Locally compact linearly Lindelöf spaces

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Abstract. There is a locally compact Hausdorff space which is linearly Lindelöf and not Lindelöf. This answers a question of Arhangel'skii and Buzyakova.

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This note is devoted to the proof of:

Theorem 1. *There is a compact Hausdorff space X and a point p in X such that:*

(1) $\chi(p, X) > \omega$;
(2) *for all regular $\kappa > \omega$, no κ-sequence of points distinct from p converges to p.*

As usual, $\chi(p, X)$, the *character* of p in X, is the least size of a local base at p. Regarding (2), if $\vec{q} = \langle q_\alpha : \alpha < \kappa \rangle$ is a κ-sequence, we say $\vec{q} \to p$ iff whenever U is a neighborhood of p, $\exists \alpha \forall \beta > \alpha \, [q_\beta \in U]$. Then, (2) asserts that $\vec{q} \not\to p$ whenever $\kappa > \omega$ is regular and all the $q_\alpha \neq p$. Observe that if $\chi(p, X) = \omega$, then (2) holds trivially.

Theorem 1 answers Question 1 of Arhangel'skii and Buzyakova [1]. They point out that given such an X, p, the space $X \backslash \{p\}$ is linearly Lindelöf (by (2)), not Lindelöf (by (1)), and locally compact.

Note that in any compact Hausdorff space X, if the point x is not isolated, then there is a sequence of type $\mathrm{cf}\,(\chi(x, X))$ converging to x. Thus, the X, p in Theorem 1 must satisfy $\mathrm{cf}\,(\chi(p, X)) = \omega$. In our example, $\chi(p, X)$ will be $\beth_\omega$.

Our X will be constructed as an inverse limit. We begin by reviewing some basic terminology:

Definition 2. *An* inverse system *is a sequence $\langle X_n, \pi_n^{n+1} : n \in \omega \rangle$, where each X_n is a compact Hausdorff space, and each π_n^{n+1} is a continuous map from X_{n+1} onto X_n.*

Such an inverse systems yields a compact Hausdorff space,

$$X_\omega = \overleftarrow{\lim}_n X_n = \{x \in \prod_n X_n : \forall n \, [x_n = \pi_n^{n+1}(x_{n+1})]\}.$$

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It also yields the obvious maps $\pi^\omega_m : X_\omega \twoheadrightarrow X_m$ for $m < \omega$ and $\pi^n_m : X_n \twoheadrightarrow X_m$ for $m \le n < \omega$.

Lemma 3. *Suppose that $\langle X_n, \pi^{n+1}_n : n \in \omega\rangle$, is an inverse system and $p = \langle p_n : n \in \omega\rangle \in X = X_\omega$ satisfies:*

(A) *each p_n is a weak $P_{\beth_n}$-point in X_n;*
(B) *each $\chi(p_n, X_n) \le \beth_{n+1}$;*
(C) *each $(\pi^n_0)^{-1}\{p_0\}$ is nowhere dense in X_n.*

Then X, p satisfies Theorem 1 with $\chi(p, X) = \beth_\omega$.

As usual, $y \in Y$ is a *weak* P_κ-point iff y is not in the closure of any subset of $Y\backslash\{y\}$ of size less than κ, and y is a P_κ-point iff the intersection of fewer than κ neighborhoods of y is always a neighborhood of y. In a Hausdorff space, every P_κ-point is a weak P_κ-point, but note that in (A), the p_n cannot all be $P_{\beth_n}$-points, as that would contradict (C). Note that (C) cannot be omitted; it is easy to construct an example satisfying (A) and (B) where each X_n is a LOTS and each π^{n+1}_n collapses an interval around p_{n+1} to the point p_n; then $\chi(p, X) = \omega$.

Proof of Lemma 3: First, note that one local base at any $x \in X$ consists of all the $(\pi^\omega_n)^{-1}(U)$ such that $n \in \omega$ and U is an open neighborhood of x_n in X_n. It follows that:

(α) $\chi(p, X_\omega) \le \sup_n \chi(p_n, X_n) = \beth_\omega$;
(β) $(\pi^\omega_0)^{-1}\{p_0\}$ is nowhere dense in X_ω.

Now, to verify (2) of Theorem 1, assume that $\vec{q} = \langle q_\alpha : \alpha < \kappa\rangle \to p$, where $\kappa > \omega$ is regular and all the $q_\alpha \ne p$. The definition of $\vec{q} \to p$ implies that $\kappa \le \chi(p, X)$, so fix m with $\kappa < \beth_m$. Now, $q_\alpha \ne p$ implies that $\pi^\omega_n(q_\alpha) \ne p_n = \pi^\omega_n(p)$ for some n, so we can fix $n \ge m$ and an $S \subseteq \kappa$ with $|S| = \kappa$ and $\pi^\omega_n(q_\alpha) \ne p_n$ for all $\alpha \in S$. But then $p_n \in \mathrm{cl}\{\pi^\omega_n(q_\alpha) : \alpha \in S\}$, contradicting (A).

In view of (α), to prove that $\chi(p, X) = \beth_\omega$, it is sufficient to fix $m < \omega$ and prove that $\chi(p, X) \ge \beth_m$. Suppose that $\mathcal{B}$ were a local base at p in X with $|\mathcal{B}| < \beth_m$. Let $F = (\pi^\omega_m)^{-1}\{p_m\}$. By ($\beta$), F is nowhere dense in X, so for each $U \in \mathcal{B}$, we can choose $y_U \in U\backslash F$. Then $p \in \mathrm{cl}\{y_U : U \in \mathcal{B}\}$, so $p_m = \pi^\omega_m(p) \in \mathrm{cl}\{\pi^\omega_m(y_U) : U \in \mathcal{B}\}$, contradicting (A). $\square$

We now need to find an inverse system satisfying the hypotheses of Lemma 3. X_n will be $\beta\beth_n$. In general, $\beta\kappa$ denotes the Čech compactification of a discrete κ; equivalently, $\beta\kappa$ is the space of ultrafilters on κ; thus, the remainder, $\kappa^* = \beta\kappa\backslash\kappa$, is the space of non-principal ultrafilters on κ.

The p_n will be good ultrafilters. Following Keisler [5], an ultrafilter x on κ is *good* (i.e., κ^+-good) iff given $A_s \in x$ for $s \in [\kappa]^{<\omega}$, there are $B_\alpha \in x$ for $\alpha < \kappa$ such that $\bigcap_{\alpha \in s} B_\alpha \subseteq A_s$ for all non-empty $s \in [\kappa]^{<\omega}$. For every infinite κ, there is a non-principal $x \in \beta\kappa$ such that x is a good ultrafilter (Keisler [5] under GCH and Kunen [7] in ZFC; see also Chang and Keisler [3, Theorem 6.1.4]). The following folklore result about such ultrafilters is proved in [2] and [4]:

Lemma 4. *If x is a good ultrafilter on κ, then x is a weak P_κ-point in $\beta\kappa$.*

Thus, fixing $p_n \in \beta\beth_n$ to be good will handle (A) of Lemma 3, but to get $p = \langle p_n : n \in \omega\rangle$ to really define a point in $X = X_\omega$, we need to choose the $\pi_n^{n+1} : \beta\beth_{n+1} \twoheadrightarrow \beta\beth_n$ such that each $p_n = \pi_n^{n+1}(p_{n+1})$. In fact, π_n^{n+1} will be $\beta(\Pi_n^{n+1})$, where $\Pi_n^{n+1} : \beth_{n+1} \twoheadrightarrow \beth_n$. Here, as usual, if $f : P \to Q$, where P, Q are Tychonov spaces, then $\beta f : \beta P \to \beta Q$ denotes its Čech extension. In the special case of discrete P, Q, where $x \in \beta P$ is an ultrafilter on P, $(\beta f)(x) \in \beta Q$ is the induced measure, $\{B \subseteq Q : f^{-1}(B) \in x\}$. Now, showing that appropriate $\Pi_n^{n+1} : \beth_{n+1} \twoheadrightarrow \beth_n$ can be chosen requires a digression:

Definition 5. *An ultrafilter x on κ is* regular *iff there are $E_\alpha \in x$ for $\alpha < \kappa$ such that $\bigcap_n E_{\alpha_n} = \emptyset$ whenever the α_n (for $n \in \omega$) are distinct.*

Clearly, such x are countably incomplete. Moreover,

Lemma 6. *If x is a countably incomplete good ultrafilter on κ, then x is regular.*

This is Exercise 6.1.3 of [3]; a proof is contained in the proof of Lemma 2.1 of Keisler [6]. The proof of universality of regular ultrapowers ([3, Theorem 4.3.12]) is easily modified to produce:

Lemma 7. *Suppose that $\kappa \geq 2^\lambda$ and y is any ultrafilter on λ. Let x be a regular ultrafilter on κ. Then there is an $f : \kappa \twoheadrightarrow \lambda$ such that $(\beta f)(x) = y$.*

Proof: Since $\kappa \geq 2^\lambda$, we may list the elements of y (possibly with repetitions) as $\{B_\alpha : \alpha < \kappa\}$. Let the $E_\alpha \subseteq \kappa$ be as in Definition 5. Choose $g : \kappa \to \lambda$ such that $g(\xi)$ is some element of $\bigcap\{B_\alpha : \xi \in E_\alpha\}$ (observe that this is a finite intersection). Then $(\beta g)(x) = y$ because each $g^{-1}(B_\alpha) \supseteq E_\alpha \in x$. This g may fail to be onto, but we may now fix a set $A \in x$ with $|\kappa \backslash A| = \kappa$, and choose $f : \kappa \twoheadrightarrow \lambda$ such that $f \restriction A = g \restriction A$, so that $(\beta f)(x) = (\beta g)(x) = y$. □

Proof of Theorem 1: We obtain the situation of Lemma 3. Fix $X_n = \beta\beth_n$, and fix $p_n \in \beta\beth_n$ to be good and non-principal (and hence countably incomplete). Applying Lemmas 6 and 7, fix $\Pi_n^{n+1} : \beth_{n+1} \twoheadrightarrow \beth_n$ so that setting $\pi_n^{n+1} = \beta(\Pi_n^{n+1})$ yields $p_n = \pi_n^{n+1}(p_{n+1})$. Then (A) follows by Lemma 4, and (B) is clear, since there is a base for the space X_n of size $2^{\beth_n} = \beth_{n+1}$. Finally, (C) holds because $(\pi_0^n)^{-1}\{p_0\} \subseteq (\beth_n)^*$, which is nowhere dense in $\beta\beth_n$. □

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