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ON DERIVATIONS OVER RINGS OF TRIANGULAR MATRICES

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A b s t r a c t. The purpose of this article is to describe the structure of derivations over general rings of upper or lower triangular matrices. Some examples are developed, together with a simple formula that gives the dimension of the module of derivations on the center of the underlying ring.

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1. *Introduction*

The determination of the structure of derivations is relevant to develop their behaviour as operators. In general it is difficult if not possible to determine such structure. Some advances can be obtained in algebras of infinite matrices corresponding to Hilbert-Schmidt operators acting on a separable Hilbert space (cf. [2]). For non existence theorems of bounded derivations in Banach algebras of weighted sequences see [3]. For examples of general derivations in non C^* algebras nor non von Neumann algebras see [4]. In some sense the lack of knowledge of such results is overtaken by appealing to alternative construccions. For instance, in the theory of C^* - algebras

J. Glimm (cf. [8]) introduced the notion of uniformly hyperfinite algebras (UHF - algebras), i.e., C^* - unitary algebras $\mathfrak{U}$ endowed with an increasing sequence $\{\mathfrak{U}_n\}$ of finite dimensional full matrix algebras containing the identity whose union is dense in $\mathfrak{U}$. The study of $*$ - derivations defined on $\cup\mathfrak{U}_n$ includes the study of general quantum lattice systems. Among other successful contributions on UHF algebras and normal $*$ - derivations we mention that of S. Sakai and H. Araki (cf. [10], [1]). The corresponding notion of UHF - algebras for VN algebras goes back to F. J. Murray and J. Von Neumann (cf. [9]). J. Dixmier considered inductive limits of matrix algebras without the demand that the embeddings preserve units (cf. [7]). Latter, O. Bratteli introduced the approximately finite dimensional C^* - algebras (or AF algebras), i.e., C^* - algebras $\mathfrak{Q}$ that have an increasing sequence of finite dimensional $*$ - subalgebras $\{\mathfrak{Q}_n\}$ whose union is dense in $\mathfrak{Q}$ (cf. [5]). The resource to consider the structure and properties of operators acting on finite matrix algebras is plainly relevant. Throughout this article $\mathfrak{R}$ will be a ring and ${}_{\mathfrak{R}}\mathfrak{R}$ and $\mathfrak{R}_{\mathfrak{R}}$ will denote the left and right module structures of $\mathfrak{R}$ over itself. If n is a positive integer, we shall be concerned with the ring $\mathbb{UT}_n(\mathfrak{R})$ of upper triangular $n \times n$ matrices over $\mathfrak{R}$. Of course, our conclusions hold under suitable modifications to the ring $\mathbb{LT}_n(\mathfrak{R})$ of lower triangular $n \times n$ on $\mathfrak{R}$. $\mathbb{UT}_n(\mathfrak{R})$ inherits left and right module structures on ${}_{\mathfrak{R}}\mathfrak{R}$ and $\mathfrak{R}_{\mathfrak{R}}$ so that

$$r \cdot (x_{i,j})_{1\leq i\leq j\leq n} = (r \cdot x_{i,j})_{1\leq i\leq j\leq n} \quad \text{or} \quad r \cdot (x_{i,j})_{1\leq i\leq j\leq n} = (x_{i,j} \cdot r)_{1\leq i\leq j\leq n}$$

whenever $r \in \mathfrak{R}$ and $(x_{i,j})_{1\leq i\leq j\leq n} \in \mathbb{UT}_n(\mathfrak{R})$. Let us denote these structures by $\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R})$ and $\mathbb{UT}_n(\mathfrak{R}_{\mathfrak{R}})$ respectively. Since $\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R}) \approx [\mathbb{UT}_n(\mathfrak{R}_{\mathfrak{R}})]^{op}$ we´ll restrict our research to $\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R})$. So let $\mathcal{D}(\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R}))$ be the set of left derivations on $\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R})$, i.e., $\Delta \in \mathcal{D}(\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R}))$ iff $\Delta \in \mathrm{Hom}({}_{\mathfrak{R}}\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R}))$ and $\Delta(\eta \cdot \lambda) = \Delta(\eta)\cdot\lambda + \eta\cdot\Delta(\lambda)$ if $\eta, \lambda \in \mathbb{UT}_n(\mathfrak{R})$ (Leibnitz rule, henceforth abbreviated by Lr). If $1 \leq k \leq h \leq n$ we´ll write $e_{k,h} = \left(\delta_{i,j}^{k,h}\right)_{1\leq i\leq j\leq n}$, where $\delta_{i,j}^{k,h}$ are the usual Kronecker´ symbols. In general $\mathbb{UT}_n(\mathfrak{R})$ is not an $\mathfrak{R}$ - algebra, but the relations $r \cdot (e_{i,j} \cdot e_{k,h}) = (r \cdot e_{i,j}) \cdot e_{k,h} = e_{i,j} \cdot (r \cdot e_{k,h})$ hold for if $r \in \mathfrak{R}$ and all $e_{i,j}$, $e_{k,h}$´s in $\mathbb{UT}_n(\mathfrak{R})$. In Section 2 weshall exhibit the structure of all derivations on $\mathbb{UT}_2({}_{\mathfrak{R}}\mathfrak{R})$, and the particular cases of $\mathbb{Z}_2$, the quaternionic ring on $\mathbb{R}$ and of a general semigroup ring of a group on a ring are considered. In Section 3, if n is a fixed positive integer in Th. 5 we develop a general structure theorem of derivations on $\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R})$. Besides, in Prop. 7 it is evaluated the dimension of $\mathcal{D}(\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R}))$ as a $Z(\mathfrak{R})$ module.

2. *Derivations on* $\mathbb{UT}_2\left({}_{\mathfrak{R}}\mathfrak{R}\right)$

Proposition 1. *If* $\mathfrak{R}$ *is an integral domain and* $\Delta \in \mathcal{D}\left(\mathbb{UT}_2\left({}_{\mathfrak{R}}\mathfrak{R}\right)\right)$ *there are unique* $a, b \in Z\left(\mathfrak{R}\right)$ *so that*

$$\Delta\left(x \cdot e_{1,1} + y \cdot e_{1,2} + z \cdot e_{2,2}\right) = \left[(x-z) \cdot a + y \cdot b\right] \cdot e_{1,2}, \quad x, y, z \in \mathfrak{R}. \quad (1)$$

P r o o f. Let $\Delta\left(e_{r,s}\right) = \left(\varrho_{i,j}^{r,s}\right)_{1 \le i \le j \le 2}$, $1 \le r \le s \le 2$. Since $e_{1,1}^2 = e_{1,1}$ by Lr we obtain $\varrho_{1,1}^{1,1} = \varrho_{2,2}^{1,1} = 0$. So there is a unique $a \in \mathfrak{R}$ so that $\Delta\left(e_{1,1}\right) = a \cdot e_{1,2}$. Since $e_{1,2} = e_{1,1} \cdot e_{1,2} = e_{1,2} \cdot e_{2,2}$ by Lr we get $\varrho_{2,2}^{1,2} = 0$ and $\varrho_{1,1}^{1,2} = \varrho_{2,2}^{2,2} = 0$ respectively. Hence there is a unique $b \in \mathfrak{R}$ so that $\Delta\left(e_{1,2}\right) = b \cdot e_{1,2}$. Indeed, $e_{2,2}^2 = e_{2,2}$ and by Lr $\varrho_{1,1}^{2,2} = 0$, i.e., $\Delta\left(e_{2,2}\right) = c \cdot e_{1,2}$ for a unique $c \in \mathfrak{R}$. Now (1) follows because $\Delta \in \operatorname{Hom}\left({}_{\mathfrak{R}}\mathbb{UT}_n\left({}_{\mathfrak{R}}\mathfrak{R}\right)\right)$. Moreover, if $\eta, \lambda \in \mathbb{UT}_n\left(\mathfrak{R}\right)$ by (1) we can write

$$\Delta\left(\eta \cdot \lambda\right) = \left[\eta_{1,1} \cdot \lambda_{1,1} \cdot a + \left(\eta_{1,1} \cdot \lambda_{1,2} + \eta_{1,2} \cdot \lambda_{2,2}\right) \cdot b + \eta_{2,2} \cdot \lambda_{2,2} \cdot c\right] \cdot e_{1,2}. \quad (2)$$

On the other hand, by (1) is

$$\begin{aligned} \Delta\left(\eta\right) \cdot \lambda + \eta \cdot \Delta\left(\lambda\right) &= \left[\left(\eta_{1,1} \cdot a + \eta_{1,2} \cdot b + \eta_{2,2} \cdot c\right) \cdot \lambda_{2,2} + \right. \\ &\quad \left. + \eta_{1,1} \cdot \left(\lambda_{1,1} \cdot a + \lambda_{1,2} \cdot b + \lambda_{2,2} \cdot c\right)\right] \cdot e_{1,2}. \end{aligned} \quad (3)$$

From (2) and (3) the equation

$$\eta_{1,1} \cdot \left(a \cdot \lambda_{2,2} + \lambda_{2,2} \cdot c\right) = \eta_{1,2} \cdot \left(\lambda_{2,2} \cdot b - b \cdot \lambda_{2,2}\right) + \eta_{2,2} \cdot \left(\lambda_{2,2} \cdot c - c \cdot \lambda_{2,2}\right) \quad (4)$$

holds for all $\eta_{1,1}, \eta_{1,2}, \eta_{2,2}, \lambda_{2,2}$ in $\mathfrak{R}$. In particular, by (4) if $\eta_{1,1} = \eta_{2,2} = 0$, $\eta_{1,2} = 1$ and $\lambda_{2,2}$ is arbitrary then $b \in Z\left(\mathfrak{R}\right)$. If $\eta_{1,1} = \eta_{1,2} = 0$, $\eta_{2,2} = 1$ and $\lambda_{2,2}$ is arbitrary then $c \in Z\left(\mathfrak{R}\right)$. If $\eta_{1,2} = \eta_{2,2} = 0$ and $\eta_{1,1} = \lambda_{2,2} = 1$ then $a + c = 0$, i.e $a \in Z\left(\mathfrak{R}\right)$. Now, (1) follows by (2) since $c = -a$. □

Example 2. $\mathcal{D}\left(\mathbb{UT}_2\left(\mathbb{Z}_2\right)\right) = \left\{0, \Delta_1, \Delta_2, \Delta_3\right\}$, *where*

$$\begin{aligned} \Delta_1\left(x \cdot e_{1,1} + y \cdot e_{1,2} + z \cdot e_{2,2}\right) &= y \cdot e_{1,2}, \\ \Delta_2\left(x \cdot e_{1,1} + y \cdot e_{1,2} + z \cdot e_{2,2}\right) &= (x + z) \cdot e_{1,2}, \\ \Delta_3\left(x \cdot e_{1,1} + y \cdot e_{1,2} + z \cdot e_{2,2}\right) &= (x + y + z) \cdot e_{1,2}. \end{aligned}$$

Example 3. *If* $\mathbb{H}\left(\mathbb{R}\right)$ *denotes the ring of real quaternions* $\mathcal{D}\left(\mathbb{UT}_2\left({}_{\mathbb{H}(\mathbb{R})}\mathbb{H}\left(\mathbb{R}\right)\right)\right)$ *consists of those elements* $\Delta = \Delta(a, b)$ *with* $a, b \in \mathbb{R}$ *and*

$$\Delta\left(a, b\right)\left(x \cdot e_{1,1} + y \cdot e_{1,2} + z \cdot e_{2,2}\right) = \left[(x-z) \cdot a + y \cdot b\right] \cdot e_{1,2}, \quad x, y, z \in \mathbb{H}\left(\mathbb{R}\right).$$

Example 4. *If $\mathfrak{G}$ is a group and $\mathfrak{R}$ is a ring let $\mathfrak{R}[\mathfrak{G}]$ be the semigroup ring of $\mathfrak{G}$ over $\mathfrak{R}$. Elements of $\mathfrak{R}[\mathfrak{G}]$ are functions $\mathfrak{G} \to \mathfrak{R}$ of finite support and*

$$(a+b)(x) = a(x) + b(x), \quad (a \cdot b)(x) = \sum_{u,v \in \mathfrak{G}: u \cdot v = x} a(u) \cdot b(v)$$

if $a, b \in \mathfrak{R}[\mathfrak{G}]$, $x \in \mathfrak{G}$. $\mathfrak{R}[\mathfrak{G}]$ becomes a non abelian ring and $\mathcal{D}\left(\mathbb{UT}_2\left({}_{\mathfrak{R}[\mathfrak{G}]}\mathfrak{R}[\mathfrak{G}]\right)\right)$ contains elements of the form (1) so that $a, b \in Z(\mathfrak{R})[\mathfrak{G}]$ that preserve conjugacy classes, i.e., $a(x) = a\left(u \cdot x \cdot u^{-1}\right)$, $b(x) = b\left(u \cdot x \cdot u^{-1}\right)$ if $u, x \in \mathfrak{G}$.

3. *A structure theorem and dimensionality*

Theorem 5. *A left homomorphism Δ on $\mathbb{UT}_n(\mathfrak{R})$ is a derivation if and only if there is a unique set of scalars $\left\{\varrho_{u,v}^{k,h}\right\}_{\substack{1 \le k \le h \le n \\ 1 \le u \le v \le n}}$ so that*

$$\left(\varrho_{k,h}^{k,h}\right)_{1 \le k,h \le n}, \left(\varrho_{j,k}^{k,k}\right)_{1 \le j,k \le n}, \left(\varrho_{h,l}^{h,h}\right)_{1 \le h,l \le n} \in \mathbb{UT}_n(Z(\mathfrak{R})), \tag{5}$$

if $1 \le k \le h \le n$ then

$$\varrho_{u,v}^{k,h} = 0 \quad if \quad k < u \quad or \quad v < h, \tag{6}$$

$$\varrho_{u,v}^{k,h} = \begin{cases} \varrho_{u,k}^{k,k} & if \quad 1 \le u < k \le n,\ 1 \le v = h \le n, \\ 0 & if \quad \begin{cases} 1 \le u < k \le h \le n,\ 1 \le u \le v \le n,\ v \ne h, \\ or \\ 1 \le k \le h < v \le n,\ 1 \le u \le v \le n,\ u \ne k, \end{cases} \\ \varrho_{h,v}^{h,h} & if \quad 1 \le h < v \le n,\ 1 \le u = k \le n, \end{cases} \tag{7}$$

$$\begin{aligned} \varrho_{u,k}^{u,u} + \varrho_{u,h}^{k,h} &= 0 \quad if \quad 1 \le u < k \le n, 1 \le k \le h \le n, \\ \varrho_{k,v}^{k,h} + \varrho_{h,v}^{v,v} &= 0 \quad if \quad 1 \le h < v \le n, 1 \le k \le h \le n. \end{aligned} \tag{8}$$

and

$$\varrho_{k,h}^{k,h} = \varrho_{k,l}^{k,l} + \varrho_{l,h}^{l,h} \quad if \quad 1 \le k \le l \le h \le n. \tag{9}$$

Thus, if $\mu \in \mathbb{UT}_n(\mathfrak{R})$ then

$$\Delta(\mu) = \sum_{1 \le k < h \le n} \left(\mu_{k,h} \cdot \varrho_{k,h}^{k,h} - \sum_{k < l \le h} \mu_{l,h} \cdot \varrho_{k,l}^{k,k} - \sum_{k \le l < h} \mu_{k,l} \cdot \varrho_{l,h}^{h,h} \right) \cdot e_{k,h}. \tag{10}$$

P r o o f. For $1 \leq i \leq j \leq n$ is $\Delta(e_{i,j}) = \sum_{1\leq u\leq v\leq n} \varrho_{u,v}^{i,j} \cdot e_{u,v}$ for some unique $\varrho_{u,v}^{i,j}$ ´s. Since $e_{i,i}^2 = e_{i,i}$ by Lr is $\varrho_{u,v}^{i,j} = 0$ if $u \neq i$ and $v \neq i$. Indeed, it is easily seeing that $\varrho_{i,i}^{i,i}$ are necessarily zero in this case and so

$$\Delta(e_{i,i}) = \sum_{u=1}^{i-1} \varrho_{u,i}^{i,i} \cdot e_{u,i} + \sum_{v=i+1}^{n} \varrho_{i,v}^{i,i} \cdot e_{i,v}, \quad 1 \leq i \leq n. \tag{11}$$

In particular, sums in (11) are assumed to be zero in case that $i = 1$ or $i = n$. Let $1 \leq k < i \leq n$, $1 \leq k \leq h \leq n$. By (11) is $\Delta(e_{i,i}) \cdot e_{k,h} = 0$ and as $e_{i,i} \cdot e_{k,h} = 0$ by Lr we get $e_{i,i} \cdot \Delta(e_{k,h}) = 0$. Analogously, if $1 \leq i < h \leq n$, $1 \leq k \leq h \leq n$ by (11) is $e_{k,h} \cdot \Delta(e_{i,i}) = 0$. Since $e_{k,h} \cdot e_{i,i} = 0$ by Lr we get $\Delta(e_{k,h}) \cdot e_{i,i} = 0$. Thus we can conclude (6). If $1 \leq k \leq h \leq n$ by (11) we write

$$\Delta(e_{k,h}) = \sum_{u=1}^{k-1} \varrho_{u,k}^{k,k} \cdot e_{u,h} + \sum_{v=k}^{n} \varrho_{k,v}^{k,h} \cdot e_{k,v} = \sum_{u=1}^{h} \varrho_{u,h}^{k,h} \cdot e_{u,h} + \sum_{v=h+1}^{n} \varrho_{h,v}^{h,h} \cdot e_{k,v}. \tag{12}$$

By (6) and (12) it follows (7). If $1 \leq u < k \leq n$, $1 \leq k \leq h \leq n$ by (11) is

$$\Delta(e_{u,u}) \cdot e_{k,h} = \varrho_{u,k}^{u,u} \cdot e_{u,h} = -e_{u,u} \cdot \Delta(e_{k,h}). \tag{13}$$

If $1 \leq h < v \leq n$, $1 \leq k \leq h \leq n$ by (11) is

$$e_{k,h} \cdot \Delta(e_{v,v}) = \varrho_{h,v}^{v,v} \cdot e_{k,v} = -\Delta(e_{k,h}) \cdot e_{v,v}. \tag{14}$$

By (13) and (14) we deduce (8). Since $\varrho_{k,k}^{k,k} = 0$, besides (8) and (11) it follows (10) for $\mu = e_{k,k}$ if $1 \leq k \leq n$. If $1 \leq k < h \leq n$ then (10) holds for $\mu = e_{k,h}$ by (6), (7) and (8). Now, the general case in (10) holds since $\varrho_{k,k}^{k,k} = 0$ if $\leq k \leq n$, (6), (7) and (8). Moreover, if $\eta, \lambda \in \mathbb{UT}_n(\mathfrak{R})$ and $1 \leq k \leq h \leq n$ by Lr the following identities hold

$$\begin{aligned}
&\sum_{c=k}^{h} \eta_{k,c} \cdot \lambda_{c,h} \cdot \varrho_{k,h}^{k,h} - \sum_{a=k+1}^{h} \sum_{b=a}^{h} \eta_{a,b} \cdot \lambda_{b,h} \cdot \varrho_{k,a}^{k,k} - \sum_{b=k}^{h-1} \sum_{a=k}^{b} \eta_{k,a} \cdot \lambda_{a,b} \cdot \varrho_{b,h}^{h,h} \\
&= \sum_{b=k}^{h} \left[\left(\eta_{k,b} \cdot \varrho_{k,b}^{k,b} - \sum_{a=k+1}^{b} \eta_{a,b} \cdot \varrho_{k,a}^{k,k} - \sum_{a=k}^{b-1} \eta_{k,a} \cdot \varrho_{a,b}^{b,b} \right) \cdot \lambda_{b,h} + \right. \\
&\left. + \eta_{k,b} \cdot \left(\lambda_{b,h} \cdot \varrho_{b,h}^{b,h} - \sum_{a=b+1}^{h} \lambda_{a,h} \cdot \varrho_{b,a}^{b,b} - \sum_{a=b}^{h-1} \lambda_{b,a} \cdot \varrho_{a,h}^{h,h} \right) \right].
\end{aligned} \tag{15}$$

If $1 \leq k \leq h \leq n$ and $\eta = e_{k,h}$ in (15) then $\lambda_{h,h} \cdot \varrho_{k,h}^{k,h} = \varrho_{k,h}^{k,h} \cdot \lambda_{h,h}$. Since λ is arbitrary then $\left(\varrho_{k,h}^{k,h}\right)_{1\leq k\leq h\leq n} \in \mathbb{UT}_n\left(Z(\mathfrak{R})\right)$. If $1 \leq k < a$, $r \in \mathfrak{R}$, on choosing $h \geq a$ we write $\eta = e_{a,a}$ and $\lambda = r\cdot e_{a,h}$. So, by (15) is $r\cdot\varrho_{k,a}^{k,k} = \varrho_{k,a}^{k,k}\cdot r$ and since r is arbitrary and $\varrho_{a,a}^{a,a} = 0$ is $\left(\varrho_{k,a}^{k,k}\right)_{1\leq k\leq a\leq n} \in \mathbb{UT}_n\left(Z(\mathfrak{R})\right)$. Finally, by (8) is $\left(\varrho_{k,a}^{a,a}\right)_{1\leq k\leq a\leq n} \in \mathbb{UT}_n\left(Z(\mathfrak{R})\right)$ and we are ready to prove (9). Indeed, we have

$$\sum_{a=k+1}^{h}\sum_{b=a}^{h}\eta_{a,b}\cdot\lambda_{b,h}\cdot\varrho_{k,a}^{k,k} = \sum_{b=k+1}^{h}\left(\sum_{a=k+1}^{b}\eta_{a,b}\cdot\varrho_{k,a}^{k,k}\right)\cdot\lambda_{b,h}, \tag{16}$$

$$\begin{aligned}
\sum_{b=k}^{h-1}\sum_{a=k}^{b}\eta_{k,a}\cdot\lambda_{a,b}\cdot\varrho_{b,h}^{h,h} &= \sum_{b=k}^{h-1}\eta_{k,b}\sum_{a=b}^{h-1}\lambda_{b,a}\cdot\varrho_{a,h}^{h,h},\\
&\sum_{b=k}^{h}\left(\sum_{a=k}^{b-1}\eta_{k,a}\cdot\varrho_{a,b}^{b,b}\cdot\lambda_{b,h}+\right.\\
\left.+\sum_{a=b+1}^{h}\eta_{k,b}\cdot\lambda_{a,h}\cdot\varrho_{b,a}^{b,b}\right) &= \sum_{a=k}^{h}\sum_{a=b+1}^{h}\eta_{k,a}\cdot\lambda_{b,h}\cdot\left(\varrho_{a,b}^{a,a}+\varrho_{a,b}^{b,b}\right) = 0.
\end{aligned}$$

Thus by (15) and (16)

$$\left(\sum_{c=k}^{h}\eta_{k,c}\cdot\lambda_{c,h}\right)\cdot\varrho_{k,h}^{k,h} = \sum_{c=k}^{h}\eta_{k,c}\cdot\lambda_{c,h}\cdot\left(\varrho_{k,c}^{k,c}+\varrho_{c,h}^{c,h}\right)$$

and (9) follows if we put $\eta = e_{k,c}$ and $\lambda = e_{c,h}$ for each $c \in \{k, k+1, ..., h\}$. By the same reasoning, if $\left\{\varrho_{u,v}^{k,h}\right\}_{\substack{1\leq k\leq h\leq n\\ 1\leq u\leq v\leq n}}$ is a subset of $\mathfrak{R}$ so that (5), (6), (7), (8) and (9) hold then (10) defines an element $\mu \in \mathcal{D}\left(\mathbb{UT}_n\left(\mathfrak{R}\right)\right)$. $\square$

Remark 6. *By (8) in Th. 5 is* $\varrho_{k,h}^{k,k} + \varrho_{k,h}^{h,h} = 0$ *if* $1 \leq k \leq h \leq n$. *By (9) the matrix* $\left(\varrho_{k,h}^{k,h}\right)_{1\leq k\leq h\leq n}$ *is determinated by* $\varrho_{i,i+1}^{i,i+1}$, $1 \leq i < n$. *Indeed, we get the following*

Proposition 7. $\dim_{Z(\mathfrak{R})}\mathcal{D}\left({}_{\mathfrak{R}}\mathbb{UT}_n\left(\mathfrak{R}\right)\right) = \left(n^2+n-2\right)/2.$

P r o o f. If $1 \leq j < n$, $\mu \in \mathbb{UT}_n\left(\mathfrak{R}\right)$ let $\Delta_j\left(\mu\right) = \sum_{1\leq k\leq n-j<h\leq n}\mu_{k,h}\cdot e_{k,h}$. Clearly each $\Delta_j \in \operatorname{Hom}\left({}_{\mathfrak{R}}\mathbb{UT}_n\left({}_{\mathfrak{R}}\mathfrak{R}\right)\right)$. Indeed, if $1 \leq k \leq n-j < h \leq n$ and

$\eta, \lambda \in \mathbb{UT}_n(\mathfrak{R})$ then

$$\begin{aligned}(\Delta_j(\eta)\cdot\lambda+\eta\cdot\Delta_j(\lambda))_{k,h} &= \sum_{l=n-j+1}^{h}\eta_{k,l}\cdot\lambda_{l,h}+\sum_{l=k}^{n-j}\eta_{k,l}\cdot\lambda_{l,h}\\ &= (\eta\cdot\lambda)_{k,h}=\Delta_j(\eta\cdot\lambda)_{k,h}.\end{aligned}$$

Since $\Delta_j(\eta\cdot\lambda)_{k,h} = (\Delta_j(\eta)\cdot\lambda)_{k,h} = (\eta\cdot\Delta_j(\lambda))_{k,h} = 0$ if $n-j<k$ or $h \le n-j$ each Δ_j becomes a derivation. Now, if $1 \le k < h \le n$ and $\mu \in \mathbb{UT}_n(\mathfrak{R})$ we´ll write $\Lambda_{k,h}(\mu) = [\mu, e_{k,h}] = \mu\cdot e_{k,h} - e_{k,h}\cdot\mu$, i.e., $\Lambda_{k,h}$ is the inner derivation defined by $e_{k,h}$ and the Lie bracket $[\cdot,\cdot]$. We´ll prove that the set $\mathcal{B} = \{\Delta_j\}_{1\le j<n} \cup \{\Lambda_{k,h}\}_{1\le k<h\le n}$ is a base of $\mathcal{D}({}_{\mathfrak{R}}\mathbb{UT}_n(\mathfrak{R}))$ over $Z(\mathfrak{R})$. For, let

$$\sum_{j=1}^{n-1} a_j\cdot\Delta_j + \sum_{1\le k<h\le n} b_{k,h}\cdot\Lambda_{k,h} = 0 \tag{17}$$

for some constants a_j ´s and $b_{k,h}$ ´s in $Z(\mathfrak{R})$. By (17) we obtain that

$$\mu_{k,h}\cdot\sum_{j=n-h+1}^{n-k} a_j + \sum_{l=k}^{h-1} b_{l,h}\cdot\mu_{k,l} - \sum_{m=k+1}^{h} b_{k,m}\cdot\mu_{m,h} = 0 \tag{18}$$

for all $\mu \in \mathbb{UT}_n(\mathfrak{R})$ and $1 \le k < h \le n$. If we put $\mu = e_{1,2}$ by (18) is $a_{n-1} = 0$. If $\mu = e_{1,l}$ with $2 < l \le n$ by (18) is $a_{n-1} + ... + a_{n-l+1} = 0$. If we already proved that $a_{n-j} = 0$ if $1 \le j < l-1$ then $a_{n-l+1} = 0$, i.e., we get $a_1 = ... = a_{n-1} = 0$. Now, if $1 \le k < l < h \le n$ and $\mu = e_{k,l}$ by (18) is $b_{l,h} = 0$, i.e., $b_{l,h} = 0$ if $2 \le l < h \le n$. If $1 < m \le h \le n$ and $\mu = e_{m,h}$ by (18) is $b_{1,m} = 0$, and so $\mathcal{B}$ is linearly independent. Finally, given $\Delta \in \mathcal{D}({}_{\mathfrak{R}}\mathbb{UT}_n(\mathfrak{R}))$ and $1 \le i \le j \le n$ we write as before $\Delta(e_{i,j}) = \sum_{1\le u\le v\le n} \varrho_{u,v}^{i,j}\cdot e_{u,v}$ for some unique $\varrho_{u,v}^{i,j}$ ´s. By Th. 5 we already know that all $\varrho_{u,v}^{i,j}$ ´s belong to $Z(\mathfrak{R})$. Further, by (6), (7), (8) and (9) it is readily seeing that

$$\Delta = \sum_{j=1}^{n-1} \varrho_{n-j,n-j+1}^{n-j,n-j+1}\cdot\Delta_j + \sum_{1\le k<h\le n} \varrho_{k,h}^{k,k}\cdot\Lambda_{k,h} \tag{19}$$

and our claim follows. □

Remark 8. *As a consequence of the Skolem-Noether theorem (cf. [6], pags. 93 and 105) every k linear derivation on a finite dimensional central simple k - algebra $\mathfrak{Q}$ is inner. This result is no longer applicable because $\mathbb{UT}_n({}_{\mathfrak{R}}\mathfrak{R})$ is not central as a $Z(\mathfrak{R})$ - algebra. For instance, the set of matrices*

$\mu \in \mathbb{UT}_n \left({}_{\mathfrak{R}}\mathfrak{R} \right)$ whose nth row is null is a non zero bilateral ideal. However, observe that $\Delta_j (\circ) = \left[\sum_{l=1}^{n-j} e_{l,l}, \circ \right]$ if $1 \leq j < n$. Since each $\Lambda_{k,h}$ is inner by (19) it follows that every derivation on $\mathbb{UT}_n \left({}_{\mathfrak{R}}\mathfrak{R} \right)$ is inner.

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