Bulletin T.CXXXI de l'Académie serbe des sciences et des arts − 2005
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 30

ON SOME PRODUCT CONFORMALLY FLAT LOCALLY DECOMPOSABLE RIEMANNIAN MANIFOLDS

MILEVA PRVANOVIĆ

(Presented at the 2nd Meeting, held on March 25, 2005)

A b s t r a c t. *We investigate product conformally flat locally decomposable Riemannian manifolds whose curvature tensor can be expressed in the terms of Ricci tensors only.*

AMS Mathematics Subject Classification (2000): 53C15
Key Words: locally decomposable Riemannian manifold, product conformally flat manifold

The object of the paper.

The Riemannian manifolds (M, g) whose Riemannian curvature tensor satisfies the condition

$$R(X,Y,Z,W) = \mathcal{L}[\rho(X,W)\rho(Y,Z) - \rho(X,Z)\rho(Y,W)], \qquad (*)$$

where $\rho(X,W)$ is the Ricci tensor and $\mathcal{L}$ is some scalar function, appears in the investigations of some authors ([1],[2],[3]). D. Kowalczyk proved in [1] that conformally flat manifold satisfies the condition (*) on $U_R = \{p \in M \mid R - \frac{\kappa}{n(n-1)}\gamma \neq 0 \text{ at } p\}$, if and only if

$$\mathcal{L} = \frac{n-1}{(n-2)\kappa}, \quad \rho = \frac{\kappa}{n-1}g + \beta w \otimes w, \quad w \in T_p(U), \ \beta \in R.$$

Here $n = \dim M$, κ is the scalar curvature and

$$\gamma(X,Y,Z,W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W).$$

The object of the present paper is to generalize this result to Riemannian manifolds endowed with product structure. More precisely, we investigate product conformally flat locally decomposable Riemannian manifolds whose curvature tensor can be expressed in the terms of the Ricci tensors only.

1. *Preliminary results*

Let (M,g,f) be a locally decomposable Riemannian manifold. This means that the Riemannian manifold (M,g) is endowed with product structure $f, f^2 = id.$, such that

$$F(X,Y) = g(fX,Y) = g(X,fY), \quad \bigtriangledown f = 0,$$

for all $X,Y \in T_p(M)$, where $T_p(M)$ is the tangent vector space of M at $p \in M$, and $\bigtriangledown$ is the operator of the Levi-Civita connection. M being the locally product manifold $M_1 \times M_2$, we denote dim $M_1 = p$, dim $M_2 = q$. Then $n = \dim M = p+q$, and tr $f = r = p-q$. We suppose $p > 2,\ q > 2$.

Let e_i, $i = 1,2,\dots,n$ be an orthonormal basis of T_p. We define the Ricci tensor ρ, $\sim$ Ricci tensor $\tilde{\rho}$ and the scalar curvatures κ and $\tilde{\kappa}$ by

$$\rho(X,Y) = \sum_{i=1}^{n} R(e_i,X,Y,e_i), \quad \tilde{\rho}(X,Y) = \sum_{i=1}^{n} R(fe_i,X,Y,e_i),$$

$$\kappa = \sum_{i=1}^{n} \rho(e_i,e_i), \quad \tilde{\kappa} = \sum_{i=1}^{n} \tilde{\rho}(e_i,e_i).$$

Both ρ and $\tilde{\rho}$ are symmetric. Moreover, as a consequence of $\bigtriangledown f = 0$, we have

$$\rho(fX,fY) = \rho(X,Y), \quad \tilde{\rho}(fX,fY) = \tilde{\rho}(X,Y), \quad \rho(fX,Y) = \tilde{\rho}(X,Y).$$

Now, let us suppose that

$$\begin{aligned} R(X,Y,Z,W) &= \\ &= \mathcal{L}_1[\rho(X,W)\rho(Y,Z) - \rho(X,Z)\rho(Y,W) + \tilde{\rho}(X,W)\tilde{\rho}(Y,Z) - \tilde{\rho}(X,Z)\tilde{\rho}(Y,W)] \quad (1.1)\\ &+ \mathcal{L}_2[\rho(X,W)\tilde{\rho}(Y,Z) + \tilde{\rho}(X,W)\rho(Y,Z) - \rho(X,Z)\tilde{\rho}(Y,W) - \tilde{\rho}(X,Z)\rho(Y,W)], \end{aligned}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are some scalar functions such that $\mathcal{L}_1^2 - \mathcal{L}_2^2 \neq 0$. With respect to the local coordinates, (1.1) can be rewritten as follows

$$\begin{aligned} R_{ijhk} = \; & \mathcal{L}_1[\rho_{ik}\rho_{jh} - \rho_{ih}\rho_{jk} + \tilde{\rho}_{ik}\tilde{\rho}_{jh} - \tilde{\rho}_{ih}\tilde{\rho}_{jk}] \\ & + \mathcal{L}_2[\rho_{ik}\tilde{\rho}_{jh} + \tilde{\rho}_{ik}\rho_{jh} - \rho_{ih}\tilde{\rho}_{jk} - \tilde{\rho}_{ih}\rho_{jk}], \end{aligned} \tag{1.2}$$

from which we have

$$\begin{aligned} \rho_{ik} - \mathcal{L}_1(\kappa\rho_{ik} + \tilde{\kappa}\tilde{\rho}_{ik}) - \mathcal{L}_2(\tilde{\kappa}\rho_{ik} + \kappa\tilde{\rho}_{ik}) &= -2\mathcal{L}_1\rho_{ia}\rho_k^a - 2\mathcal{L}_2\rho_{ia}\tilde{\rho}_k^a, \\ \tilde{\rho}_{ik} - \mathcal{L}_1(\kappa\tilde{\rho}_{ik} + \tilde{\kappa}\rho_{ik}) - \mathcal{L}_2(\kappa\rho_{ik} + \tilde{\kappa}\tilde{\rho}_{ik}) &= -2\mathcal{L}_2\rho_{ia}\rho_k^a - 2\mathcal{L}_1\rho_{ia}\tilde{\rho}_k^a. \end{aligned} \tag{1.3}$$

The relations (1.3) imply

$$\begin{aligned} \rho_{ia}\rho_k^a &= -\frac{1}{2}\left(\frac{\mathcal{L}_1}{\mathcal{L}_1^2 - \mathcal{L}_2^2} - \kappa\right)\rho_{ik} + \frac{1}{2}\left(\frac{\mathcal{L}_2}{\mathcal{L}_1^2 - \mathcal{L}_2^2} + \tilde{\kappa}\right)\tilde{\rho}_{ik}, \\ \tilde{\rho}_{ia}\rho_k^a &= -\frac{1}{2}\left(\frac{\mathcal{L}_1}{\mathcal{L}_1^2 - \mathcal{L}_2^2} - \kappa\right)\tilde{\rho}_{ik} + \frac{1}{2}\left(\frac{\mathcal{L}_2}{\mathcal{L}_1^2 - \mathcal{L}_2^2} + \tilde{\kappa}\right)\rho_{ik}. \end{aligned} \tag{1.4}$$

The product conformal curvature tensor is defined by Tachibana in [4]. If this tensor vanishes, then

$$\begin{aligned} & R(X,Y,Z,W) = \\ & \tfrac{1}{4}\{g(X,W)[A\rho(Y,Z) - B\tilde{\rho}(Y,Z)] + g(Y,Z)[A\rho(X,W) - B\tilde{\rho}(X,W)] \\ & -g(X,Z)[A\rho(Y,W) - B\tilde{\rho}(Y,W)] - g(Y,W)[A\rho(X,Z) - B\tilde{\rho}(X,Z)] \\ & +F(X,W)[A\tilde{\rho}(Y,Z) - B\rho(Y,Z)] + F(Y,Z)[A\tilde{\rho}(X,W) - B\rho(X,W)] \\ & -F(X,Z)[A\tilde{\rho}(Y,W) - B\rho(Y,W)] - F(Y,W)[A\tilde{\rho}(X,Z) - B\rho(X,Z)]\} \\ & -\tfrac{1}{2}(C\kappa - D\tilde{\kappa})G(X,Y,Z,W) + \tfrac{1}{2}(D\kappa - C\tilde{\kappa})G(fX,Y,Z,W), \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} & A = \frac{4(n-4)}{(n-4)^2 - r^2}, \quad B = \frac{4r}{(n-4)^2 - r^2}, \\ & C = \frac{2[(n-2)(n-4) + r^2]}{[(n-4)^2 - r^2][(n-2)^2 - r^2]}, \; D = \frac{4r(n-3)}{[(n-4)^2 - r^2][(n-2)^2 - r^2]}, \\ & G(X,Y,Z,W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ & \qquad + F(X,W)F(Y,Z) - F(X,Z)F(Y,W). \end{aligned} \tag{1.6}$$

We note that $(n-4)^2 - r^2 \neq 0$, $(n-2)^2 - r^2 \neq 0$, because of $p > 2$, $q > 2$.

Thus, if (M, g, f) satisfies (1.1) and its product conformal curvature tensor vanishes, then

$$
\begin{aligned}
&4\mathcal{L}_1[\rho(X,W)\rho(Y,Z)-\rho(X,Z)\rho(Y,W)+\tilde\rho(X,W)\tilde\rho(Y,Z)-\tilde\rho(X,Z)\tilde\rho(Y,W)]\\
&+4\mathcal{L}_2[\rho(X,W)\tilde\rho(Y,Z)+\tilde\rho(X,W)\rho(Y,Z)-\rho(X,Z)\tilde\rho(Y,W)-\tilde\rho(X,Z)\rho(Y,W)] =\\
&= A[g(X,W)\rho(Y,Z)+g(Y,Z)\rho(X,W)-g(X,Z)\rho(Y,W)-g(Y,W)\rho(X,Z)\\
&+F(X,W)\tilde\rho(Y,Z)+F(Y,Z)\tilde\rho(X,W)-F(X,Z)\tilde\rho(Y,W)-F(Y,W)\tilde\rho(X,Z)]\\
&-B[g(X,W)\tilde\rho(Y,Z)+g(Y,Z)\tilde\rho(X,W)-g(X,Z)\tilde\rho(Y,W)-g(Y,W)\tilde\rho(X,Z)\\
&+F(X,W)\rho(Y,Z)+F(Y,Z)\rho(X,W)-F(X,Z)\rho(Y,W)-F(Y,W)\rho(X,Z)]\\
&-2(C\kappa-D\tilde\kappa)G(X,Y,Z,W)+2(D\kappa-C\tilde\kappa)G(fX,Y,Z,W).
\end{aligned}
\tag{1.7}
$$

Let us put

$$
\begin{aligned}
&\Gamma(X,Y,Z,W) =\\
&= \rho(X,W)\rho(Y,Z)-\rho(X,Z)\rho(Y,W)+\tilde\rho(X,W)\tilde\rho(Y,Z)-\tilde\rho(X,Z)\tilde\rho(Y,W),
\end{aligned}
$$

$$
\begin{aligned}
&M(X,Y,Z,W) =\\
&= g(X,W)\rho(Y,Z)+g(Y,Z)\rho(X,W)-g(X,Z)\rho(Y,W)-g(Y,W)\rho(X,Z)\\
&+F(X,W)\tilde\rho(Y,Z)+F(Y,Z)\tilde\rho(X,W)-F(X,Z)\tilde\rho(Y,W)-F(Y,W)\tilde\rho(X,Z).
\end{aligned}
$$

Then (1.7) can be rewritten in the form

$$
\begin{aligned}
&4\mathcal{L}_1\Gamma(X,Y,Z,W)+4\mathcal{L}_2\Gamma(fX,Y,Z,W) = A\,M(X,Y,Z,W)\\
&-B\,M(fX,Y,Z,W)-2(C\kappa-D\tilde\kappa)G(X,Y,Z,W)+2(D\kappa-C\tilde\kappa)G(fX,Y,Z,W).
\end{aligned}
\tag{1.8}
$$

Putting into (1.8) fX instead of X, we obtain

$$
\begin{aligned}
&4\mathcal{L}_2\Gamma(X,Y,Z,W)+4\mathcal{L}_1\Gamma(fX,Y,Z,W) = -B\,M(X,Y,Z,W)\\
&+A\,M(fX,Y,Z,W)+2(D\kappa-C\tilde\kappa)G(X,Y,Z,W)-2(C\kappa-D\tilde\kappa)G(fX,Y,Z,W).
\end{aligned}
$$

The last two equations imply

$$
\begin{aligned}
&\Gamma(X,Y,Z,W)-\frac{\alpha}{4}M(X,Y,Z,W)-\frac{\beta}{4}M(fX,Y,Z,W) =\\
&=-\frac{\mathcal{L}_1(C\kappa-D\tilde\kappa)+\mathcal{L}_2(D\kappa-C\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}\;G(X,Y,Z,W)\\
&+\frac{\mathcal{L}_1(D\kappa-C\tilde\kappa)+\mathcal{L}_2(C\kappa-D\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}\;G(fX,Y,Z,W),
\end{aligned}
\tag{1.9}
$$

where

$$\alpha=\frac{A\mathcal{L}_1+B\mathcal{L}_2}{\mathcal{L}_1^2-\mathcal{L}_2^2},\quad \beta=-\frac{A\mathcal{L}_2+B\mathcal{L}_1}{\mathcal{L}_1^2-\mathcal{L}_2^2}. \tag{1.10}$$

2. *The operator* $Q(X,Y,Z,W,U,V)$

We define the operator $Q(X,Y,Z,W,U,V)$ by

$$\begin{aligned}
&Q(X,Y,Z,W,U,V)=\\
&\rho(X,U)R(V,Y,Z,W)+\rho(Y,U)R(X,V,Z,W)+\rho(Z,U)R(X,Y,V,W)\\
&+\rho(W,U)R(X,Y,Z,V)+\rho(X,fU)R(fV,Y,Z,W)+\rho(Y,fU)R(X,fV,Z,W)\\
&+\rho(Z,fU)R(X,Y,fV,W)+\rho(W,fU)R(X,Y,Z,fV)-\rho(X,V)R(U,Y,Z,W)\\
&-\rho(Y,V)R(X,U,Z,W)-\rho(Z,V)R(X,Y,U,W)-\rho(W,V)R(X,Y,Z,U)\\
&-\rho(X,fV)R(fU,Y,Z,W)-\rho(Y,fV)R(X,fU,Z,W)-\rho(Z,fV)R(X,Y,fU,W)\\
&\qquad-\rho(W,fV)R(X,Y,Z,fU).
\end{aligned} \tag{2.1}$$

Applying (2.1) to (1.7) and using (1.2) and (1.5), we get, after some simple but long calculation,

$$\begin{aligned}
&[P\rho(X,U)-S\tilde\rho(X,U)]G(V,Y,Z,W)+[P\rho(Y,U)-S\tilde\rho(Y,U)]G(X,V,Z,W)\\
&+[P\rho(Z,U)-S\tilde\rho(Z,U)]G(X,Y,V,W)+[P\rho(W,U)-S\tilde\rho(W,U)]G(X,Y,Z,V)\\
&-[S\rho(X,U)-P\tilde\rho(X,U)]G(fV,Y,Z,W)-[S\rho(Y,U)-P\tilde\rho(Y,U)]G(X,fV,Z,W)\\
&-[S\rho(Z,U)-P\tilde\rho(Z,U)]G(X,Y,fV,W)-[S\rho(W,U)-P\tilde\rho(W,U)]G(X,Y,Z,fV)\\
&-[P\rho(X,V)-S\tilde\rho(X,V)]G(U,Y,Z,W)-[P\rho(Y,V)-S\tilde\rho(Y,V)]G(X,U,Z,W)\\
&-[P\rho(Z,V)-S\tilde\rho(Z,V)]G(X,Y,U,W)-[P\rho(W,V)-S\tilde\rho(W,V)]G(X,Y,Z,U)\\
&+[S\rho(X,V)-P\tilde\rho(X,V)]G(fU,Y,Z,W)+[S\rho(Y,V)-P\tilde\rho(Y,V)]G(X,fU,Z,W)\\
&+[S\rho(Z,V)-P\tilde\rho(Z,V)]G(X,Y,fU,W)+[S\rho(W,V)-P\tilde\rho(W,V)]G(X,Y,Z,fU)=0,
\end{aligned} \tag{2.2}$$

where

$$P=\alpha A-\beta B-8(C\kappa-D\tilde\kappa),\qquad S=\alpha B-\beta A-8(D\kappa-C\tilde\kappa). \tag{2.3}$$

We put $W = V = e_i$ into (2.2). Summing up, we get

$$\begin{aligned}
&[P\rho(X,U) - S\tilde{\rho}(X,U)][(n-2)g(Y,Z) + rF(Y,Z)]\\
&-[P\rho(Y,U) - S\tilde{\rho}(Y,U)][(n-2)g(X,Z) + rF(X,Z)]\\
&+[P\tilde{\rho}(X,U) - S\rho(X,U)][(n-2)F(Y,Z) + rg(Y,Z)]\\
&-[P\tilde{\rho}(Y,U) - S\rho(Y,U)][(n-2)F(X,Z) + rg(X,Z)]\\
&-2[P\rho(X,Z) - S\tilde{\rho}(X,Z)]g(Y,U) + 2[P\rho(Y,Z) - S\tilde{\rho}(Y,Z)]g(X,U)\\
&-2[P\tilde{\rho}(X,Z) - S\rho(X,Z)]F(Y,U) + 2[P\tilde{\rho}(Y,Z) - S\rho(Y,Z)]F(X,U)\\
&-(P\kappa - S\tilde{\kappa})G(X,Y,Z,U) - (P\tilde{\kappa} - S\kappa)G(fX,Y,Z,U) = 0.
\end{aligned} \tag{2.4}$$

Now, we put $Y = Z = e_i$ into (2.4). Summing up, we obtain

$$\begin{gathered}
[n(n-4) + r^2][P\rho(X,U) - S\tilde{\rho}(X,U)]\\
+2r(n-2)[P\tilde{\rho}(X,U) - S\rho(X,U)]\\
-[(n-4)(P\kappa - S\tilde{\kappa}) + r(P\tilde{\kappa} - S\kappa)]g(X,U)\\
-[(n-4)(P\tilde{\kappa} - S\kappa) + r(P\kappa - S\tilde{\kappa})]F(X,U) = 0.
\end{gathered} \tag{2.5}$$

Putting into (2.5) fX instead of X, we find

$$\begin{gathered}
[n(n-4) + r^2][P\tilde{\rho}(X,U) - S\rho(X,U)]\\
+2r(n-2)[P\rho(X,U) - S\tilde{\rho}(X,U)]\\
-[(n-4)(P\kappa - S\tilde{\kappa}) + r(P\tilde{\kappa} - S\kappa)]F(X,U)\\
-[(n-4)(P\tilde{\kappa} - S\kappa) + r(P\kappa - S\tilde{\kappa})]g(X,U) = 0.
\end{gathered} \tag{2.6}$$

According to our supposition, $(n-4)^2 - r^2 \neq 0$. Thus, from (2.5) and (2.6), we get

$$(P^2 - S^2)\left\{\rho(X,U) - \frac{1}{n^2 - r^2}[(n\kappa - r\tilde{\kappa})g(X,U) - (r\kappa - n\tilde{\kappa})F(X,U)]\right\} = 0. \tag{2.7}$$

The relation (2.7) shows that

$$\rho(X,U) = \frac{1}{n^2 - r^2}[(n\kappa - r\tilde{\kappa})g(X,U) - (r\kappa - n\tilde{\kappa})F(X,U)], \tag{2.8}$$

or

$$P^2 - S^2 = 0. \tag{2.9}$$

The condition (2.8) can be rewritten in the form

$$\rho(X,U) = \lambda g(X,U) + \mu F(X,U).$$

Substituting this into (1.1), we find

$$\begin{aligned} R(X,Y,Z,W) &= [(\lambda^2+\mu^2)\mathcal{L}_1 + 2\lambda\mu\mathcal{L}_2]G(X,Y,Z,W) \\ &+[2\lambda\mu\mathcal{L}_1 + (\lambda^2+\mu^2)\mathcal{L}_2]G(fX,Y,Z,W), \end{aligned}$$

that is, (M,g,f) is manifold of almost constant curvature.

The condition (2.9) will be examined in the sections 3, 4 and 5.

3. *The case* $P = S \neq 0$

According to (2.3), this condition implies

$$(\alpha+\beta)(A-B) = 8(C-D)(\kappa+\tilde{\kappa}).$$

But, in view of (1.6), we have

$$A - B = \frac{4}{(n-4)+r}, \quad C - D = \frac{2}{[(n-4)+r][(n-2)+r]}.$$

Thus

$$\alpha+\beta = \frac{4(\kappa+\tilde{\kappa})}{(n-2)+r}. \tag{3.1}$$

On the other hand, in view of (1.10), we have

$$\alpha+\beta = \frac{A\mathcal{L}_1 + B\mathcal{L}_2 - A\mathcal{L}_2 - B\mathcal{L}_1}{\mathcal{L}_1^2 - \mathcal{L}_2^2} = \frac{4}{[(n-4)+r](\mathcal{L}_1+\mathcal{L}_2)}.$$

This and (3.1) give

$$\mathcal{L}_1 + \mathcal{L}_2 = \frac{1}{(\kappa+\tilde{\kappa})}\frac{[(n-2)+r]}{[(n-4)+r]}. \tag{3.2}$$

In the case $P = S$, (2.5) reduces to

$$\rho(X,U) - \tilde{\rho}(X,U) = \frac{\kappa-\tilde{\kappa}}{n-r}[g(X,U) - F(X,U)] \tag{3.3}$$

or, in the local coordinates

$$\rho_{ij} - \tilde{\rho}_{ij} = \frac{\kappa-\tilde{\kappa}}{n-r}(g_{ij} - F_{ij}). \tag{3.4}$$

From (3.4), we find

$$(\rho_{ia} - \tilde{\rho}_{ia})\rho^a_j = \left(\frac{\kappa - \tilde{\kappa}}{n - r}\right)^2 (g_{ij} - F_{ij}). \tag{3.5}$$

On the other hand, from (1.4) we get

$$(\rho_{ia} - \tilde{\rho}_{ia})\rho^a_j = -\frac{1}{2}\left[\frac{1}{\mathcal{L}_1 - \mathcal{L}_2} - (\kappa - \tilde{\kappa})\right]\left(\frac{\kappa - \tilde{\kappa}}{n - r}\right)(g_{ij} - F_{ij}).$$

This, together with (3.5) gives

$$\mathcal{L}_1 - \mathcal{L}_2 = \frac{n - r}{[(n-2) - r]} \cdot \frac{1}{(\kappa - \tilde{\kappa})}. \tag{3.6}$$

Finally, (3.2) and (3.6) imply

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{2}\left\{\frac{1}{(\kappa + \tilde{\kappa})}\frac{[(n-2)+r]}{[(n-4)+r]} + \frac{1}{(\kappa - \tilde{\kappa})}\frac{(n-r)}{[(n-2)-r]}\right\}, \\ \mathcal{L}_2 &= \frac{1}{2}\left\{\frac{1}{(\kappa + \tilde{\kappa})}\frac{[(n-2)+r]}{[(n-4)+r]} - \frac{1}{(\kappa - \tilde{\kappa})}\frac{(n-r)}{[(n-2)-r]}\right\} \end{aligned} \tag{3.7}$$

Now, let us consider the left hand side of (1.9). Explicitly, it is

$$\begin{aligned} &\rho(X,W)\rho(Y,Z) - \rho(X,Z)\rho(Y,W) + \tilde{\rho}(X,W)\tilde{\rho}(Y,Z) - \tilde{\rho}(X,Z)\tilde{\rho}(Y,W) \\ &-\frac{\alpha}{2}[g(X,W)\rho(Y,Z) + g(Y,Z)\rho(X,W) - g(X,Z)\rho(Y,W) - g(Y,W)\rho(X,Z) \\ &+F(X,W)\tilde{\rho}(Y,Z) + F(Y,Z)\tilde{\rho}(X,W) - F(X,Z)\tilde{\rho}(Y,W) - F(Y,W)\tilde{\rho}(X,Z)] \\ &-\frac{\beta}{2}[F(X,W)\rho(Y,Z) + g(Y,Z)\tilde{\rho}(X,W) - F(X,Z)\rho(Y,W) - g(Y,W)\tilde{\rho}(X,Z) \\ &+g(X,W)\tilde{\rho}(Y,Z) + F(Y,Z)\rho(X,W) - g(X,Z)\tilde{\rho}(Y,W) - F(Y,W)\rho(X,Z)]. \end{aligned} \tag{3.8}$$

But, in view of (3.3), we have

$$\tilde{\rho}(X,Y) = \rho(X,Y) - \frac{\kappa - \tilde{\kappa}}{n - r}[g(X,Y) - F(X,Y)].$$

Substituting this into (3.8), we obtain, after some calculation, the following expression for (1.9):

$$
\begin{aligned}
&2\left\{\rho(X,W)-\frac{\kappa-\tilde\kappa}{2(n-r)}[g(X,W)-F(X,W)]-\frac{\alpha+\beta}{8}[g(X,W)+F(X,W)]\right\}\times\\
&\times\left\{\rho(Y,Z)-\frac{\kappa-\tilde\kappa}{2(n-r)}[g(Y,Z)-F(Y,Z)]-\frac{\alpha+\beta}{8}[g(Y,Z)+F(Y,Z)]\right\}\\
&-2\left\{\rho(X,Z)-\frac{\kappa-\tilde\kappa}{2(n-r)}[g(X,Z)-F(X,Z)]-\frac{\alpha+\beta}{8}[g(X,Z)+F(X,Z)]\right\}\times\\
&\times\left\{\rho(Y,W)-\frac{\kappa-\tilde\kappa}{2(n-r)}[g(Y,W)-F(Y,W)]-\frac{\alpha+\beta}{8}[g(Y,W)+F(Y,W)]\right\}=\\
&=\left\{-\frac{\mathcal{L}_1(C\kappa-D\tilde\kappa)+\mathcal{L}_2(D\kappa-C\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}-\frac{1}{2}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)^2+\frac{(\alpha+\beta)^2}{32}\right.\\
&\qquad\left.+\frac{\alpha-\beta}{4}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)\right\}G(X,Y,Z,W)\\
&+\left\{\frac{\mathcal{L}_1(D\kappa-C\tilde\kappa)+\mathcal{L}_2(C\kappa-D\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}+\frac{1}{2}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)^2+\frac{(\alpha+\beta)^2}{32}-\right.\\
&\qquad\left.-\frac{\alpha-\beta}{4}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)\right\}G(fX,Y,Z,W).
\end{aligned}
\tag{3.9}
$$

In view of (1.6), (1.10) and (3.7), we can see that

$$
\begin{aligned}
\frac{\mathcal{L}_1(C\kappa-D\tilde\kappa)+\mathcal{L}_2(D\kappa-C\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}&=\frac{(\kappa-\tilde\kappa)^2}{2(n-r)[(n-4)-r]}+\frac{(\kappa+\tilde\kappa)^2}{2[(n-2)+r]^2},\\
\frac{\mathcal{L}_1(D\kappa-C\tilde\kappa)+\mathcal{L}_2(C\kappa-D\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}&=\frac{(\kappa-\tilde\kappa)^2}{2(n-r)[(n-4)-r]}-\frac{(\kappa+\tilde\kappa)^2}{2[(n-2)+r]^2},\\
\frac{1}{4}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)(\alpha-\beta)&=\left(\frac{\kappa-\tilde\kappa}{n-r}\right)^2\frac{[(n-2)-r]}{[(n-4)-r]}.
\end{aligned}
$$

These relations, together with (3.1), imply

$$
\begin{aligned}
&\frac{\mathcal{L}_1(C\kappa-D\tilde\kappa)+\mathcal{L}_2(D\kappa-C\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}+\frac{1}{2}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)^2-\frac{(\alpha+\beta)^2}{32}-\frac{\alpha-\beta}{4}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)=0,\\
&\frac{\mathcal{L}_1(D\kappa-C\tilde\kappa)+\mathcal{L}_2(C\kappa-D\tilde\kappa)}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}+\frac{1}{2}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)^2+\frac{(\alpha+\beta)^2}{32}-\frac{\alpha-\beta}{4}\left(\frac{\kappa-\tilde\kappa}{n-r}\right)=0,
\end{aligned}
$$

because of which, (3.9) reduces to

$$\begin{aligned}
&\left\{\rho(X,W)-\frac{\kappa-\tilde{\kappa}}{2(n-r)}[g(X,W)-F(X,W)]-\frac{\alpha+\beta}{8}[g(X,W)+F(X,W)]\right\}\times\\
&\times\left\{\rho(Y,Z)-\frac{\kappa-\tilde{\kappa}}{2(n-r)}[g(Y,Z)-F(Y,Z)]-\frac{\alpha+\beta}{8}[g(Y,Z)+F(Y,Z)]\right\}\\
&-\left\{\rho(X,Z)-\frac{\kappa-\tilde{\kappa}}{2(n-r)}[g(X,Z)-F(X,Z)]-\frac{\alpha+\beta}{8}[g(X,Z)+F(X,Z)]\right\}\times\\
&\times\left\{\rho(Y,W)-\frac{\kappa-\tilde{\kappa}}{2(n-r)}[g(Y,W)-F(Y,W)]-\frac{\alpha+\beta}{8}[g(Y,W)+F(Y,W)]\right\}=0.
\end{aligned}$$

This yields

$$\rho(X,Y)-\frac{\kappa-\tilde{\kappa}}{2(n-r)}[g(X,Y)-F(X,Y)]-\frac{\alpha+\beta}{8}[g(X,Y)+F(X,Y)]=\theta v(X)v(Y),$$

where $v(X)$ is l-form such that v(fX)=v(X) and θ is a scalar function. Without loss of the generality, we can suppose that $\sum_{i=1}^{n} v(e_i)v(e_i)=1$. Then

$$\theta=-\frac{\kappa+\tilde{\kappa}}{(n-2)+r},$$

and therefore

$$\begin{aligned}
\rho(X,Y)= &\ \frac{1}{2}\left[\frac{\kappa+\tilde{\kappa}}{(n-2)+r}+\frac{\kappa-\tilde{\kappa}}{n-r}\right]g(X,Y)\\
&+\frac{1}{2}\left[\frac{\kappa+\tilde{\kappa}}{(n-2)+r}-\frac{\kappa-\tilde{\kappa}}{n-r}\right]F(X,Y)-\frac{\kappa+\tilde{\kappa}}{(n-2)+r}v(X)v(Y).
\end{aligned}\tag{3.10}$$

4. *The case* $P=-S\neq 0$

This condition, in view of (2.3) and (1.6), gives

$$\alpha-\beta=\frac{4(\kappa-\tilde{\kappa})}{(n-2)-r}.\tag{4.1}$$

On the other hand, in view of (1.10),

$$\alpha-\beta=\frac{4}{[(n-4)-r]}\cdot\frac{1}{(\mathcal{L}_1-\mathcal{L}_2)}.$$

Thus

$$\mathcal{L}_1 - \mathcal{L}_2 = \frac{(n-2)-r}{(n-4)-r} \cdot \frac{1}{(\kappa - \tilde{\kappa})}. \tag{4.2}$$

If $P = -S$, (2.5) reduces to

$$\rho(X,U) + \tilde{\rho}(X,U) = \frac{\kappa + \tilde{\kappa}}{n+r}[g(X,U) + F(X,U)], \tag{4.3}$$

or, in the local coordinates,

$$\rho_{ij} + \tilde{\rho}_{ij} = \frac{\kappa + \tilde{\kappa}}{n+r}(g_{ij} + F_{ij}).$$

Therefore,

$$(\rho_{ia} + \tilde{\rho}_{ia})\rho^a_j = \left(\frac{\kappa + \tilde{\kappa}}{n+r}\right)^2 (g_{ij} + F_{ij}). \tag{4.4}$$

On the other hand, using (1.4), we find

$$(\rho_{ia} + \tilde{\rho}_{ia})\rho^a_j = \frac{1}{2}\left[-\frac{1}{\mathcal{L}_1 + \mathcal{L}_2} + (\kappa + \tilde{\kappa})\right]\left(\frac{\kappa + \tilde{\kappa}}{n+r}\right)(g_{ij} + F_{ij}). \tag{4.5}$$

The equations (4.4) and (4.5) show that

$$\left(\frac{\kappa + \tilde{\kappa}}{n+r}\right)(g_{ij} + F_{ij}) = \frac{1}{2}\left[-\frac{1}{\mathcal{L}_1 + \mathcal{L}_2} + (\kappa + \tilde{\kappa})\right](g_{ij} + F_{ij}),$$

from which we find

$$\mathcal{L}_1 + \mathcal{L}_2 = \frac{n+r}{[(n-2)+r]}\frac{1}{(\kappa + \tilde{\kappa})}. \tag{4.6}$$

From (4.2) and (4.6), we get

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{2}\left[\frac{1}{(\kappa - \tilde{\kappa})}\frac{(n-2)-r}{[(n-4)-r]} + \frac{1}{(\kappa + \tilde{\kappa})}\frac{n+r}{[(n-2)+r]}\right], \\ \mathcal{L}_2 &= \frac{1}{2}\left[-\frac{1}{(\kappa - \tilde{\kappa})}\frac{(n-2)-r}{[(n-4)-r]} + \frac{1}{(\kappa + \tilde{\kappa})}\frac{n+r}{[(n-2)+r]}\right]. \end{aligned} \tag{4.7}$$

According (4.3),

$$\tilde{\rho}(X,U) = -\rho(X,U) + \frac{\kappa + \tilde{\kappa}}{n+r}[g(X,U) + F(X,U)].$$

Substituting this into (3.8), we find that (1.9), in the case $P = -S \neq 0$, can be expressed in the form

$$\begin{aligned}&\left\{\rho(X,W)-\frac{\kappa+\tilde\kappa}{2(n+r)}[g(X,W)+F(X,W)]-\frac{\alpha-\beta}{8}[g(X,W)-F(X,W)]\right\}\times\\ &\times\left\{\rho(Y,Z)-\frac{\kappa+\tilde\kappa}{2(n+r)}[g(Y,Z)+F(Y,Z)]-\frac{\alpha-\beta}{8}[g(Y,Z)-F(Y,Z)]\right\}\\ &-\left\{\rho(X,Z)-\frac{\kappa+\tilde\kappa}{2(n+r)}[g(X,Z)+F(X,Z)]-\frac{\alpha-\beta}{8}[g(X,Z)-F(X,Z)]\right\}\times\\ &\times\left\{\rho(Y,W)-\frac{\kappa+\tilde\kappa}{2(n+r)}[g(Y,W)+F(Y,W)]-\frac{\alpha-\beta}{8}[g(Y,W)-F(Y,W)]\right\}=0.\end{aligned}$$

This equation implies

$$\rho(X,Y)-\frac{\kappa+\tilde\kappa}{2(n+r)}[g(X,Y)+F(X,Y)]-\frac{\alpha-\beta}{8}[g(X,Y)-F(X,Y)]=\varphi w(X)w(Y),$$

where $w(X)$ is a l-form and φ is a scalar function. Without loss of the generality, we can suppose that $\sum_{i+1}^{n} w(e_i)w(e_i) = 1$. Then

$$\varphi = -\frac{\kappa-\tilde\kappa}{(n-2)-r}$$

and therefore

$$\begin{aligned}\rho(X,Y) = {}& \frac{1}{2}\left[\frac{\kappa+\tilde\kappa}{n+r}+\frac{\kappa-\tilde\kappa}{(n-2)-r}\right]g(X,Y)\\ &+\frac{1}{2}\left[\frac{\kappa+\tilde\kappa}{n+r}-\frac{\kappa-\tilde\kappa}{(n-2)-r}\right]F(X,Y)-\frac{\kappa-\tilde\kappa}{(n-2)-r}w(X)w(Y).\end{aligned} \tag{4.8}$$

5. *The case* $P = S = 0$

In this case we obtain from (2.3)

$$\alpha=\frac{4[(n-2)\kappa-r\tilde\kappa]}{(n-2)^2-r^2},\quad \beta=\frac{4[(n-2)\tilde\kappa-r\kappa]}{(n-2)^2-r^2}. \tag{5.1}$$

Thus

$$\alpha+\beta=\frac{4}{(n-2)+r}(\kappa+\tilde\kappa),\quad \alpha-\beta=\frac{4}{(n-2)-r}(\kappa-\tilde\kappa). \tag{5.2}$$

On the other hand, in view of (1.10), we have

$$\mathcal{L}_1+\mathcal{L}_2=\frac{A-B}{\alpha+\beta},\quad \mathcal{L}_1-\mathcal{L}_2=\frac{A+B}{\alpha-\beta}.$$

Therefore

$$\mathcal{L}_1=\frac{1}{2}\left[\frac{A-B}{\alpha+\beta}+\frac{A+B}{\alpha-\beta}\right],\quad \mathcal{L}_2=\frac{1}{2}\left[\frac{A-B}{\alpha+\beta}-\frac{A+B}{\alpha-\beta}\right].$$

Finally, taking into account (1.10) and (5.2), we obtain

$$\begin{aligned}\mathcal{L}_1&=\frac{1}{2}\left[\frac{1}{(\kappa+\tilde{\kappa})}\frac{(n-2)+r}{[(n-4)+r]}+\frac{1}{(\kappa-\tilde{\kappa})}\frac{(n-2)-r}{[(n-4)-r]}\right]\\ \mathcal{L}_2&=\frac{1}{2}\left[\frac{1}{(\kappa+\tilde{\kappa})}\frac{(n-2)+r}{[(n-4)+r]}-\frac{1}{(\kappa-\tilde{\kappa})}\frac{(n-2)-r}{[(n-4)-r]}\right].\end{aligned}\tag{5.3}$$

Now, we note that (1.9) can be rewritten in the form

$$\begin{aligned}&[\rho(X,W)-\frac{\alpha}{4}g(X,W)-\frac{\beta}{4}F(X,W)][\rho(Y,Z)-\frac{\alpha}{4}g(Y,Z)-\frac{\beta}{4}F(Y,Z)]\\ &-[\rho(X,Z)-\frac{\alpha}{4}g(X,Z)-\frac{\beta}{4}F(X,Z)][\rho(Y,W)-\frac{\alpha}{4}g(Y,W)-\frac{\beta}{4}F(Y,W)]\\ &+[\tilde{\rho}(X,W)-\frac{\alpha}{4}F(X,W)-\frac{\beta}{4}g(X,W)][\tilde{\rho}(Y,Z)-\frac{\alpha}{4}F(Y,Z)-\frac{\beta}{4}g(Y,Z)]\\ &-[\tilde{\rho}(X,Z)-\frac{\alpha}{4}F(X,Z)-\frac{\beta}{4}g(X,Z)][\tilde{\rho}(Y,W)-\frac{\alpha}{4}F(Y,W)-\frac{\beta}{4}g(Y,W)]=\\ &=\left[\frac{\alpha^2+\beta^2}{16}-\frac{\mathcal{L}_1(C\kappa-D\tilde{\kappa})+\mathcal{L}_2(D\kappa-C\tilde{\kappa})}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}\right]G(X,Y,Z,W)\\ &\ +\left[\frac{\alpha\beta}{8}+\frac{\mathcal{L}_1(D\kappa-C\tilde{\kappa})+\mathcal{L}_2(C\kappa-D\tilde{\kappa})}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}\right]G(fX,Y,Z,W).\end{aligned}\tag{5.4}$$

But, in the case $P=S=0$, (5.1) and (5.3) hold good, because of which we have

$$\frac{\alpha^2+\beta^2}{16}-\frac{\mathcal{L}_1(C\kappa-D\tilde{\kappa})+\mathcal{L}_2(D\kappa-C\tilde{\kappa})}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}=0$$

$$\frac{\alpha\beta}{8}+\frac{\mathcal{L}_1(D\kappa-C\tilde{\kappa})+\mathcal{L}_2(C\kappa-D\tilde{\kappa})}{2(\mathcal{L}_1^2-\mathcal{L}_2^2)}=0$$

In other words, the equation (5.4), in the case $P=S=0$, reduces to

$$\begin{aligned}
&[\rho(X,W)-\frac{\alpha}{4}g(X,W)-\frac{\beta}{4}F(X,W)][\rho(Y,Z)-\frac{\alpha}{4}g(Y,Z)-\frac{\beta}{4}F(Y,Z)]\\
&-[\rho(X,Z)-\frac{\alpha}{4}g(X,Z)-\frac{\beta}{4}F(X,Z)][\rho(Y,W)-\frac{\alpha}{4}g(Y,W)-\frac{\beta}{4}F(Y,W)]\\
&+[\tilde\rho(X,W)-\frac{\alpha}{4}F(X,W)-\frac{\beta}{4}g(X,W)][\tilde\rho(Y,Z)-\frac{\alpha}{4}F(Y,Z)-\frac{\beta}{4}g(Y,Z)]\\
&-[\tilde\rho(X,Z)-\frac{\alpha}{4}F(X,Z)-\frac{\beta}{4}g(X,Z)][\tilde\rho(Y,W)-\frac{\alpha}{4}F(Y,W)-\frac{\beta}{4}g(Y,W)]=0.
\end{aligned}\tag{5.5}$$

In view of (5.1), the equation (5.5) yields

$$\begin{aligned}
\rho(X,Y)=&\frac{(n-2)\kappa-r\tilde\kappa}{(n-2)^2-r^2}g(X,Y)-\frac{r\kappa-(n-2)\tilde\kappa}{(n-2)^2-r^2}F(X,Y)\\
&+\eta[u(X)u(Y)+\tilde u(X)\tilde u(Y)]+\psi[\tilde u(X)u(Y)+u(X)\tilde u(Y)],
\end{aligned}\tag{5.6}$$

where η and ψ are some scalar functions, while $u(X)$ is a l-form and $\tilde u(X)=u(fX)$. Without loss of the generality, we can suppose that $\sum_{i=1}^{n}u(e_i)u(e_i)=2$. Then, putting into (5.6) $X=Y=e_i$ and summing up, we find

$$\eta=-\frac{(n-2)\kappa-r\tilde\kappa}{2[(n-2)^2-r^2]}.$$

In a similar way, using $\tilde\rho(X,Y)$, we get

$$\psi=\frac{r\kappa-(n-2)\tilde\kappa}{2[(n-2)^2-r^2]}.$$

Thus

$$\begin{aligned}
\rho(X,Y)=&\frac{(n-2)\kappa-r\tilde\kappa}{(n-2)^2-r^2}\left\{g(X,Y)-\frac{1}{2}[u(X)u(Y)+\tilde u(X)\tilde u(Y)]\right\}\\
&-\frac{r\kappa-(n-2)\tilde\kappa}{(n-2)^2-r^2}\left\{F(X,Y)-\frac{1}{2}[\tilde u(X)u(Y)+u(X)\tilde u(Y)]\right\}.
\end{aligned}\tag{5.7}$$

6. *Conclusion*

Summing up the results obtained in the sections 2, 3, 4 and 5, we can state the following

Theorem. *Let (M,g,f) be a locally decomposable Riemannian manifold,* dim $M = n = p + q$, tr $f = r = p - q$, $p > 2$, $q > 2$. *If* (M, g, f) *is product conformally flat and the Riemannian curvature tensor satisfies the condition (1.1) where* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are some scalar functions such that* $\mathcal{L}_1^2 - \mathcal{L}_2^2 \neq 0$, *then there occurs one of the following cases:*

(i) (M, g, f) *is a manifold of almost constant curvature at the points where* $\kappa^2 - \tilde{\kappa}^2 \neq 0$,

(ii) the relations (3.7) and (3.10) are fulfilled, where $v(X)$ *is 1-form such that* $\sum_{i=1}^{n} v(e_i)v(e_i) = 1$*;*

(iii) the relations (4.7) and (4.8) are fulfilled, where $w(X)$ *is 1-form such that* $\sum_{i=1}^{n} w(e_i)w(e_i) = 1$*;*

(iv) the relations (5.3) and (5.7) are fulfilled, where $u(X)$ *is 1-form such that* $\sum_{i=1}^{n} u(e_i)u(e_i) = 2$ *and* $\tilde{u}(X) = u(fX)$.

REFERENCES

[1] D. K o w a l c z y k, *On some subclass of semisymmetric manifolds,* Soochow Journal of Mathematics, vol. 27 (2001), 445–461.

[2] C. M u r a t h a n, K. A r s l a n, R. D e s z c z, R. E z e n t o ş and C. Ö z g ü r, *On some class of hypersurfaces of semi-Euclidean spaces,* Publ. Math. Debrecen, 58 (2001), 587–604.

[3] W. R o t e r, *A note on second order recurrent spaces,* Bull. Acad. Polon. Sci., Ser. sci., math., astr. et phys. 12 (1964), 621–626.

[4] S. T a c h i b a n a, *Some theorems on locally product Riemannian spaces,* Tôhoku Math. J. 12 (1960), 281–292.

Mathematical Institute SANU
Knez Mihailo 35
11001 Belgrade
Serbia and Montenegro