one determined by the first Stiefel-Whitney class of the universal vector bundle over $BO(n)$. The cohomology ring of $BO(n)$ with both systems of local coefficients was described by M. Čadek in 1999 [Č].

It is easy to show that classifying spaces for the groups $O(n_1) \times \cdots \times O(n_m)$ and $SO(n_1) \times \cdots \times SO(n_m)$ are homotopy equivalent to the spaces $BO(n_1) \times \cdots \times BO(n_m)$ and $BSO(n_1) \times \cdots \times BSO(n_m)$, respectively. Cohomology rings of these spaces with $\mathbb{Z}_2$ coefficients can be easily obtained using Künneth formula. Extending methods of Brown and Feshbach, in 1985 M. Markl [M] described cohomology rings of $BO(n_1) \times \cdots \times BO(n_m)$ and $BSO(n_1) \times \cdots \times BSO(n_m)$ with integer coefficients.

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Since $\pi_1(BO(n_1)\times\cdots\times BO(n_m)) = (\mathbb{Z}_2)^m$, there are just 2^m systems of local integer coefficients over $BO(n_1)\times\cdots\times BO(n_m)$. In this paper we generalize the result from [Č] and describe the cohomology ring of $BO(n_1)\times\cdots\times BO(n_m)$ with all possible local integer coefficients via generators and relations.

2. Preliminaries

We use the definition of local coefficients and singular cohomology groups with local coefficients from [S] (exercises F in Chapter 1 and J in Chapter 5). In [S] there is also a theorem on the existence of the Thom class with local integer coefficients and a version of the Thom isomorphism in the context of local coefficients.

In the sequel, by X we denote a connected CW-complex. Let n be a positive integer and let $\xi = (E\xrightarrow{p}X)$ be an n-dimensional vector bundle over X. Denote $\overline{E}$ the total space without the zero section, $E_x = p^{-1}(x)$ the fiber over $x\in X$, $\overline{E_x}$ the fiber without the zero element and $i_x : E_x\longrightarrow E$ the inclusion.

Then $\{H_n(E_x,\overline{E_x})\}$ forms a system of local integer coefficients over X, let us denote it by $\mathbb{Z}_\xi$. An element $t\in H^n(E,\overline{E};p^*\mathbb{Z}_\xi)$ such that

$$i_x^*t\in H^n(E_x,\overline{E_x};H_n(E_x,\overline{E_x}))$$

corresponds to the identity in $\mathrm{Hom}(H_n(E_x,\overline{E_x}),H_n(E_x,\overline{E_x}))$ for every $x\in X$ is called the *Thom class* of the vector bundle ξ.

Lemma 1. *Let $\xi = (E\xrightarrow{p}X)$ be an n-dimensional vector bundle over X and let $\mathcal{G}$ be an arbitrary system of local coefficients over X. Then there is a unique Thom class t and it determines an isomorphism*

$$\Phi_t : H^q(X;\mathcal{G})\longrightarrow H^{q+n}(E,\overline{E};p^*\mathcal{G}\otimes\mathbb{Z}_\xi)$$

defined by the multiplication of t as $\Phi_t(x) = p^*(x)\cup t$.

Let $o : (E,\emptyset)\longrightarrow(E,\overline{E})$ be an inclusion. Similarly as for group coefficients we can define the *Euler class* of the vector bundle ξ to be a class $e\in H^n(X;\mathbb{Z}_\xi)$ such that $p^*(e) = o^*(t)$. Using this definition, the Thom isomorphism and the isomorphism p^* induced by the homotopy equivalence $p : E\longrightarrow X$ and substituting them into the long exact sequence of the pair $(E,\overline{E})$, we get the long exact Gysin sequence with local coefficients

$$\xrightarrow{\Delta} H^{q-n}(X;\mathcal{G}\otimes\mathbb{Z}_\xi)\xrightarrow{\cup e} H^q(X;\mathcal{G})\xrightarrow{p^*} H^q(\overline{E};p^*\mathcal{G})\xrightarrow{\Delta} H^{q-n+1}(X;\mathcal{G}\otimes\mathbb{Z}_\xi)\xrightarrow{\cup e}$$

Since $\pi_1(BO(n)) = \mathbb{Z}_2$, we have two nonequivalent systems of local integer coefficients over $BO(n)$ — the trivial one, denoted by $\mathbb{Z}$, and the nontrivial one, which we call *twisted* and denote by $\mathbb{Z}^t$.

In the case of the universal vector bundle $\gamma_n = (E_n \xrightarrow{p} BO(n))$ over classifying space $BO(n)$ the system of local coefficients $\mathbb{Z}_{\gamma_n}$ is equivalent to the system of twisted integer coefficients $\mathbb{Z}^t$. Moreover, $\mathbb{Z}^t \otimes \mathbb{Z}^t = \mathbb{Z}$, $\mathbb{Z} \otimes \mathbb{Z}^t = \mathbb{Z}^t$. Since $\overline{E}_n$ is homotopically equivalent to the total space of the sphere bundle SE_n, which is homotopically equivalent to $BO(n-1)$ and the inclusion $SE_n \hookrightarrow E_n$ corresponds to $\iota : BO(n-1) \hookrightarrow BO(n)$, we can substitute $BO(n-1)$ for $\overline{E}_n$ in the long exact Gysin sequence for the bundle γ_n and compute cohomology inductively.

Now we generalize this idea. The twisting of integer coefficients over the space $BO(n_1)\times\cdots\times BO(n_m)$ is more complicated, having $\pi_1(BO(n_1)\times\cdots\times BO(n_m)) = (\mathbb{Z}_2)^m$. Hence there are 2^m nonequivalent systems of local integer coefficients over $BO(n_1) \times \cdots \times BO(n_m)$. For $a \in (\mathbb{Z}_2)^m$ we denote by $\mathbb{Z}_a$ the system of local coefficients in which i-th generator of the $\pi_1(BO(n_1) \times \cdots \times BO(n_m))$ acts as multiplication by -1 if and only if $a_i = 1$. The formula for the tensor product of the systems of local coefficients then has a form $\mathbb{Z}_a \otimes \mathbb{Z}_b = \mathbb{Z}_{a+b}$.

For two systems of local coefficients $\mathcal{G}_1$, $\mathcal{G}_2$ over a space X there is a cup product in cohomology with local coefficients defined as follows

$$\cup : H^k(X;\mathcal{G}_1) \times H^l(X;\mathcal{G}_2) \longrightarrow H^{k+l}(X;\mathcal{G}_1 \otimes \mathcal{G}_2).$$

Hence we can construct a ring

$$h^*(n_1,\ldots,n_m) = \bigoplus_{a\in(\mathbb{Z}_2)^m} H^*(BO(n_1) \times \cdots \times BO(n_m);\mathbb{Z}_a),$$

which is a $\mathbb{Z}\oplus(\mathbb{Z}_2)^m$-graded ring describing the cohomology of the space $BO(n_1)\times \cdots \times BO(n_m)$ with all possible systems of local coefficients.

We can see immediately, that the cohomology ring with the system of local coefficients $\mathbb{Z}_{(0,\ldots,0)}$ is isomorphic to the singular cohomology ring. So, we have $H^*(BO(n_1) \times \cdots \times BO(n_m);\mathbb{Z})$ contained as a subring in $h^*(n_1,\ldots,n_m)$.

Now let γ_{n_m} denotes the universal vector bundle over $BO(n_m)$. Then $\pi_m^*\gamma_{n_m}$ is a vector bundle over $BO(n_1) \times \cdots \times BO(n_m)$, where π_m denotes a projection on m-th factor. It is easy to show that the system of the local integer coefficients $\mathbb{Z}_{\pi_m^*\gamma_{n_m}}$ is equivalent to $\mathbb{Z}_{(0,\ldots,0,1)}$ over $BO(n_1) \times \cdots \times BO(n_m)$.

Let us denote by e_m the Euler class of the vector bundle $\pi_m^*\gamma_{n_m}$. In this case the long exact Gysin sequence for $h^*(n_1,\ldots,n_m)$ can be written in the following form:

$$\text{(G)}\qquad \xrightarrow{\Delta} h^{q-n_m}(n_1,\ldots,n_m) \xrightarrow{\cup e_m} h^q(n_1,\ldots,n_m) \xrightarrow{\iota^*} h^q(n_1,\ldots,n_m-1) \xrightarrow{\Delta} h^{q-n_m+1}(n_1,\ldots,n_m) \longrightarrow$$

This long exact sequence is a principal tool for the proof of our main result.

In the description of the cohomology ring $h^*(n_1, \dots, n_m)$ the following facts about Bockstein homomorphisms play a key role.

Let X be a connected CW-complex with $\pi_1(X) = (\mathbb{Z}_2)^m$. The Bockstein homomorphism associated with the short exact sequence of local coefficients

$$0 \longrightarrow \mathbb{Z}_a \xrightarrow{2\times} \mathbb{Z}_a \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

denote by $\delta_a : H^q(X;\mathbb{Z}_2) \longrightarrow H^{q+1}(X;\mathbb{Z}_a)$ for arbitrary $a \in (\mathbb{Z}_2)^m$.

Write $\rho_2 : H^*(X;\mathbb{Z}_a) \longrightarrow H^*(X;\mathbb{Z}_2)$ the homomorphism induced from the reduction modulo 2. We get the following induced long exact sequence

$$\text{(B)} \qquad \longrightarrow H^q(X;\mathbb{Z}_a) \xrightarrow{2\times} H^q(X;\mathbb{Z}_a) \xrightarrow{\rho_2} H^q(X;\mathbb{Z}_2) \xrightarrow{\delta_a} H^{q+1}(X;\mathbb{Z}_a) \longrightarrow$$

The immediate consequence of this long exact sequence is the following

Lemma 2. *Let X be a path connected topological space with $\pi_1(X) = (\mathbb{Z}_2)^m$ and $a \in (\mathbb{Z}_2)^m$. Suppose that all elements of finite order in $H^*(X;\mathbb{Z}_a)$ are of order 2.*

Then for $x \in H^*(X;\mathbb{Z}_2)$ the equality $\delta_a x = 0$ holds if and only if $\rho_2 \circ \delta_a x = 0$. Furthermore, for the torsion subgroup T of $H^*(X;\mathbb{Z}_a)$ we obtain $T = \delta_a(H^*(X;\mathbb{Z}_2))$ and the homomorphism ρ_2 restricted on T is an injection into $H^*(X;\mathbb{Z}_2)$.

Let $p_{i,s}$, $w_{i,s}$ and e_s denote the i-th Pontrjagin, the i-th Stiefel-Whitney and the Euler class of the vector bundle $\pi_s^*\gamma_{n_s}$ over $BO(n_1) \times \cdots \times BO(n_m)$, respectively. Note that we have $\rho_2 p_{i,s} = w_{2i,s}^2$, $\rho_2 e_s = w_{n_s,s}$ by the definition of the characteristic classes.

In [Č, Lemma 2], it is proved that the system of twisted coefficients over $BO(n)$ is uniquely determined by the first Stiefel-Whitney class
$w_1 \in H^1(BO(n);\mathbb{Z}_2)$. Using the obvious generalization of this fact, we can state following

Lemma 3. *The system of twisted coefficients $\mathbb{Z}_a$ over $BO(n_1) \times \cdots \times BO(n_m)$ is uniquely determined by the sum of the first Stiefel-Whitney classes*

$$\sum_{a_s=1} w_{1,s} \in H^1(BO(n_1) \times \cdots \times BO(n_m);\mathbb{Z}_2).$$

Let $\delta_a : H^q(BO(n_1) \times \cdots \times BO(n_m);\mathbb{Z}_2) \longrightarrow H^{q+1}(BO(n_1) \times \cdots \times BO(n_m);\mathbb{Z}_a)$ be a Bockstein homomorphism. Then

$$\rho_2 \delta_a x = (\rho_2 \delta_a 1)x + \mathrm{Sq}^1 x$$

for all $x\in H^q(BO(n_1)\times\cdots\times BO(n_m);\mathbb{Z}_2)$ *and moreover,*

$$\rho_2\delta_a 1=\sum_{a_s=1} w_{1,s}.$$

So we have a correspondence between Bockstein homomorphisms δ_a, the systems of local coefficients $\mathbb{Z}_a$ and the elements $\sum w_{1,\lambda}$.

Notation. In the sequel we abbreviate $\delta_{(0,\dots,0)}$ to δ_0. Note that $\rho_2\delta_0=\mathrm{Sq}^1$.

Denote by $\mathcal{M}=\bigcup_{s=1}^m\{1,2,\dots,[(n_s-1)/2]\}\times\{s\}$. We consider this set ordered by the lexicographic ordering $(i,r)<(j,s)$ iff $i<j$ or $i=j$ and $r<s$.

Denote by c_λ the element of $(\mathbb{Z}_2)^m$ with 1 at the λ-th position and zero otherwise. By the symbol $I\triangle J$ we denote symmetric difference of the sets I and J, $I\triangle J=(I\cup J)-(I\cap J)$, by the symbol $[x]$ we denote the integer part of x.

Denote for $I\subseteq\mathcal{M}$, $1\le s\le m$ by

$$w_{0,s}=w_\emptyset=p_{0,s}=p_\emptyset=1,\qquad w_I=\prod_{(i,\tau)\in I} w_{2i,\tau},\qquad p_I=\prod_{(i,\tau)\in I} p_{i,\tau}.$$

Let $\iota^*: h(n_1,\dots,n_m)\longrightarrow h(n_1,\dots,n_m-1)$ denote the homomorphism induced by the natural inclusion $\iota: BO(n_1)\times\cdots\times BO(n_m-1)\longrightarrow BO(n_1)\times\cdots\times BO(n_m)$.

In the proof of the Theorem we use following statements:

Lemma 4. *In* $H^*(BO(n_1)\times\cdots\times BO(n_m);\mathbb{Z}_2)$, *the following relations are satisfied:*

$$\begin{aligned}\rho_2(\delta_a w_I\delta_b w_J)&=\sum_{(i,\tau)\in I}\mathrm{Sq}^1 w_{2i,\tau}\rho_2\delta_{a+b}w_{(I-\{(i,\tau)\})\triangle J}\rho_2 p_{(I-\{(i,\tau)\})\cap J}\\&\quad+(\rho_2\delta_a 1)\rho_2\delta_b w_{I\triangle J}\rho_2 p_{I\cap J},\\ \rho_2\big((\delta_a 1)(\delta_b w_J)\big)&=\rho_2\Big(\sum_{a_\lambda=1}(\delta_{c_\lambda}1)\cdot(\delta_{a+b+c_\lambda}w_I)\Big),\end{aligned}$$

where $I\subseteq\mathcal{M}$ *and* $a,b\in(\mathbb{Z}_2)^m$.

Proof: Using the relations

$$\begin{aligned}\mathrm{Sq}^1 w_I\,\mathrm{Sq}^1 w_J&=\sum_{(i,\tau)\in I}\mathrm{Sq}^1 w_{2i,\tau}\,\mathrm{Sq}^1 w_{(I-\{(i,\tau)\})\triangle J}\rho_2 p_{(I-\{(i,\tau)\})\cap J},\\ w_I\,\mathrm{Sq}^1 w_J&=\sum_{(i,\tau)\in J}\mathrm{Sq}^1 w_{2i,\tau}w_{(J-\{(i,\tau)\})\triangle I}\rho_2 p_{(J-\{(i,\tau)\})\cap I},\end{aligned}$$

proved in [B] and Lemma 3, one can show the first formula. The second one can be obtained by an elementary calculation. □

3. Main result

Theorem. *Let $n_1, \dots, n_m$ be positive integers. The cohomology ring $h^*(n_1, \dots, n_m)$ is the polynomial ring*

$$\mathcal{R}_{n_1,\dots,n_m} = \mathbb{Z}\left[p_{i,\tau}, e_s, \delta_a w_J \mid (i,\tau) \in \mathcal{M}, J \subseteq \mathcal{M}, a \in (\mathbb{Z}_2)^m, s = 1 \dots m\right]$$

modulo the ideal $\mathcal{I}_{n_1,\dots,n_m}$ generated by the following relations:

$$\delta_0 1 = 0, \tag{1}$$

$$2\delta_a w_J = 0, \tag{2}$$

$$e_s = \delta_{c_s} w_{n_s-1,s} \quad \text{for } n_s \text{ odd}, \tag{3}$$

(4)

$$\delta_a w_I \delta_b w_J = \sum_{(i,\tau)\in I} (\delta_0 w_{2i,\tau}) p_{(I-\{(i,\tau)\})\cap J} \delta_{a+b} w_{(I-\{(i,\tau)\})\triangle J} + \delta_a 1 \delta_b w_{I\triangle J} p_{I\cap J},$$

$$(\delta_a 1)(\delta_b w_J) = \sum_{a_\lambda=1} (\delta_{c_\lambda} 1)(\delta_{a+b+c_\lambda} w_J), \tag{5}$$

where $I, J \subseteq \mathcal{M}$, $I \neq 0$, J can be empty.

Moreover, the following relation is satisfied in $\mathcal{R}_{n_1,\dots,n_m}$ for n_s even:

$$p_{n_s/2,s} = e_s^2. \tag{6}$$

Remark 1. The relations (4) and (5) can be written together as the relation

$$\begin{aligned}\delta_a w_I \delta_b w_J = &\sum_{(i,\tau)\in I} (\delta_0 w_{2i,\tau}) p_{(I-\{(i,\tau)\})\cap J} \delta_{a+b} w_{(I-\{(i,\tau)\})\triangle J} \\ &+ p_{I\cap J}\left(\sum_{a_\lambda=1} (\delta_{c_\lambda} 1)(\delta_{a+b+c_\lambda} w_{I\triangle J})\right),\end{aligned}$$

where both $I, J \subseteq \mathcal{M}$ can be empty.

Remark 2. The space $BO(0)$ is homotopy equivalent to the space SE_1, the total space of the associated sphere bundle over $BO(1)$. However, $SE(1)$ is homotopy equivalent to the contractible space S^∞. Hence $\pi_1(BO(0)) = 0$ and there is only one possible system of local integer coefficients over $BO(0)$. Consequently, for $n_m = 1$ and $a \in (\mathbb{Z}_2)^m$, $a_m = 1$ we do not have $H^*(BO(n_1) \times \cdots \times BO(n_m - 1); \mathbb{Z}_a)$. Let us note that its role in the long exact sequence (G) is played by $H^*(BO(n_1) \times \cdots \times BO(n_m - 1); \mathbb{Z}_{a+c_m})$, since for the Bockstein homomorphism $\delta_a : H^*(BO(n_1) \times \cdots \times BO(n_m); \mathbb{Z}_2) \longrightarrow H^*(BO(n_1) \times \cdots \times BO(n_m); \mathbb{Z}_a)$ we have $\iota^* \circ \delta_a = \delta_{a+c_m}$. In particular, $\iota^* \circ \delta_{c_m} = \delta_0$.

4. Proof

Define a ring homomorphism $\theta_{n_1,\ldots,n_m} : \mathcal{R}_{n_1,\ldots,n_m} \longrightarrow h^*(n_1,\ldots,n_m)$ in the following way: to each formal generator of $\mathcal{R}_{n_1,\ldots,n_m}$ we assign the corresponding characteristic class in $h^*(n_1,\ldots,n_m)$ as follows:

$$\theta_{n_1,\ldots,n_m}(e_s) = e_s\big(\pi_s^*(\gamma_{n_s})\big),$$
$$\theta_{n_1,\ldots,n_m}(p_{i,\tau}) = p_i\big(\pi_\tau^*(\gamma_{n_\tau})\big) \text{ and}$$
$$\theta_{n_1,\ldots,n_m}(\delta_a w_J) = \delta_a\Big(\prod_{(j,\zeta)\in J} w_{j,\zeta}\big(\pi_\zeta^*(\gamma_{n_\zeta})\big)\Big).$$

We show by induction on m and n_m that $\theta_{n_1,\ldots,n_m}(\mathcal{I}_{n_1,\ldots,n_m}) = 0$ and then we prove that the induced ring homomorphism $\overline{\theta}_{n_1,\ldots,n_m} : \mathcal{R}_{n_1,\ldots,n_m}/\mathcal{I}_{n_1,\ldots,n_m} \longrightarrow h^*(n_1,\ldots,n_m)$ is an isomorphism.

We start the induction for $m = 1$ and $n_1 = 0$, using Remark 2. Due to the contractibility of $BO(0)$, an inductive step from $m-1$ to m, $n_m = 1$ can be carried out by considering $BO(n_1) \times \cdots \times BO(n_{m-1})$ as $BO(n_1) \times \cdots \times BO(n_{m-1}) \times BO(0)$. Note that the inductive step to $BO(n_1) \times \cdots \times BO(n_{m-1}) \times BO(1)$ is slightly different from the inductive step for $n_m > 1$ odd.

Now suppose that Theorem holds for $n_m - 1$ and n_m is even.
We have $\mathcal{R}_{n_1,\ldots,n_m} = \mathcal{R}_{n_1,\ldots,n_m-1}[e_m]$, $\mathcal{I}_{n_1,\ldots,n_m} = \mathcal{I}_{n_1,\ldots,n_m-1}$, hence

$$\mathcal{R}_{n_1,\ldots,n_m}/\mathcal{I}_{n_1,\ldots,n_m} = (\mathcal{R}_{n_1,\ldots,n_m-1}/\mathcal{I}_{n_1,\ldots,n_m-1})[e_m].$$

Let $\psi : \mathcal{R}_{n_1,\ldots,n_m-1} \longrightarrow \mathcal{R}_{n_1,\ldots,n_m}$ be the natural inclusion. Then, by the inductive hypothesis, $\theta_{n_1,\ldots,n_m-1} = \iota^* \circ \theta_{n_1,\ldots,n_m} \circ \psi$ is an epimorphism, hence $\iota^* : h^*(n_1,\ldots,n_m) \longrightarrow h^*(n_1,\ldots,n_m-1)$ is also an epimorphism and from (G) we get a short exact sequence

$$(7) \quad \xrightarrow{0} h^{q-n_m}(n_1,\ldots,n_m) \xrightarrow{\cup e_m} h^q(n_1,\ldots n_m) \xrightarrow{\iota^*} h^q(n_1,\ldots,n_m-1) \xrightarrow{0}$$

In particular, we get $\operatorname{im}(\iota^* \circ \theta_{n_1,\ldots,n_m}) = \operatorname{im}\iota^*$ and one can show that, under this condition, $\theta_{n_1,\ldots,n_m}$ is an epimorphism. Using this condition we will prove later the surjectivity of $\theta_{n_1,\ldots,n_m}$ for the case n_m odd.

Now denote by $\operatorname{Tor} h^*(n_1,\ldots,n_m)$ the subgroup of torsion elements of $h^*(n_1,\ldots,n_m)$. Since the multiplication by e_m is injective and $\theta_{n_1,\ldots,n_m}$ is an epimorphism, we get isomorphism of graded groups

$$h^*(n_1,\ldots,n_m) = \mathbb{Z}[p_{i,\tau}, e_s \mid (i,\tau) \in \mathcal{M}, n_s \text{ even}, s \leq m] \oplus \operatorname{Tor} h^*(n_1,\ldots,n_m).$$

Since for each $a \in (\mathbb{Z}_2)^m$ we have $2\delta_a = 0$, all torsion elements are of order 2. Using Lemma 4 and Lemma 2 we obtain $\theta_{n_1,\ldots,n_m}(\mathcal{I}_{n_1,\ldots,n_m}) = 0$.

So the induced ring homomorphism
$\overline{\theta}_{n_1,\dots,n_m} : \mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m} \longrightarrow h^*(n_1,\dots,n_m)$ is an epimorphism.

Now since $\mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m} \simeq (\mathcal{R}_{n_1,\dots,n_m-1}/\mathcal{I}_{n_1,\dots,n_m-1})[e_m]$, we have the short exact sequence

$$\xrightarrow{0} \mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m} \xrightarrow{\cup e_m} \mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m} \xrightarrow{\Psi} \mathcal{R}_{n_1,\dots,n_m-1}/\mathcal{I}_{n_1,\dots,n_m-1} \xrightarrow{0}$$

where $\Psi = \operatorname{coker}(\cup e_m)$. This exact sequence and (7) fit together into diagram connected by $\overline{\theta}_{n_1,\dots,n_m}$ and 5-lemma applied inductively on this diagram yields that $\overline{\theta}_{n_1,\dots,n_m}$ is an isomorphism for n_m even.

The fact that $p_{n_m/2,m} = e_m^2$ can be proved analogously to [Č].

Now suppose Theorem holds for $n_m - 1$ and n_m is odd. By the definition of Thom and Euler classes, we get $\rho_2 e_m = w_{n_m,m}$. According to Lemma 3,

$$\rho_2 \delta_{c_m} w_{n_m-1,m} = w_{1,m} w_{n_m-1,m} + \mathrm{Sq}^1\, w_{n_m-1,m} = w_{n_m,m} = \rho_2 e_m.$$

Let us note that with respect to the notation $w_{0,m} = 1$, this equality holds for $n_m = 1$ as well.

Using the Thom isomorphism and the definition of the Euler class, we get $2e_m = 0$.

Now using the long exact sequences (G) and (B) one can show

$$e_m = \delta_{c_m} w_{n_m-1,m}. \tag{8}$$

To prove that $\theta_{n_1,\dots,n_m} : \mathcal{R}_{n_1,\dots,n_m} \longrightarrow h^*(n_1,\dots,n_m)$ is an epimorphism we show that $\operatorname{im}(\iota^* \circ \theta_{n_1,\dots,n_m}) = \ker \Delta = \operatorname{im} \iota^*$.

Let $n_m > 1$ and consider $a \in (\mathbb{Z}_2)^m$, $s < m$, n_s even and $I \subseteq \mathcal{M}$ such that $((n_m-1)/2, m) \notin I$. Since the elements p_I, e_s and $\delta_a w_I$ are contained in $\mathcal{R}_{n_1,\dots,n_m-1}$ by the inductive hypothesis, they are contained in $\operatorname{im}(\iota^* \circ \theta_{n_1,\dots,n_m})$ as well.

Denote by e the Euler class of the bundle $\pi_m^* \gamma_{n_m-1}$ over $BO(n_1) \times \cdots \times BO(n_m - 1)$. We have

$$\begin{aligned} \iota^* \theta_{n_1,\dots,n_m}(\delta_a(w_I w_{n_m-1,m})) &= \delta_a w_I \cup e, \\ \iota^* \theta_{n_1,\dots,n_m}(p_{(n_m-1)/2,m}) &= e^2, \\ \iota^* \theta_{n_1,\dots,n_m}(e_m) = \iota^* e_m &= \iota^*(1 \cup e_m) = 0 \end{aligned} \tag{9}$$

by the description of $h^*(n_1,\dots,n_m-1)$ and Lemma 2, hence $e \notin \operatorname{im}(\iota^* \circ \theta_{n_1,\dots,n_m})$.

Since $e_m \neq 0$ and $2e_m = 0$, exactness of (G) yields $\Delta e = \pm 2$. So every element of $h^*(n_1,\dots,n_m-1)$ has the form $x = x_1 + x_2 \cup e$, where $x_1, x_2 \in \operatorname{im}(\iota^* \circ \theta_{n_1,\dots,n_m})$. Thus $\Delta x = \pm 2x_2$. Hence $x \in \ker\Delta$ if and only if $2x_2 = 0$, which implies $\ker\Delta = \operatorname{im}(\iota^* \circ \theta_{n_1,\dots,n_m})$.

Let $n_m = 1$. Due to Remark 2, the Euler class e of the 0-dimensional vector bundle $\pi_m^*\gamma_0$ over $BO(n_1)\times\cdots\times BO(n_{m-1})\times BO(0)$ is equal to ± 1 and the first relation of (9) has the form

$$\iota^*\theta_{n_1,\dots,n_m}(\delta_a w_I) = \delta_{a+c_m} w_I$$

for $a \in (\mathbb{Z}_2)^m$, $a_m = 1$. By the same considerations as above we obtain $\operatorname{im}\iota^* = \ker\Delta = \operatorname{im}(\iota^* \circ \theta_{n_1,\dots,n_m})$ as well.

Hence $\theta_{n_1,\dots,n_m} : \mathcal{R}_{n_1,\dots,n_m} \longrightarrow h^*(n_1,\dots,n_m)$ is an epimorphism.

Using (9) and inductive hypothesis we get isomorphism of graded groups

$$h^*(n_1,\dots,n_m) = \mathbb{Z}[p_{i,\tau}, e_s \mid (i,\tau)\in\mathcal{M}, n_s \text{ even}, s<m] \oplus \operatorname{Tor} h^*(n_1,\dots,n_m).$$

By the inductive hypothesis and (8), each torsion element of $h^*(n_1,\dots,n_m)$ is of order 2. Applying Lemma 4 and Lemma 2, we obtain $\theta_{n_1,\dots,n_m}(\mathcal{I}_{n_1,\dots,n_m}) = 0$.

The induced ring homomorphism $\overline{\theta}_{n_1,\dots,n_m} : \mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m} \longrightarrow h^*(n_1,\dots,n_m)$ is an epimorphism.

To complete the whole proof, it suffices to show that $\overline{\theta}_{n_1,\dots,n_m} : \mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m} \longrightarrow h^*(n_1,\dots,n_m)$ is a monomorphism. Immediately we get that it is injective on the free part of $\mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m}$ generated by $p_{i,\tau}$ and e_s for $(i,\tau)\in\mathcal{M}$, n_s even and $s<m$.

To prove injectivity of $\overline{\theta}_{n_1,\dots,n_m}$ on the torsion subgroup of $\mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m}$, we describe an additive basis of $\operatorname{Tor}\mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m}$ and show that $\rho_2\overline{\theta}_{n_1,\dots,n_m}$ is injective on each part of the graduation of $\operatorname{Tor}\mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m}$, which proves the injectivity of $\overline{\theta}_{n_1,\dots,n_m}$ due to Lemma 2.

Consider the following slight modification of the Stiefel-Whitney classes introduced in [B]. Let

$$v_{1,\tau} = w_{1,\tau}, \qquad v_{2i,\tau} = w_{2i,\tau}, \qquad v_{2i+1,\tau} = w_{2i+1,\tau} + w_{1,\tau}w_{2i,\tau}$$

for $(i,\tau)\in\mathcal{M}$, so we have

$$\operatorname{Sq}^1 v_{1,\tau} = v_{1,\tau}^2, \qquad \operatorname{Sq}^1 v_{2i+1,\tau} = 0, \qquad \operatorname{Sq}^1 v_{2i,\tau} = v_{2i+1,\tau}.$$

The monomials $\prod_{(i,\tau)\in\mathcal{M}} v_{i,\tau}^{k_{i,\tau}}$ obviously form a basis of $H^*(BO(n_1)\times\cdots\times BO(n_m);\mathbb{Z}_2)$. For $I \subseteq \mathcal{M}$ put $v_I = w_I$ and particularly $v_\emptyset = 1$.

Then the torsion part of the group $h^*(n_1,\dots,n_m)$ contains elements

$$\sum \prod_{(i,\tau)\in\mathcal{M}} p_{i,\tau}^{\chi_{i,\tau}} \prod_{s=1}^{m-1} e_s^{\nu_s} \prod_l \delta_{\mu_l} v_{I_l} \tag{10}$$

for $\chi_{i,\tau}$ and ν_s nonnegative integers, μ_l in $(\mathbb{Z}_2)^m$, $I_l \subseteq \mathcal{M}$ such that if each $\mu_l = 0$ then at least one of I_l is nonempty.

Consider the following elements of $\operatorname{Tor} h^*(n_1,\dots,n_m)$:

$$X(A,K,T,R,a,I) = \prod_{\lambda=1}^{m} (\delta_{c_\lambda} 1)^{\alpha_\lambda} \prod_{(i,\tau)\in\mathcal{M}} p_{i,\tau}^{k_{i,\tau}} \prod_{s=1}^{m-1} e_s^{t_s} \prod_{(j,\zeta)\in\mathcal{M}} (\delta_0 v_{2j,\zeta})^{r_{j,\zeta}} \delta_a v_I,$$

$$Y(A,b,K,T) = \prod_{\lambda=1}^{m} (\delta_{c_\lambda} 1)^{\alpha_\lambda} (\delta_b 1) \prod_{(i,\tau)\in\mathcal{M}} p_{i,\tau}^{k_{i,\tau}} \prod_{s=1}^{m-1} e_s^{t_s},$$

$$Z(A,K,T) = \prod_{\lambda=1}^{m} (\delta_{c_\lambda} 1)^{\alpha_\lambda} \prod_{(i,\tau)\in\mathcal{M}} p_{i,\tau}^{k_{i,\tau}} \prod_{s=1}^{m-1} e_s^{t_s},$$

where α_λ are nonnegative integers, a, $b \in (\mathbb{Z}_2)^m$, b different from any c_λ and nonzero, $I \subseteq \mathcal{M}$ nonempty, A, K, T, R are ordered systems of the powers of $\delta_{c_\lambda} 1$, $p_{i,\tau}$, e_s, $\delta_0 v_{2j,\zeta}$ respectively, satisfying following conditions: $t_s = 1$ or 0 for n_s even and $t_s = 0$ for n_s odd. Moreover, in $X(A,K,T,R,a,I)$ we have $\max I \geq (j_0,\zeta_0)$, where $(j_0,\zeta_0) \in I$ is the highest index in $\mathcal{M}$ such that $r_{j_0,\zeta_0} > 0$. In $Y(A,b,K,T)$ we suppose $\alpha_\kappa = 0$ for each $\kappa > \max\{\lambda \mid b_\lambda = 1\}$.

Now we prove that these elements form an additive basis of $\operatorname{Tor} h^*(n_1,\dots,n_m)$.

One can decompose a polynomial of the form (10) by successive use of the relations (3)–(6) to show that the monomials $X(A,K,T,R,a,I)$, $Y(A,b,K,T)$ and $Z(A,K,T)$ are generators of $\operatorname{Tor} h^*(n_1,\dots,n_m)$.

To show linear independence of these generators of $\operatorname{Tor} h^*(n_1,\dots,n_m)$ it is sufficient to prove that the images in the reduction modulo 2 of the elements with the same graduation are linearly independent in $H^*(BO(n_1)\times\cdots\times BO(n_m);\mathbb{Z}_2)$.

The group homomorphism ρ_2 maps $X(A,K,T,R,a,I)$ into

$$\prod_{\lambda=1}^{m} v_{1,\lambda}^{\alpha_\lambda} \prod_{(i,\tau)\in\mathcal{M}} v_{2i,\tau}^{2k_{i,\tau}} \prod_{s=1}^{m-1} v_{n_s,s}^{t_s} \prod_{(j,\zeta)\in\mathcal{M}} v_{2j+1,\zeta}^{r_{j,\zeta}} \, \rho_2 \delta_a v_I.$$

Denote (i_0,τ_0) the highest index in I and let $I_0 = I - \{(i_0,\tau_0)\}$. Then we have

$$\begin{aligned}\rho_2 \delta_a v_I &= (\rho_2 \delta_a 1) v_I + v_{2i_0,\tau_0} \operatorname{Sq}^1 v_{I_0} + v_{2i_0+1,\tau_0} v_{I_0} \\ &= (\rho_2 \delta_a 1) v_{2i_0,\tau_0} v_{I_0} + v_{2i_0,\tau_0} \Big(\sum_{(i,\tau)\in I_0} v_{2i+1,\tau} v_{I_0-(i,\tau)} \Big) + v_{2i_0+1,\tau_0} v_{I_0}\end{aligned}$$

where the last monomial contains v_{2i_0+1,τ_0} and does not contain v_{2i_0,τ_0}, the other monomials contain v_{2i_0,τ_0} and do not contain v_{2i_0+1,τ_0}.

Then the last monomial of $\rho_2 X(A,K,T,R,a,I)$ has the form

$$\prod_{\lambda=1}^{m} v_{1,\lambda}^{\alpha_\lambda} \prod_{(i,\tau)\in\mathcal{M}} v_{2i,\tau}^{\widehat{k}_{i,\tau}} \prod_{s=1}^{m-1} v_{n_s,s}^{t_s} \prod_{(j,\zeta)\in\mathcal{M}} v_{2j+1,\zeta}^{\widehat{r}_{j,\zeta}} \tag{11}$$

where $\widehat{k}_{i,\tau} = 2k_{i,\tau}+1$ for $(i,\tau) \in I_0$, $\widehat{k}_{i,\tau} = 2k_{i,\tau}$ otherwise, $\widehat{r}_{j,\zeta} = r_{j,\zeta}$ except $\widehat{r}_{i_0,\tau_0} = r_{i_0,\tau_0}+1$.

The powers of $v_{i,\tau}$, $(i,\tau) \in \mathcal{M}$, of the monomial (11) determine the sets A, K, T, R, I of $X(A,K,T,R,a,I)$. One can show by comparing the powers that $X(A,K,T,R,a,I)$ is determined uniquely up to graduation, hence the elements $X(A,K,T,R,a,I)$ form a linearly independent set.

Since $\rho_2 Y(A,b,K,T)$ and $\rho_2 Z(A,K,T)$ have no nonzero powers of $v_{2i+1,\tau}$ for $i > 0$, $(i,\tau) \in \mathcal{M}$, they cannot be any linear combinations of the elements $\rho_2 X(A,K,T,R,a,I)$ and vice versa.

Hence it remains to show linear independence of the elements $Y(A,b,K,T)$ and $Z(A,K,T)$. We get

$$\rho_2 Y(A,b,K,T) = \prod_{\lambda=1}^{m} v_{1,\lambda}^{\alpha_\lambda} \left(\sum_{b_\kappa=1} v_{1,\kappa} \right) \cdot \prod_{(i,\tau)\in\mathcal{M}} v_{2i,\tau}^{2k_{i,\tau}} \cdot \prod_{s=1}^{m-1} v_{n_s,s}^{t_s},$$

$$\rho_2 Z(A,K,T) = \prod_{\lambda=1}^{m} v_{1,\lambda}^{\alpha_\lambda} \cdot \prod_{(i,\tau)\in\mathcal{M}} v_{2i,\tau}^{2k_{i,\tau}} \cdot \prod_{s=1}^{m-1} v_{n_s,s}^{t_s}.$$

Obviously, the elements $Z(A,K,T)$ are uniquely determined by the monomials $\rho_2 Z(A,K,T)$, hence they form a linearly independent set.

Set $\lambda_0 = \max\{\lambda \mid b_\lambda = 1\}$. Then the last summand of $\rho_2 Y(A,b,K,T)$ is

$$v_{1,\lambda_0}^{\alpha_{\lambda_0}+1} \cdot \prod_{\lambda=1}^{\lambda_0-1} v_{1,\lambda}^{\alpha_\lambda} \cdot \prod_{(i,\tau)\in\mathcal{M}} v_{2i,\tau}^{2k_{i,\tau}} \cdot \prod_{s=1}^{m-1} v_{n_s,s}^{t_s} \tag{12}$$

which cannot be reduction modulo 2 of any $Z(\overline{A},\overline{K},\overline{T})$ with the same graduation as $Y(A,b,K,T)$.

By comparing appropriate powers of the monomial (12), one can prove that the elements $Y(A,b,K,T)$ form linearly independent set.

So the elements $X(A,K,T,R,a,I)$, $Y(A,b,K,T)$ and $Z(A,K,T)$ form a basis of the group $\operatorname{Tor} h^*(n_1,\dots,n_m)$ and the ring homomorphism $\overline{\theta}_{n_1,\dots,n_m}$ is injective on $\operatorname{Tor} h^*(n_1,\dots,n_m)$.

This completes the proof that $\overline{\theta}_{n_1,\dots,n_m} : \mathcal{R}_{n_1,\dots,n_m}/\mathcal{I}_{n_1,\dots,n_m} \longrightarrow h^*(n_1,\dots,n_m)$ is an isomorphism.

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