

## Weighted Miranda–Talenti inequality and applications to equations with discontinuous coefficients

S. LEONARDI

*Abstract.* Let  $\Omega$  be an open bounded set in  $\mathbb{R}^n$  ( $n \geq 2$ ), with  $C^2$  boundary, and  $N^{p,\lambda}(\Omega)$  ( $1 < p < +\infty$ ,  $0 \leq \lambda < n$ ) be a weighted Morrey space.

In this note we prove a weighted version of the Miranda-Talenti inequality and we exploit it to show that, under a suitable condition of Cordes type, the Dirichlet problem:

$$\begin{cases} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique strong solution in the functional space

$$\left\{ u \in W^{2,p} \cap W_o^{1,p}(\Omega) : \frac{\partial^2 u}{\partial x_i \partial x_j} \in N^{p,\lambda}(\Omega), \quad i, j = 1, 2, \dots, n \right\}.$$

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### 1. Introduction

Let  $\Omega$  be an open bounded set in  $\mathbb{R}^n$  ( $n \geq 2$ ), with  $C^2$  boundary, and  $N^{p,\lambda}(\Omega)$  ( $1 < p < +\infty$ ,  $0 \leq \lambda < n$ ) be the weighted Morrey space formed by the functions  $u : \Omega \rightarrow \mathbb{R}$  for which

$$\|u\|_{N^{p,\lambda}(\Omega)} = \sup_{x_o \in \Omega} \left\{ \int_{\Omega} |x - x_o|^{-\lambda} |u(x)|^p dx \right\}^{1/p} < +\infty.$$

Also, let  $W^{k,p,\lambda}(\Omega)$  be the linear space of functions  $u \in W^{k,p}(\Omega)$  such that  $D^\alpha u \in N^{p,\lambda}(\Omega)$  for  $|\alpha| = k$ .

In this note we will prove, at first, a weighted version of the Miranda-Talenti inequality (see [35]), namely we will demonstrate the following

**Theorem.** *Let  $1 < p < +\infty$  and  $0 \leq \lambda < n$ . Then there exists a constant  $C_{MT} = C_{MT}(n, p, \lambda, \partial\Omega) > 0$  such that, for any  $u \in W^{2,p} \cap W_o^{1,p}(\Omega)$  for which  $\Delta u \in N^{p,\lambda}(\Omega)$ , we have*

$$\|u\|_{W^{2,p,\lambda}(\Omega) \cap W_o^{1,p}(\Omega)} \leq C_{MT} \|\Delta u\|_{N^{p,\lambda}(\Omega)}.$$

Next, we exploit the previous result to show that, under a suitable condition of Cordes type, the Dirichlet problem:

$$\begin{cases} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique strong solution in the functional space  $W^{2,p,\lambda} \cap W_0^{1,p}(\Omega)$ .

## 2. Notations, assumptions, auxiliary results

In  $\mathbb{R}^n$  ( $n \geq 2$ ), with a generic point  $x = (x_1, x_2, \dots, x_n)$ , we shall denote by  $\Omega$  an open nonempty bounded set with  $C^2$ -boundary  $\partial\Omega$  <sup>(1)</sup>.

For  $\rho > 0$  we define

$$\begin{aligned} B(x_0, \rho) &= \{x \in \mathbb{R}^n : |x - x_0| < \rho\} \\ \Omega(x_0, \rho) &= \Omega \cap B(x_0, \rho). \end{aligned}$$

If  $u \in L^1(A)$ ,  $A$  being an open nonempty bounded set of  $\mathbb{R}^n$ , then we will set

$$u_A = \frac{1}{|A|} \int_A u(x) dx \quad (2),$$

if moreover  $u \in L^1(\mathbb{R}^n)$  we recall the definition of the Hardy-Littlewood maximal function

$$Mu(x) = \sup_{\rho > 0} u_{B(x,\rho)}.$$

If  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  is a multiindex we set

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} \quad (3).$$

Moreover let  $p \in ]1, +\infty[$  and  $\lambda \in [0, n[$  <sup>(1)</sup>.

**Definition 2.1.** Let  $k \in \mathbb{N}$ . By  $W^{k,p}(\Omega)$  (respectively  $W_0^{k,p}(\Omega)$ ) we denote the closure of  $C^\infty(\Omega)$  (respectively  $C_0^\infty(\Omega)$ ) with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left\| \left( \sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{1/2} \right\|_{L^p(\Omega)}.$$

<sup>1</sup> This hypothesis will always be implicitly used.

<sup>2</sup>  $|A|$  is the  $n$ -dimensional Lebesgue measure of  $A$ .

<sup>3</sup> For the sake of simplicity we will denote the gradient  $(D^\alpha u)_{|\alpha|=1}$  by  $Du$  and the Hessian matrix  $(D^\alpha u)_{|\alpha|=2}$  by  $H(u)$ .

**Definition 2.2** (Morrey’s space). By  $L^{p,\lambda}(\Omega)$  we denote the linear space of functions  $u \in L^p(\Omega)$  such that

$$(1) \quad \|u\|_{L^{p,\lambda}(\Omega)} = \sup_{x_0 \in \Omega, \rho > 0} \left\{ \rho^{-\lambda} \int_{\Omega(x_0, \rho)} |u(x)|^p dx \right\}^{1/p} < +\infty.$$

$L^{p,\lambda}(\Omega)$  equipped with the norm (1) is a Banach space.

**Definition 2.3** (Weighted Morrey’s space [27]). By  $N^{p,\lambda}(\Omega)$  we denote the linear space of functions  $u \in L^p(\Omega)$  such that

$$(2) \quad \|u\|_{N^{p,\lambda}(\Omega)} = \left\{ \sup_{x_0 \in \Omega} \int_{\Omega} |x - x_0|^{-\lambda} |u|^p dx \right\}^{1/p} < +\infty.$$

**Remark 2.1.** Fixed  $x_o \in \mathbb{R}^n$ , set

$$\nu_{x_o}(x) = |x - x_o|^{-\lambda}.$$

The weight  $\nu_{x_o}(x)$  satisfies the following properties:

- (i)  $\frac{1}{\nu_{x_o}(x)} \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ ,
- (ii)  $\nu_{x_o}(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$ ,
- (iii)  $\nu_{x_o}(x)$  is an  $A_p$  (or Muckenhoupt) weight i.e.  $\nu_{x_o}(x)$  satisfies the condition

$$\sup_Q \left( \frac{1}{|Q|} \int_Q \nu_{x_o}(x) dx \right) \left( \frac{1}{|Q|} \int_Q \nu_{x_o}(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < +\infty$$

where the supremum is taken over all cubes  $Q$  (see [37, Corollary 4.4, p. 236 and Proposition 3.2, p. 229]).

Properties (i), (ii) imply respectively that  $N^{p,\lambda}(\Omega)$  equipped with the norm (2) is a Banach space and that  $C^\infty_0(\Omega)$  is dense in  $N^{p,\lambda}(\Omega)$ .

**Proposition 2.1** ([22]). *It holds*

$$N^{p,\lambda}(\Omega) \subset L^{p,\lambda}(\Omega).$$

**Proposition 2.2** ([22]). *If*

$$\frac{\lambda_2 - n}{p} \leq \frac{\lambda_1 - n}{q},$$

with  $1 \leq p \leq q < \infty$ , then

$$N^{q,\lambda_1}(\Omega) \subset N^{p,\lambda_2}(\Omega).$$

**Definition 2.4.** By  $W^{k,p,(\lambda)}(\Omega)$  we denote the linear space of functions  $u \in W^{k,p}(\Omega)$  such that  $D^\alpha u \in N^{p,\lambda}(\Omega)$  for  $|\alpha| = k$ .

$W^{k,p,(\lambda)}(\Omega)$  equipped with the norm

$$(3) \quad \|u\|_{W^{k,p,(\lambda)}(\Omega)} = \|u\|_{L^p(\Omega)} + \left\| \left( \sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{1/2} \right\|_{N^{p,\lambda}(\Omega)}$$

is a Banach space.

**Proposition 2.3** (Weighted Poincaré's inequality). *Let  $u \in W^{1,p,(\lambda)}(\Omega)$ . Then there exists a constant  $C = C(n, p, \lambda, |\Omega|) > 0$  such that*

$$\|u - u_\Omega\|_{N^{p,\lambda}(\Omega)} \leq C \|Du\|_{N^{p,\lambda}(\Omega)}.$$

PROOF: From Lemma 3.4 in [26] (see also [15, p. 162]) we deduce

$$(4) \quad |u(x) - u_\Omega| \leq C(n) \int_\Omega |x - y|^{1-n} |Du(y)| dy =: C(n) I(x) \quad \text{a.a. } x \in \Omega.$$

After extending  $Du$  to the whole  $\mathbb{R}^n$  by assuming  $Du = 0$  in  $\mathbb{R}^n \setminus \Omega$  we get

$$(5) \quad \begin{aligned} I(x) &\leq \int_{|x-y| \leq d_\Omega} |x - y|^{1-n} |Du(y)| dy \\ &\leq \sum_{j=0}^{+\infty} \int_{d_\Omega 2^{-j-1} \leq |x-y| < d_\Omega 2^{-j}} |x - y|^{1-n} |Du(y)| dy \\ &\leq \sum_{j=0}^{+\infty} (d_\Omega 2^{-j-1})^{1-n} \int_{|x-y| < d_\Omega 2^{-j}} |Du(y)| dy \\ &\leq C(n, d_\Omega) M |Du(x)| \sum_{j=0}^{+\infty} 2^{-j}. \end{aligned}$$

The thesis now follows from the weighted norm estimate for the maximal function (see [24] or Theorem 1 from [9]) and Remark 2.1(iii); indeed we have

$$\|u - u_\Omega\|_{N^{p,\lambda}(\Omega)} \leq C \|I\|_{N^{p,\lambda}(\mathbb{R}^n)} \leq C \|Du\|_{N^{p,\lambda}(\mathbb{R}^n)} = C \|Du\|_{N^{p,\lambda}(\Omega)}.$$

□

**Proposition 2.4.** *Let  $u \in W^{2,p,\lambda}(\Omega)$ . Then  $D^\alpha u \in N^{p,\lambda}(\Omega)$  for  $|\alpha| \leq 1$ .*

PROOF: Poincaré’s inequality gives

$$(6) \quad \|Du - (Du)_\Omega\|_{N^{p,\lambda}(\Omega)} \leq C(n, p, \lambda, |\Omega|) \|H(u)\|_{N^{p,\lambda}(\Omega)}.$$

On the other hand by Hölder’s inequality and (6) we infer

$$(7) \quad \begin{aligned} \|Du\|_{N^{p,\lambda}(\Omega)} &\leq \left[ \|Du - (Du)_\Omega\|_{N^{p,\lambda}(\Omega)} + |(Du)_\Omega| \sup_{x_o \in \Omega} \left( \int_{\Omega} |x - x_o|^{-\lambda} dx \right)^{1/p} \right] \\ &\leq C(n, p, \lambda, |\Omega|) \left( \|H(u)\|_{N^{p,\lambda}(\Omega)} + \|Du\|_{L^p(\Omega)} \right) < +\infty \end{aligned}$$

whence, using again Poincaré’s inequality,

$$(8) \quad \begin{aligned} \|u\|_{N^{p,\lambda}(\Omega)} &\leq \left[ \|u - u_\Omega\|_{N^{p,\lambda}(\Omega)} + |u_\Omega| \sup_{x_o \in \Omega} \left( \int_{\Omega} |x - x_o|^{-\lambda} dx \right)^{1/p} \right] \\ &\leq C(n, p, \lambda, |\Omega|) \left( \|Du\|_{N^{p,\lambda}(\Omega)} + \|u\|_{L^p(\Omega)} \right) \\ &\leq C(n, p, \lambda, |\Omega|) \left( \|H(u)\|_{N^{p,\lambda}(\Omega)} + \|Du\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right). \end{aligned}$$

□

A consequence of the above proposition is the following interpolation inequality.

**Proposition 2.5.** *Let  $u \in W^{2,p,\lambda}(\Omega)$ . Then for any  $\varepsilon > 0$  one has*

$$(9) \quad \|Du\|_{N^{p,\lambda}(\Omega)} \leq C(\varepsilon) \|u\|_{N^{p,\lambda}(\Omega)} + \varepsilon \|H(u)\|_{N^{p,\lambda}(\Omega)}$$

where  $C(\varepsilon) > 0$  is independent of  $u$ .

PROOF: It is enough to establish (9) for  $u \in C^2(\Omega)$ .

For  $y \in \Omega$  fixed, let us introduce radial and angular coordinates  $\rho = |x - y|$ ,  $\omega = \frac{x-y}{\rho}$ .

Then we have for  $x \in \Omega$ ,

$$Du(y) = Du(x) - \int_0^\rho D_r^2 u(y + r\omega) dr$$

whence

$$|Du(y)|^p \leq 2^{p-1} \left[ |Du|^p dx + \left| \int_0^\rho D_r^2 u(y + r\omega) dr \right|^p \right].$$

Fixing  $\delta_o > 0$  and integrating with respect to  $x$  over  $\Omega(y, \delta_o)$  we obtain

$$\begin{aligned}
& |Du(y)|^p \leq 2^{p-1} C(n) \delta_o^{-n} \left[ \int_{\Omega} |Du|^p dx \right. \\
& \quad \left. + \int_{\Omega(y, \delta_o)} \left| \int_0^\rho D_r^2 u(y + r\omega) dr \right|^p dx \right] \\
& = 2^{p-1} C(n) \delta_o^{-n} \left[ \int_{\Omega} |Du|^p dx \right. \\
& \quad \left. + \int_0^{\delta_o} \int_{|\omega|=1} \left| \int_0^\rho D_r^2 u(y + r\omega) dr \right|^p \rho^{n-1} d\omega d\rho \right] \\
& = 2^{p-1} C(n) \delta_o^{-n} \left[ \int_{\Omega} |Du|^p dx \right. \\
& \quad \left. + \int_0^{\delta_o} \int_{|\omega|=1} \rho^\lambda \rho^{n-1-\lambda} \left| \int_0^\rho D_r^2 u(y + r\omega) dr \right|^p d\omega d\rho \right] \\
(10) \quad & \leq 2^{p-1} C(n) \delta_o^{-n} \left[ \int_{\Omega} |Du|^p dx \right. \\
& \quad \left. + \delta_o^\lambda \int_0^{\delta_o} \int_{|\omega|=1} \rho^{p-1} \rho^{n-1-\lambda} \int_0^\rho |D_r^2 u(y + r\omega)|^p dr d\omega d\rho \right] \\
& \leq 2^{p-1} C(n) \delta_o^{-n} \left[ \int_{\Omega} |Du|^p dx \right. \\
& \quad \left. + \delta_o^{\lambda+p} \int_0^{\delta_o} \int_{|\omega|=1} \rho^{n-1-\lambda} |D_r^2 u(y + \rho\omega)|^p d\omega d\rho \right] \\
& = 2^{p-1} C(n) \delta_o^{-n} \left[ \int_{\Omega} |Du|^p dx + \delta_o^{\lambda+p} \int_{\Omega} |x - y|^{-\lambda} |H(u)|^p dx \right] \\
& \leq 2^{p-1} C(n) \delta_o^{-n} \left[ \|Du\|_{L^p(\Omega)}^p + \delta_o^{\lambda+p} \|H(u)\|_{N^{p,\lambda}(\Omega)}^p \right].
\end{aligned}$$

Multiplying both sides of (10) by  $|y - x_o|^{-\lambda}$  and integrating with respect to  $y$  over  $\Omega(x_o, \delta_o)$ , for fixed  $x_o \in \Omega$ , we get

$$\begin{aligned}
& \int_{\Omega(x_o, \delta_o)} |Du(y)|^p |y - x_o|^{-\lambda} dy \\
& \leq C(n, p, \lambda) \delta_o^{-\lambda} \left[ \|Du\|_{L^p(\Omega)}^p + \delta_o^{\lambda+p} \|H(u)\|_{N^{p,\lambda}(\Omega)}^p \right]
\end{aligned}$$

whence, using Theorem 7.28 from [15],

$$\begin{aligned}
 & \sup_{\delta \leq \delta_o} \int_{\Omega(x_o, \delta)} |Du(y)|^p |y - x_o|^{-\lambda} dy \\
 (11) \quad & \leq C(n, p, \lambda, |\Omega|) \left[ \delta_o^{-p-2\lambda} \|u\|_{L^p(\Omega)}^p + \delta_o^p \|H(u)\|_{L^p(\Omega)}^p + \delta_o^p \|H(u)\|_{N^{p,\lambda}(\Omega)}^p \right] \\
 & \leq C(n, p, \lambda, |\Omega|) \left[ \delta_o^{-p-2\lambda} \|u\|_{N^{p,\lambda}(\Omega)}^p + \delta_o^p \|H(u)\|_{N^{p,\lambda}(\Omega)}^p \right].
 \end{aligned}$$

The thesis now follows from the equivalence of norms as in [20, p. 25].  $\square$

### 3. Weighted Miranda-Talenti inequality

Before proving a weighted version of the Miranda-Talenti inequality we will premise some useful propositions.

**Proposition 3.1.** *Let  $u \in W_o^{2,p}(\Omega)$  such that  $\Delta u \in N^{p,\lambda}(\Omega)$ . Then  $H(u) \in N^{p,\lambda}(\Omega)$  and there exists a constant  $C = C(n, p, \lambda) > 0$  such that*

$$(12) \quad \|H(u)\|_{N^{p,\lambda}(\Omega)} \leq C \|\Delta u\|_{N^{p,\lambda}(\Omega)}.$$

PROOF: We will proceed as in the proof of Proposition 3, p. 57 from [32].

Denoted by  $R_j(v)$ ,  $j = 1, \dots, n$ , the  $j$ -th Riesz transform of a function  $v \in C_o^2(\mathbb{R}^n)$  (see [32, pp. 57 and 68]). By a density argument and Theorem 3, p. 39 from [32] we get the identity

$$(13) \quad H(u) = -R_i(R_j(\Delta u)), \quad \forall u \in W_o^{2,p}(\Omega).$$

If we now extend  $\Delta u$  to the whole  $\mathbb{R}^n$  by setting  $\Delta u = 0$  in  $\mathbb{R}^n \setminus \Omega$ , the thesis is then an immediate consequence of (13), the properties of the kernel of the Riesz transform (see also [34, pp. 220 and 243]) and the weighted  $L^p$  inequality from [9, p. 244] (see also [25] and [31]).

Namely we have

$$\|H(u)\|_{N^{p,\lambda}(\Omega)} \leq C \|\Delta u\|_{N^{p,\lambda}(\mathbb{R}^n)} = C \|\Delta u\|_{N^{p,\lambda}(\Omega)}.$$

$\square$

The above proposition allows us to prove the following interior estimate.

**Theorem 3.1.** *Let  $u \in W^{2,p}(\Omega)$  such that  $\Delta u \in N^{p,\lambda}(\Omega)$ . Then, for any domains  $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ ,  $H(u) \in N^{p,\lambda}(\Omega')$  and there exists a constant  $C = C(n, p, \lambda, \text{dist}(\Omega', \partial\Omega'')) > 0$  such that*

$$(14) \quad \|H(u)\|_{N^{p,\lambda}(\Omega')} \leq C \left( \|u\|_{N^{p,\lambda}(\Omega'')} + \|\Delta u\|_{N^{p,\lambda}(\Omega)} \right).$$

PROOF: Suppose  $0 < \lambda < n$  (if  $\lambda = 0$  see e.g. Theorem 9.11 from [15]).

Let  $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ ,  $0 < R \leq \text{dist}(\Omega', \partial\Omega'')$ ; set  $B_R \equiv B(y_o, R)$ ,  $y_o \in \overline{\Omega'}$  and, for  $\sigma \in ]0, 1[$ , let us introduce a cutoff function  $\eta \in C_o^2(B_R)$  satisfying

$$\begin{aligned} 0 &\leq \eta \leq 1, \quad \forall x \in B_R \\ \eta &= 1 \quad \text{in } B_{\sigma R} \\ \eta &= 0 \quad \text{for } |x - y_o| \geq \sigma' R, \quad \sigma' = \frac{1 + \sigma}{2} \\ |D\eta| &\leq \frac{4}{(1 - \sigma)R}, \quad |H(\eta)| \leq \frac{16}{(1 - \sigma)^2 R^2}. \end{aligned}$$

Then, if  $v = \eta u$  we also have  $v \in W_o^{2,p}(B_R)$ . We want to prove that  $\Delta v \in N^{p,\lambda}(B_R)$ .

As a matter of fact, being  $u \in W^{2,p}(\Omega)$ , one obtains  $u, Du \in N^{p,\mu}(\Omega)$  for some  $\mu > 0$  <sup>(4)</sup>. Thus, since  $\Delta u \in N^{p,\lambda}(\Omega)$  it follows  $\Delta v \in N^{p,\mu}(B_R)$  for some  $\mu \in ]0, \lambda]$ .

Let us suppose  $\mu \in ]0, \lambda[$ .

In this case the previous observations together with Proposition 3.1 imply  $H(v) \in N^{p,\mu}(B_R)$  and thus  $H(u) \in N^{p,\mu}(B_{\sigma R})$ ,  $\mu \in ]0, \lambda[$ .

Starting now from the fact that  $u \in W^{2,p,(\mu)}(B_{\sigma R})$  and repeating the above argument we get  $u, Du \in N^{p,\mu_1}(B_{\sigma R})$ , for some  $\mu_1 \in ]\mu, \lambda]$  <sup>(4)</sup>, and  $\Delta v \in N^{p,\mu_1}(B_R)$ .

If still  $\mu_1 \neq \lambda$  we iterate a finite number of times the previous procedure up obtaining  $\Delta v \in N^{p,\lambda}(B_R)$ .

Thus another application of Proposition 3.1 gives

$$H(v) \in N^{p,\lambda}(B_R) \Rightarrow H(u) \in N^{p,\lambda}(B_{\sigma R})$$

and

$$(15) \quad \begin{aligned} \|H(u)\|_{N^{p,\lambda}(B_{\sigma R})} &= \|H(v)\|_{N^{p,\lambda}(B_R)} \leq C \|\Delta v\|_{N^{p,\lambda}(B_R)} \\ &\leq C \left[ \frac{1}{(1 - \sigma)^2 R^2} \|u\|_{N^{p,\lambda}(B_R)} + \frac{1}{(1 - \sigma)R} \|Du\|_{N^{p,\lambda}(B_{\sigma' R})} + \|\Delta u\|_{N^{p,\lambda}(B_R)} \right]. \end{aligned}$$

Proceeding now as in the proof of Theorem 9.11 from [15] and taking into account Proposition 2.5, we then obtain, for  $\sigma = 1/2$ ,

$$\|H(u)\|_{N^{p,\lambda}(B_{R/2})} \leq \frac{C}{R^2} \left[ \|u\|_{N^{p,\lambda}(B_R)} + R^2 \|\Delta u\|_{N^{p,\lambda}(B_R)} \right].$$

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<sup>4</sup> Using Sobolev and Hölder inequalities and Proposition 2.2.



The required estimate follows once more from the above one by covering  $\Omega'$  with a finite number of balls of radius  $R/2$ .  $\square$

In order to extend Theorem 3.1 to the boundary  $\partial\Omega$  we first consider the case of a flat boundary portion.

If  $y_o \equiv (y_{o1}, \dots, y_{on-1}, 0)$ , we set

$$\begin{aligned} B_R^+ &= (B(y_o, R))^+ = B(y_o, R) \cap \mathbb{R}_+^n \\ &= B(y_o, R) \cap \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}. \end{aligned}$$

**Proposition 3.2.** *Let  $u \in W^{2,p}(B_1^+)$ ,  $u = 0$  on  $B_1 \cap \partial\mathbb{R}_+^n$ , such that  $\Delta u \in N^{p,\lambda}(B_1^+)$ . Then, for every  $R \in ]0, 1[$ ,  $H(u) \in N^{p,\lambda}(B_R^+)$  and there exists a constant  $C = C(n, p, \lambda) > 0$  such that*

$$(16) \quad \|H(u)\|_{N^{p,\lambda}(B_R^+)} \leq C \left[ \|u\|_{N^{p,\lambda}(B_1^+)} + \|\Delta u\|_{N^{p,\lambda}(B_1^+)} \right].$$

PROOF: We extend  $u$  and the weight  $\nu_{x_o}(x) = |x - x_o|^{-\lambda}$ ,  $x_o \in B_1^+$ , to all of  $B_1$  (see [2, Lemma IX.2]) by setting

$$\begin{aligned} \tilde{\nu}_{x_o}(x', x_n) &= \begin{cases} \nu_{x_o}(x', x_n) & \text{for } (x', x_n) \in B_1^+ \\ \nu_{x_o}(x', -x_n) & \text{for } (x', -x_n) \in B_1 \setminus B_1^+, \end{cases} \\ \tilde{u}(x', x_n) &= \begin{cases} u(x', x_n) & \text{for } (x', x_n) \in B_1^+ \\ 0 & \text{for } (x', x_n) \in B_1 \cap \partial\mathbb{R}_+^n \\ u(x', -x_n) & \text{for } (x', -x_n) \in B_1^+. \end{cases} \end{aligned}$$

It can be readily checked that the function  $\tilde{u} \in W^{2,p}(B_1)$  and moreover

$$\|\Delta \tilde{u}\|_{N^{p,\lambda}(B_1)} \leq C \|\Delta u\|_{N^{p,\lambda}(B_1^+)} < +\infty.$$

Arguing as in the previous theorem, for  $R \in ]0, 1[$ , let us introduce a cutoff function  $\eta \in C_o^2(B_1)$  satisfying

$$\begin{aligned} 0 &\leq \eta \leq 1, \quad \forall x \in B_1 \\ \eta &= 1 \quad \text{in } B_R \\ \eta &= 0 \quad \text{for } |x - y_o| \geq R', \quad R' = \frac{1+R}{2} \\ |D\eta| &\leq \frac{4}{(1-R)}, \quad |H(\eta)| \leq \frac{16}{(1-R)^2} \end{aligned}$$

and consider the function  $v = \eta\tilde{u} \in W_o^{2,p}(B_1)$ .

Then, since  $\Delta v \in N^{p,\lambda}(B_1)$ , we have  $H(\tilde{u}) \in N^{p,\lambda}(B_R)$  and

$$\|H(\tilde{u})\|_{N^{p,\lambda}(B_R)} \leq C \left[ \|\tilde{u}\|_{N^{p,\lambda}(B_1)} + \|\Delta \tilde{u}\|_{N^{p,\lambda}(B_1)} \right].$$

The estimate (16) follows now in the standard way:

$$\begin{aligned} \|H(u)\|_{N^{p,\lambda}(B_R^+)} &\leq \|H(\tilde{u})\|_{N^{p,\lambda}(B_R)} \\ &\leq C \left[ \|\tilde{u}\|_{N^{p,\lambda}(B_1)} + \|\Delta \tilde{u}\|_{N^{p,\lambda}(B_1)} \right] \\ &\leq C \left[ \|u\|_{N^{p,\lambda}(B_1^+)} + \|\Delta u\|_{N^{p,\lambda}(B_1^+)} \right]. \end{aligned}$$

□

With the aid of the previous propositions we derive a global estimate.

**Proposition 3.3.** *Let  $u \in W^{2,p} \cap W_o^{1,p}(\Omega)$  such that  $\Delta u \in N^{p,\lambda}(\Omega)$ . Then  $H(u) \in N^{p,\lambda}(\Omega)$  and there exists a constant  $C = C(n, p, \lambda, \partial\Omega) > 0$  such that*

$$(17) \quad \|H(u)\|_{N^{p,\lambda}(\Omega)} \leq C \left( \|u\|_{N^{p,\lambda}(\Omega)} + \|\Delta u\|_{N^{p,\lambda}(\Omega)} \right).$$

PROOF: Since  $\partial\Omega \in C^2$ , for each point  $y_o \in \partial\Omega$  there is a neighborhood  $\mathcal{N} = \mathcal{N}_{y_o}$  and a corresponding diffeomorphism  $\psi = \psi_{y_o}$  from  $\mathcal{N}$  onto the unit ball  $B = B(0, 1)$  in  $\mathbb{R}^n$  such that

- (i)  $\psi \in C^2(\mathcal{N})$ ,  $\psi^{-1} \in C^2(B)$ ,
- (ii)  $\psi(\mathcal{N} \cap \Omega) = B^+$ ,
- (iii)  $\psi(\mathcal{N} \cap \partial\Omega) = B \cap \partial\mathbb{R}_+^n$ .

Writing

$$\tilde{u}(x) = u(\psi(x)), \quad x \in \mathcal{N}$$

we have  $\tilde{u} \in W^{2,p}(B^+)$ ,  $\Delta \tilde{u} \in N^{p,\lambda}(B^+)$  and  $\tilde{u} = 0$  on  $B \cap \partial\mathbb{R}_+^n$ .

By Proposition 3.2 we thus obtain the estimate

$$\|H(\tilde{u})\|_{N^{p,\lambda}(B_R^+)} \leq C \left[ \|\tilde{u}\|_{N^{p,\lambda}(B_1)} + \|\Delta \tilde{u}\|_{N^{p,\lambda}(B_1)} \right], \quad R \in ]0, 1[.$$

Taking  $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}_{y_o} = \psi^{-1}(B_{R/2})$  and returning back to our original coordinates, we obtain

$$\|H(u)\|_{N^{p,\lambda}(\tilde{\mathcal{N}})} \leq C \left[ \|u\|_{N^{p,\lambda}(\mathcal{N})} + \|\Delta u\|_{N^{p,\lambda}(\mathcal{N})} \right].$$

Finally, by covering  $\partial\Omega$  with a finite number of such neighborhoods  $\tilde{\mathcal{N}}$  and using also the interior estimate (14) we obtain the thesis. □

The following inequality of Miranda-Talenti type holds (see Talenti [35], Grisvard [18, Section 2.3] and also Gilbarg, Trudinger [15, Chapter 9]).

**Theorem 3.2.** *There exists a constant  $C_{MT} = C_{MT}(n, p, \lambda, \partial\Omega) > 0$  such that, for any  $u \in W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)$  (5), we have*

$$(18) \quad \|u\|_{W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)} \leq C_{MT} \|\Delta u\|_{N^{p,\lambda}(\Omega)}.$$

PROOF: Since  $N^{p,\lambda}(\Omega) \subset L^p(\Omega)$  (6) the Laplace operator

$$\Delta : W^{2,p} \cap W_o^{1,p}(\Omega) \rightarrow N^{p,\lambda}(\Omega)$$

is a bijection. Moreover, by virtue of Proposition 3.3

$$\Delta : W^{2,p,\lambda} \cap W_o^{1,p}(\Omega) \rightarrow N^{p,\lambda}(\Omega),$$

is also a bijection.

On the other hand, being

$$\|\Delta u\|_{N^{p,\lambda}(\Omega)} \leq \|u\|_{W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)},$$

it follows that

$$\Delta : W^{2,p,\lambda} \cap W_o^{1,p}(\Omega) \rightarrow N^{p,\lambda}(\Omega)$$

is continuous and thus, by the “open mapping” Theorem, also  $\Delta^{-1}$  is continuous, i.e.

$$\|\Delta^{-1}(\Delta u)\|_{W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)} \leq C_{MT} \|\Delta u\|_{N^{p,\lambda}(\Omega)}.$$

□

#### 4. Applications to elliptic equations

Let us now consider the question of existence and uniqueness in  $W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)$  of the solution to the Dirichlet problem:

$$(19) \quad \begin{cases} E(u) \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The structural hypotheses on the operator  $E$  (see Cordes [10], [11], [12], Talenti [35], Giusti [16], Campanato, Cannarsa [8], Campanato [6] and also Guglielmino [19], Nicolosi [28]) are:

<sup>5</sup> Due to inequality (11) we can equip  $W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)$  with the norm (3).

<sup>6</sup> See Proposition 2.1.

- (a)  $a_{ij}(x) \in L^\infty(\Omega)$ ,  $a_{ij}(x) = a_{ji}(x)$   $i, j = 1, 2, \dots, n$ ;  
 (b) (Strong ellipticity condition) there exists a constant  $\nu > 0$  such that

$$(20) \quad \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2 \quad \text{a.a. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n;$$

- (c) (Cordes-type condition) there exists a constant  $K \in [0, 1[$  such that

$$(21) \quad \frac{(\sum_{i=1}^n a_{ii}(x))^2}{\sum_{i,j=1}^n (a_{ij}(x))^2} \geq n - \frac{K^2}{C_{MT}^2} \quad \text{a.a. } x \in \Omega.$$

Existence-uniqueness of the solution in the space  $W^{2,2} \cap W_o^{1,2}(\Omega)$  and regularity of its second derivatives in the classical Morrey space  $L^{2,\lambda}(\Omega)$  for such a class of elliptic equations have been studied respectively by Talenti [35] and by Talenti [36], Giusti [16], [17]; while in the case of a generic  $p \in ]1, +\infty[$ , as far as the author is aware, until now only existence-uniqueness of the solution in the space  $W^{2,p} \cap W_o^{1,p}(\Omega)$  have been studied by Pucci [29] and Campanato [4], [5], [6] (see also Pucci, Talenti [30]).

It is our aim to prove global regularity in  $N^{p,\lambda}(\Omega)$  of the second derivatives of the solution to the problem (19).

Before proving the above stated result we will premise some remarks.

**Remark 4.1.** Hypothesis (20) implies that

$$(22) \quad \sum_{i=1}^n a_{ii}(x) \geq n \nu.$$

Moreover, by Cauchy-Schwartz inequality we infer

$$(23) \quad \sum_{i=1}^n a_{ii}(x) = \sum_{i,j=1}^n a_{ij}(x) \delta_{ij} \leq \sqrt{n} \left( \sum_{i,j=1}^n (a_{ij}(x))^2 \right)^{1/2}.$$

The above two inequalities yield

$$(24) \quad \sum_{i,j=1}^n (a_{ij}(x))^2 \geq n \nu^2.$$

From (a), (22), (23) and (24) we deduce that the function

$$(25) \quad a(x) = \frac{\sum_{i=1}^n a_{ii}(x)}{\sum_{i,j=1}^n (a_{ij}(x))^2}$$

is measurable, strictly positive and bounded a.e. in  $\Omega$  (<sup>7</sup>) (see also Giusti [16, p. 368] and Campanato, Cannarsa [8, pp. 1378–1379]).

Now, using the Lax-Milgram type Theorem of [21] (see also Campanato [5], [6], [7]) we prove the following theorem:

**Theorem 4.1.** *Let  $f \in N^{p,\lambda}(\Omega)$  and let conditions (a),(b),(c) be satisfied. Then there exists a unique solution  $u$  of the problem*

$$(26) \quad \begin{cases} u \in W^{2,p,\lambda} \cap W_o^{1,p}(\Omega) \\ \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x). \end{cases}$$

Moreover we have the estimate

$$(27) \quad \|u\|_{W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)} \leq \frac{C_{MT}}{\nu(1-K)} \|f\|_{N^{p,\lambda}(\Omega)}.$$

PROOF: Fixed  $f \in N^{p,\lambda}(\Omega)$ , let us observe that, by virtue of Remark 4.1, problem (26) is equivalent to problem

$$(28) \quad \begin{cases} u \in W^{2,p,\lambda} \cap W_o^{1,p}(\Omega) \\ A(u) \equiv \sum_{i,j=1}^n a(x) a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = a(x) f(x). \end{cases}$$

We will prove that the operator  $A$  is “near” by the Laplace operator

$$\Delta : W^{2,p,\lambda} \cap W_o^{1,p}(\Omega) \rightarrow N^{p,\lambda}(\Omega).$$

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<sup>7</sup> By (23) and (24) we get

$$a(x) = \frac{\sum_{i=1}^n a_{ii}(x)}{(\sum_{i,j=1}^n (a_{ij}(x))^2)^{1/2}} \frac{1}{(\sum_{i,j=1}^n (a_{ij}(x))^2)^{1/2}} \leq \frac{1}{\nu}.$$

In fact, for any  $u \in W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)$ , we have

$$\begin{aligned}
(29) \quad & \left\| \Delta u - \sum_{i,j=1}^n a(x) a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{N^{p,\lambda}(\Omega)} \\
&= \left\| \sum_{i,j=1}^n (\delta_{ij} - a(x) a_{ij}(x)) \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{N^{p,\lambda}(\Omega)} \\
&\leq \left\| \left[ \sum_{i,j=1}^n (\delta_{ij} - a(x) a_{ij}(x))^2 \right]^{1/2} \left[ \sum_{i,j=1}^n \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \right]^{1/2} \right\|_{N^{p,\lambda}(\Omega)} \\
&= \left\| \left[ n - 2 \sum_{i=1}^n a(x) a_{ii}(x) + \sum_{i,j=1}^n (a(x) a_{ij}(x))^2 \right]^{1/2} \left[ \sum_{i,j=1}^n \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \right]^{1/2} \right\|_{N^{p,\lambda}(\Omega)} \\
&= \left\| \left[ n - \frac{\left( \sum_{i=1}^n a_{ii}(x) \right)^2}{\sum_{i,j=1}^n (a_{ij}(x))^2} \right]^{1/2} \left[ \sum_{i,j=1}^n \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \right]^{1/2} \right\|_{N^{p,\lambda}(\Omega)} \\
&\leq \frac{K}{C_{MT}} \|H(u)\|_{N^{p,\lambda}(\Omega)}
\end{aligned}$$

where we have exploited Cauchy-Schwartz inequality, the definition of  $a(x)$  and hypotheses (a), (21).

From (29) and (18) we deduce

$$(30) \quad \|\Delta u - A(u)\|_{N^{p,\lambda}(\Omega)} \leq K \|\Delta u\|_{N^{p,\lambda}(\Omega)}.$$

Thus from the Theorem in [21] it follows that there exists a unique  $u \in W^{2,p,\lambda} \cap W_o^{1,p}(\Omega)$  which satisfies equation (26).

To prove the required estimate for the solution  $u$  we will argue in the following way:

$$\begin{aligned}
(31) \quad \|\Delta u\|_{N^{p,\lambda}(\Omega)} &\leq \|\Delta u - A(u)\|_{N^{p,\lambda}(\Omega)} \\
&\quad + \left\| \sum_{i,j=1}^n a(x) a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{N^{p,\lambda}(\Omega)} \\
&\leq K \|\Delta u\|_{N^{p,\lambda}(\Omega)} + 1/\nu \|f\|_{N^{p,\lambda}(\Omega)}
\end{aligned}$$

from which it follows

$$(32) \quad \|\Delta u\|_{N^{p,\lambda}(\Omega)} \leq \frac{1}{\nu(1-K)} \|f\|_{N^{p,\lambda}(\Omega)}.$$

Combining together (18) and (32) we get (27).  $\square$

**Corollary 4.1.** *Let the hypotheses of Theorem 4.1 be satisfied.*

- (i) *If  $1 < p \leq n$ ,  $n - p < \lambda < n$ , then  $Du \in C^{0,\mu}(\bar{\Omega})$  with  $\mu = 1 - \frac{n-\lambda}{p}$ ;*
- (ii) *if  $p > n$ ,  $0 \leq \lambda < n$ , then  $Du \in C^{0,\mu}(\bar{\Omega})$  with  $\mu = 1 - \frac{n}{p}$ .*

**Remark 4.2.** Given a function  $\psi \in W^{2,p,(\lambda)}(\Omega)$ , the result of Theorem 4.1 can be readily extended to the nonhomogeneous Dirichlet problem

$$\begin{cases} u \in W^{2,p,(\lambda)}(\Omega) \\ \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) \\ u - \psi \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \end{cases}$$

just observing that the previous problem is equivalent to the following one

$$\begin{cases} w \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \\ \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} = f(x) - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 \psi}{\partial x_i \partial x_j}. \end{cases}$$

**Remark 4.3.** Let us consider the fully nonlinear second order elliptic operator of “quasi-basic” type

$$A(u) = a(x, H(u))$$

where

$$x \in \Omega, \quad u : \Omega \rightarrow \mathbb{R}^N \quad (N \in \mathbb{N}),$$

$a(x, \xi)$  is a vector of  $\mathbb{R}^N$ , measurable in  $x$  and continuous in  $\xi$  such that  $a(x, 0) = 0$ , elliptic in the sense of the definition  $(A_q)$  of Campanato [6], i.e. there exist three constants  $\alpha, \gamma, \delta$ , with  $\gamma + \delta < 1$ , such that  $\forall x \in \Omega$  and  $\forall \xi, \tau \in \mathbb{R}^{n^2 N}$

$$\left\| \sum_i \xi_{ii} - \frac{\alpha}{C(q)} [a(x, \xi + \tau) - a(x, \tau)] \right\|_{\mathbb{R}^N} \leq \frac{\gamma^{q+1}}{C(q)} \|\xi\| + \delta^{\frac{q+1}{q}} \left\| \sum_i \xi_{ii} \right\| \quad (8)$$

With a few formal adjustments the above result can as well be extended to quasi-basic operators just substituting the constant  $C(q)$  by the constant  $C_{MT}$  from Theorem 3.2.

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<sup>8</sup>  $C(q)$  is the constant of the unweighted (i.e.  $\lambda = 0$ ) Miranda-Talenti inequality.

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DIPARTIMENTO DI MATEMATICA ED INFORMATICA, VIALE A. DORIA 6, 95125 CATANIA, ITALY  
E-mail: leonardi@dmi.unict.it

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