The Linear Arboricity of the Schrijver Graph $SG(2k + 2, k)$

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Abstract. The linear arboricity $la(G)$ of a graph $G$ is the minimum number of linear forests which partition the edge set $E(G)$ of $G$. The vertex linear arboricity $vla(G)$ of a graph $G$ is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that every subset induces a linear forest. The Schrijver graph $SG(n,k)$ is the graph whose vertex set consists of all 2-stable $k$-subsets of the set $\{0, 1, \ldots, n-1\}$ and two vertices $A$ and $B$ are adjacent if and only if $A \cap B = \phi$. In this paper, it is proved that $la(SG(2k + 2, k)) = \lceil (k + 2)/2 \rceil$ for $k \geq 3$ and $vla(SG(2k + 2, k)) = va(SG(2k + 2, k)) = 2$ for $k \geq 2$.

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1. Introduction

Throughout this paper, all graphs considered are finite, undirected and simple. For a real number $x$, $\lceil x \rceil$ is the least integer not less than $x$ and $\lfloor x \rfloor$ is the most integer not more than $x$. For a graph $G$, we use $V(G), E(G), \Delta(G)$ to denote the vertex set, the edge set and the maximum degree, respectively. $N_G(v)$ denotes the set of vertices adjacent to the vertex $v$ in $G$. $G[W]$ denotes the subgraph induced by $W \subseteq V(G)$ (or $W \subseteq E(G)$) in $G$. For disjoint subsets $S$ and $S'$ of $V(G)$, we denote the set of edges with one end in $S$ and the other in $S'$ by $[S, S']$, which is called an edge cut if $S' = \overline{S}$, where $\overline{S} = V(G) \setminus S$ is the subset obtained by removing all vertices of $S$ from $V(G)$. Let $G \setminus H$ be the graph $G - E(H)$ that is obtained by taking away all edges of $H$ from $G$. A $k$-path is a path with length $k$.

A linear forest is a graph in which each component is a path. The linear arboricity $la(G)$ of a graph as defined by Harary [11] is the minimum number of linear forests which partition the edge set $E(G)$ of $G$. Akiyama $et$ $al.$ [1] conjectured that $la(G) = \lceil (\Delta(G) + 1)/2 \rceil$ for any regular graph $G$, and proved that the conjecture is true for complete graphs and graphs with $\Delta = 3, 4$ [1, 2]. Enomoto and Péroche [7] proved that the conjecture is true for graphs with $\Delta = 5, 6, 8$. Guldan [10] proved that the conjecture is true for graphs with $\Delta = 10$. It is
obvious that \( \text{la}(G) \geq \lceil (\Delta(G))/2 \rceil \) for every graph \( G \) and \( \text{la}(G) \geq \lceil (\Delta(G)+1)/2 \rceil \) for every regular graph \( G \). So the conjecture is equivalent to the following conjecture.

**Conjecture 1.1** (Linear Arboricity Conjecture(LAC)). [1] For any graph \( G \),
\[
\lceil \Delta(G)/2 \rceil \leq \text{la}(G) \leq \lceil (\Delta(G)+1)/2 \rceil.
\]

Akiyama et al. [1] determined the linear arboricity of complete bipartite graphs and trees. Martinov [12] determined the linear arboricity of extremal locally-tree-like graphs which have a minimal number of edges according to the number of vertices. Martinova [13] determined the linear arboricity of maximal outerplanar graphs. Wu [19] determined the linear arboricity of series-parallel graphs, moreover, Wu [20] proved the conjecture is true for a planar graph \( G \) with \( \Delta(G) \neq 7 \), and the case \( \Delta(G) = 7 \) was also settled in Wu [21]. Tan et al. [18] determined the linear arboricity of planar graphs with maximum degree at least five.

The vertex linear arboricity \( \text{vla}(G) \) of a graph is the minimum number of subsets into which the vertex set \( V(G) \) can be partitioned so that every subset induces a linear forest. The vertex arboricity \( \text{va}(G) \) of a graph \( G \) can be defined similarly. Matsumoto [14] proved that for any finite graph \( G \), \( \text{vla}(G) \leq \lceil (\Delta(G)+1)/2 \rceil \), moreover, if \( \Delta(G) \) is even, then \( \text{vla}(G) = \lceil (\Delta(G)+1)/2 \rceil \) if and only if \( G \) is the complete graph of order \( \Delta(G) + 1 \) or a cycle. Goddard [9] and Poh [15] proved that \( \text{vla}(G) \leq 3 \) for a planar graph \( G \). Akiyama [3] proved \( \text{vla}(G) \leq 2 \) if \( G \) is an outerplanar graph. Alavi [4] proved that \( \text{vla}(G) + \text{vla}(G^c) \leq 1 + \lceil (n+1)/2 \rceil \) for any graph \( G \) of order \( n \), where \( G^c \) is the complement of \( G \). Zuo [22, 23] determined the vertex linear arboricity of distance graphs and a class of integer distance graphs with special distance sets, respectively. Raspaud and Wang [16] discussed the vertex arboricity of planar graphs, and Borodin and Ivanova [5] proved that planar graphs without 4-cycles adjacent to 3-cycles are list vertex 2-arborable. The following result is obvious.

**Lemma 1.1.** If \( G = G_1 \cup G_2 \cup \cdots \cup G_n \), then \( \text{la}(G) \leq \text{la}(G_1) + \text{la}(G_2) + \cdots + \text{la}(G_n) \). In particular, \( \text{la}(G) = \max\{\text{la}(G_1),\text{la}(G_2),\ldots,\text{la}(G_n)\} \), where \( G_i (i=1,2,\ldots,n) \) are connected components of \( G \).

The Kneser graph \( KG(n,k) \) is the graph whose vertex set consists of all \( k \)-subsets of an \( n \)-set, and two vertices are adjacent if and only if they are disjoint. A subset \( S \) of \( [n] = \{0,1,\ldots,n-1\} \) is said to be 2-stable if \( 2 \leq |x-y| \leq n-2 \) for any two distinct elements \( x \) and \( y \), i.e., \( S \) does not contain two consecutive numbers in the cyclic ordering of \( [n] \).

**Definition 1.1.** [17] The Schrijver graph \( SG(n,k) \) is defined as follows. Its vertices are those \( k \)-element subsets of the set \( [n] = \{0,1,\ldots,n-1\} \) that do not contain cyclically consecutive elements \( i,i+1 \) or \( n-1,0 \). Two such vertices are adjacent if they represent disjoint \( k \)-subsets.

Equivalently, the Schrijver graph \( SG(n,k) \) is the graph whose vertex set consists of all 2-stable \( k \)-subsets of the set \( [n] = \{0,1,\ldots,n-1\} \) and two vertices \( A \) and \( B \) are adjacent if and only if \( A \cap B = \emptyset \). Clearly, the Schrijver graph \( SG(n,k) \) is the subgraph of \( KG(n,k) \) induced by all vertices that are 2-stable subsets. The structure of Schrijver graph \( SG(2k+2,k) \) was studied in [6]. Now we recall some results that will be used here.

The vertex set of the Schrijver graph \( SG(n,k) \) has cardinality \( \frac{n}{k} \binom{n-k-1}{k-1} \). In particular, \( SG(2k+2,k) \) has \( (k+1)^2 \) vertices. For \( 0 \leq i \leq 2k+1 \), let \( v(0,i) = \{i,i+2,\ldots,i+2k-2\} \), in which each element is taken modulo \( 2k+2 \). We make the convention that all indices
and elements are taken modulo $2k + 2$ in the following except special instruction. We also regard $v(0, i)$ as a sequence with the elements ordered in the above manner.

A sequence is called a $k$-sequence if it has $k$ elements. Let $m = \lfloor k/2 \rfloor$. For $1 \leq j \leq m$, let $A_j$ be the $k$-sequence in which the $(k - j + 1)$-th entry is equal to 2, and the other $k - 1$ entries are equal to 1. Clearly, $A_j$ can be viewed as a row vector with $k$ components, and $v(0, i)$ and $A_j$ can be added to $v(0, i) + A_j$. In fact, when a $k$-set $A$, regarded as a row vector, and a $k$-sequence $B$ are added to get $A + B$, we just add the two sequences entry-wise to get a $k$-sequence if all the sums are distinct. For the sake of convenience, in addition operation, one can view $v(0, i)$ as a row vector with $k$ components $(i, i+2, \ldots, i+2k-2)$ in $R^k$ over real number field $R$, in which each element is taken modulo $2k + 2$.

Now for $0 \leq i \leq 2k + 1$ and $1 \leq j \leq m$, let

$$v(j, i) = v(j - 1, i) + A_j$$

be the recursion formula, where $v(0, i) = (i, i+2, \ldots, i+2k-2)$ and the addition is taken modulo $2k + 2$. Let $V_0 = \{v(0, i) \mid i = 0, 1, \ldots, 2k + 1\}$, and $V_j = \{v(j, i) \mid i = 0, 1, \ldots, 2k + 1\}$ for $1 \leq j \leq m$. We need the following lemmas for the proof of our main results.

**Lemma 1.2.** [6] For $0 \leq j \leq m - 1$, $|V_j| = 2k + 2$, and

$$|V_m| = \begin{cases} 2k + 2, & \text{if } k \text{ is odd}, \\ k + 1, & \text{otherwise}. \end{cases}$$

Note that $|V_m| = k + 1$ when $k$ is even. Thus, in this case, the index $i$ of $v(m, i)$ is taken modulo $k + 1$ for even $k$ henceforth.

**Lemma 1.3.** [6] For each $v(0, i) \in V_0$, and $v(j, i) \in V_j$, we have

$$N_G(v(0, i)) = \{v(0, i + p) \mid p = 1, 3, \ldots, 2k + 1\} \cup \{v(1, i)\},$$

$$N_G(v(j, i)) = \{v(j, i - 1), v(j, i + 1), v(j - 1, i), v(j + 1, i)\}$$

for $1 \leq j \leq m - 1$,

$$N_G(v(m, i)) = \{v(m, i - 1), v(m, i + 1), v(m, i + k + 1), v(m - 1, i)\}$$

for $k$ is odd, and

$$N_G(v(m, i)) = \{v(m, i - 1), v(m, i + 1), v(m - 1, i), v(m - 1, i + k + 1)\}$$

for $k$ is even.

By Lemma 1.3, $\Delta(G) = k + 2$ for $k \geq 3$, and the following two results are obtained immediately.

**Corollary 1.1.** [6] The graph $G[V_0]$ is a complete bipartite graph with two partite subsets

$$X = \{v(0, 0), v(0, 2), \ldots, v(0, 2k)\} \quad \text{and} \quad Y = \{v(0, 1), v(0, 3), \ldots, v(0, 2k + 1)\}.$$

**Corollary 1.2.** [6] The graph $G[V_j]$ is a cycle with length $2k + 2$ for $1 \leq j \leq m - 1$, $G[V_m]$ is a cycle with length $k + 1$ for even $k$, and $G[V_m]$ is a 3-regular graph for odd $k$. 

2. The linear arboricity of $SG(2k + 2, k)$

Let $G(X, Y)$ be a balanced bipartite graph with partite sets $X = \{x_i \mid i \in \mathbb{Z}_n\}$ and $Y = \{y_i \mid i \in \mathbb{Z}_n\}$. In [8], it was defined that the bipartite difference $\alpha$ of an edge $x_py_q$ in $G(X, Y)$ by the value $(q - p) \mod n$, i.e., $\alpha = (q - p) \mod n$. It is obvious that an edge subsets in $G(X, Y)$ containing the edges with the same bipartite difference must be a matching. In particular, this edge subset is also a perfect matching if $G(X, Y)$ is $K_{n,n}$.

Let $M_\alpha$ be the edge set consisting of edges with bipartite difference $\alpha$. The following lemmas give a decomposition of $K_{n,n}$.

**Lemma 2.1.** Let $K_{n,n}$ be a balanced complete bipartite graph with partite sets $X = \{x_i \mid i = 0, 1, \ldots, n - 1\}$ and $Y = \{y_i \mid i = 0, 1, \ldots, n - 1\}$, then $K_{n,n}$ can be decomposed into the union of $n/2$ Hamiltonian paths and a matching for even $n$, and decomposed into the union of $(n - 1)/2$ Hamiltonian paths and a linear forest for odd $n$.

**Proof.** If $n$ is even, then $K_{n,n}$ can be decomposed into the union of $n/2$ Hamiltonian cycles $M_\alpha \cup M_{\alpha+1}(\alpha = 0, 2, \ldots, n - 2)$. Next, we take away one edge $x_{\alpha/2}y_{n-\alpha/2-1}$ from each $M_\alpha \cup M_{\alpha+1}(\alpha = 0, 2, \ldots, n - 2)$. Then

$$H_{\alpha/2} = M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\} \quad (\alpha = 0, 2, \ldots, n - 2)$$

are $n/2$ Hamiltonian paths of $K_{n,n}$, and $M = \{x_{\alpha/2}y_{n-\alpha/2-1} \mid \alpha = 0, 2, \ldots, n - 2\}$ is a matching.

Similarly, for odd $n$, each $M_\alpha \cup M_{\alpha+1}(\alpha = 0, 2, \ldots, n - 3)$ generates a Hamiltonian cycle. Therefore $M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\}$ is a Hamiltonian path. Let

$$H_{\alpha/2} = M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\} \quad (\alpha = 0, 2, \ldots, n - 3).$$

Moreover, it is clear that $M = M_{n-1} \cup \{x_{\alpha/2}y_{n-\alpha/2-1} \mid \alpha = 0, 2, \ldots, n - 3\}$ forms a linear forest.

Therefore, $la(K_{n,n}) \leq \lceil (n + 1)/2 \rceil$ and $la(K_{n,n} \setminus M) \leq \lceil (n - 1)/2 \rceil$.

**Lemma 2.2.** [1] The linear arboricity of every 3-regular graph is 2.

**Lemma 2.3.** [2] The linear arboricity of every 4-regular graph is 3.

Now we give the main result of this paper.

**Theorem 2.1.** Let $G = SG(2k + 2, k)$ ($k \geq 2$) be a Schrijver graph, then

$$la(G) = \begin{cases} 
3, & \text{for } k = 2, \\
\lceil (k + 2)/2 \rceil, & \text{for } k \geq 3.
\end{cases}$$

**Proof.** For $k = 2$, $SG(2k + 2, k) = SG(6, 2)$ is a 4-regular graph, and the result holds by Lemma 2.3. So, in this section, suppose that $k \geq 3$ hereafter. It is obvious that $la(G) \geq \lceil (k + 2)/2 \rceil$ since $\Delta(G) = k + 2$. So it suffices to show that $la(G) \leq \lceil (k + 2)/2 \rceil$. By Lemma 1.3,

$$[V_j, V_{j+1}] = \{v(j, i)v(j+1, i) \mid i = 0, 1, \ldots, 2k + 1\}$$

for $0 \leq j \leq m - 2$,

$$[V_{m-1}, V_m] = \{v(m - 1, i)v(m, i) \mid i = 0, 1, \ldots, 2k + 1\}$$

which is a matching if $k$ is odd, and

$$G[[V_{m-1}, V_m]] = \{v(m - 1, i)v(m, i)v(m - 1, i + k + 1) \mid i = 0, 1, \ldots, k\}$$

for $0 \leq j \leq m - 2$, and

$$[V_{m-1}, V_m] = \{v(m - 1, i)v(m, i) \mid i = 0, 1, \ldots, 2k + 1\}$$

which is a matching if $k$ is even, and

$$G[[V_{m-1}, V_m]] = \{v(m - 1, i)v(m, i)v(m - 1, i + k + 1) \mid i = 0, 1, \ldots, k\}$$
if $k$ is even, in which each component is a 2-path.

By Corollary 1.1, $G[V_0]$ is a balanced complete bipartite graph with two partite subsets

$$X = \{v(0,0), v(0,2), \ldots, v(0,2k)\} \quad \text{and} \quad Y = \{v(0,1), v(0,3), \ldots, v(0,2k+1)\}.$$ 

Let $v(0,2i) = x_i$ and $v(0,2i+1) = y_i$. Then $G[V_0] = K_{k+1,k+1}$ is a balanced complete bipartite graph with two partite subsets

$$X = \{x_i \mid i = 0, 1, \ldots, k\} \quad \text{and} \quad Y = \{y_i \mid i = 0, 1, \ldots, k\}.$$ 

**Case 1.** $k \geq 3$ is odd.

It is not difficult to see that

$$G[\cup_{j=0}^{m-1} [V_j, V_{j+1}]] = \{v(0,i)\cdot v(1,i) \cdots v(m,i) \mid i = 0, 1, \ldots, 2k+1\}$$

is a linear forest where each component is an $m$-path. Let $B = \cup_{j=0}^{m-1} [V_j, V_{j+1}]$ and $S_j = V_0 \cup V_1 \cup \cdots \cup V_j$. By Lemma 1.3, it is not difficult to see that every $[V_j, V_{j+1}] = [S_j, S_j^c]$ is an edge cut of $G$. Hence $G \setminus B$ is a graph whose components are $G[V_0], G[V_1], \ldots, G[V_m]$. Next, we will take away a matching from $G[V_0]$. By Lemma 2.1, $G[V_0] = K_{k+1,k+1}^m$ can be decomposed into the union of $(k+1)/2$ Hamiltonian paths and a matching $M = \{x_{\alpha/2} y_{\alpha/2} \mid \alpha = 0, 2, \ldots, k-1\}$. Then $M \cup B$ forms a linear forest. Moreover, we have $G = (G[V_0] \setminus M) \cup G[V_1] \cup \cdots \cup G[V_m] \cup (M \cup B)$. Thus by Lemma 1.1, Corollary 1.2 and Lemma 2.2, $\text{la}(G) \leq \text{la}((G[V_0] \setminus M) \cup G[V_1] \cup \cdots \cup G[V_m]) + 1 = \text{la}(G[V_0] \setminus M) + 1 \leq (k+1)/2 + 1 = [(k+2)/2].$

**Case 2.** $k \geq 4$ is even.

Let

$$B' = G[\cup_{j=0}^{m-2} [V_j, V_{j+1}]] = \{v(0,i)\cdot v(1,i) \cdots v(m-1,i) \mid i = 0, 1, 3, \ldots, 2k+1\},$$

then

$$G = G[V_0] \cup G[V_1] \cup \cdots \cup G[V_m] \cup B' \cup G[[V_{m-1}, V_m]].$$

In the following, we first decompose $G[V_j] \setminus \{0, 1, \ldots, m\}$ and $B'$. Let

$$P_i = v(0,i)\cdot v(1,i) \cdots v(m-1,i) \quad \text{for} \quad 0 \leq i \leq 2k+1.$$ 

By Lemma 2.1, $G[V_0] = K_{k+1,k+1}$ can be decomposed into the union of $k/2$ Hamiltonian paths

$$H_{\alpha/2} = M_{\alpha} \cup M_{\alpha+1} \setminus \{x_{\alpha/2} y_{\alpha/2}\} \quad \text{for} \quad \alpha = 0, 2, 4, \ldots, k-2$$

and a linear forest

$$M = M_k \cup \{x_{\alpha/2} y_{\alpha/2} \mid \alpha = 0, 2, 4, \ldots, k-2\}.$$ 

Hence $G[V_0] = H_0 \cup H_1 \cup \cdots \cup H_{k/2-1} \cup M$. For $1 \leq j \leq k/2-1$, let $G[V_j] = P_{j,1} \cup P_{j,2}$, where

$$P_{j,1} = v(j,0)\cdot v(j,1) \cdots v(j,2k) \quad \text{and} \quad P_{j,2} = v(j,0)\cdot v(j,2k+1)\cdot v(j,2k).$$

For $j = m$, let $G[V_m] = P_{m,1} \cup P_{m,2}$, where

$$P_{m,1} = v(m,0)\cdot v(m,1) \cdots v(m,k-1) \quad \text{and} \quad P_{m,2} = v(m,0)\cdot v(m,k)\cdot v(m,k-1).$$

**Subcase 2.1.** $k \equiv 0 \pmod{4}$.

Let $P_0 = M_0 \cup M_0'$, where

$$M_0 = \{v(2t,0)\cdot v(2t+1,0) \mid t = 0, 1, \ldots, k/4-1\},$$
and
\[ M'_0 = \{ v(2t + 1, 0)v(2t + 2, 0) \mid t = 0, 1, \ldots, k/4 - 2 \}. \]

And let \( P_{2k} = M_{2k} \cup M'_{2k} \), where
\[ M_{2k} = \{ v(2t, 2k)v(2t + 1, 2k) \mid t = 0, 1, \ldots, k/4 - 1 \}, \]
and
\[ M'_{2k} = \{ v(2t + 1, 2k)v(2t + 2, 2k) \mid t = 0, 1, \ldots, k/4 - 2 \}. \]

Then
\[ H_0 \cup M_0 \cup M'_{2k} \cup P_{2k+1} \cup (\bigcup_{j=1}^{m} P_{j, 1}) \cup \{ v(m - 1, 2k)v(m, k - 1), v(m - 1, 2k + 1)v(m, k) \} \]
forms a Hamiltonian path of \( G \). Let
\[ T = [V_{m-1}, V_m] \setminus \{ v(m - 1, 2k + 1)v(m, k), v(m - 1, 2k)v(m, k - 1) \}. \]
Then
\[ H_1 \cup P_2 \cup P_{2k-1} \cup (\bigcup_{j=1}^{m-1} P_{j, 2}) \cup T \]
forms a linear forest. For \( 2 \leq j \leq m - 1 \), each \( H_j \cup P_{2j} \cup P_{2k-2j+1} \) forms a linear forest. Finally,
\[ M \cup M'_0 \cup M_{2k} \cup P_{m, 2} \cup (\bigcup_{j=0}^{m} P_{2j+1}) \cup (\bigcup_{j=0}^{m-1} P_{2j+k}) \]
forms a linear forest. Thus, the edge set \( E(G) \) is partitioned into \( (k + 2)/2 \) linear forests.
Hence \( \text{la}(G) \leq (k + 2)/2 \).

**Subcase 2.2.** \( k \equiv 2 \pmod{4} \).

Similar to Subcase 2.1, let \( P_0 = N_0 \cup N'_0 \), where
\[ N_0 = \{ v(2t, 0)v(2t + 1, 0) \mid t = 0, 1, \ldots, (k - 6)/4 \}, \]
and
\[ N'_0 = \{ v(2t + 1, 0)v(2t + 2, 0) \mid t = 0, 1, \ldots, (k - 6)/4 \}. \]

Let \( P_{2k} = N_{2k} \cup N'_{2k} \), where
\[ N_{2k} = \{ v(2t, 2k)v(2t + 1, 2k) \mid t = 0, 1, \ldots, (k - 6)/4 \}, \]
and
\[ N'_{2k} = \{ v(2t + 1, 2k)v(2t + 2, 2k) \mid t = 0, 1, \ldots, (k - 6)/4 \}. \]

Then it is not difficult to see that
\[ H_0 \cup N_0 \cup N'_{2k} \cup P_{2k+1} \cup (\bigcup_{j=1}^{m} P_{j, 1}) \cup \{ v(m - 1, 0)v(m, 0), v(m - 1, 2k + 1)v(m, k) \}, \]
forms a Hamiltonian path of \( G \). Let
\[ T' = [V_{m-1}, V_m] \setminus \{ v(m - 1, 0)v(m, 0), v(m - 1, 2k + 1)v(m, k) \}. \]
Clearly,
\[ H_1 \cup P_2 \cup P_{2k-1} \cup (\bigcup_{j=1}^{m-1} P_{j, 2}) \cup T' \]
forms a linear forest, and for \( 2 \leq j \leq m - 1 \), each \( H_j \cup P_{2j} \cup P_{2k-2j+1} \) forms a linear forest. Finally, it is not difficult to verify that
\[ M \cup N'_0 \cup N_{2k} \cup P_{m, 2} \cup (\bigcup_{j=0}^{m} P_{2j+1}) \cup (\bigcup_{j=0}^{m-1} P_{2j+k}) \]
forms a linear forest. Thus, the edge set \( E(G) \) is partitioned into \( (k + 2)/2 \) linear forests.
Hence we have \( \text{la}(G) \leq (k + 2)/2 \), too.

Up to now, we have shown that \( \text{la}(G) \leq \lceil (k + 2)/2 \rceil \), and then the theorem holds.
Therefore, the linear arboricity conjecture holds for Schrijver graph $SG(2k+2,k)$ for $k \geq 2$.

3. The vertex linear arboricity and vertex arboricity of Schrijver graph $SG(2k+2,k)$

In this section, we discuss the vertex linear arboricity and the vertex arboricity of the Schrijver graph.

**Theorem 3.1.** The vertex linear arboricity for the Schrijver graph $G = SG(2k+2,k)$, $(k \geq 2)$, is two.

**Proof.** The proof will be split into three cases. The main idea is to partition the vertex set $V(G)$ into two subsets such that every subset induces a linear forest.

Case 1. $k \geq 3$ is odd.

Let

$$Q = \{v(j,i) \mid 0 \leq j \leq m, \ i = 0, 2, \ldots, 2k\},$$

and

$$R = \{v(j,i) \mid 0 \leq j \leq m, \ i = 1, 3, \ldots, 2k + 1\}.$$  

By Lemma 1.3,

$$G[Q] = \{v(0,0)v(1,i)\cdots v(m,i)v(m,i+k+1)v(m−1,i+k+1)\cdots v(0,i+k+1)\mid i = 0, 2, \ldots, k − 1\}$$

and

$$G[R] = \{v(0,0)v(1,i)\cdots v(m,i)v(m,i+k+1)v(m−1,i+k+1)\cdots v(0,i+k+1)\mid i = 1, 3, \ldots, k\}$$

are two linear forests in which every component is a $k$-path.

Case 2. $k \geq 4$ is even.

Let

$$Q' = \{v(j,i) \mid 0 \leq j \leq m−1, \ i = 0, 2, \ldots, 2k\} \cup \{v(m,i) \mid i = 0, 2, \ldots, k\},$$

and

$$R' = \{v(j,i) \mid 0 \leq j \leq m−1, \ i = 1, 3, \ldots, 2k + 1\} \cup \{v(m,i) \mid i = 1, 3, \ldots, k − 1\}.$$  

By Lemma 1.3,

$$G[Q'] = \{v(0,0)v(1,0)\cdots v(m,0)v(m,k)v(m−1,k)\cdots v(0,k)\} \cup \{v(0,i)v(1,i)\cdots v(m,i) \mid i = 2, 4, \ldots, k − 2\} \cup \{v(0,i)v(1,i)\cdots v(m−1,i) \mid i = k + 2, k + 4, \ldots, 2k\}$$

and

$$G[R'] = \{v(0,i)v(1,i)\cdots v(m,i) \mid i = 1, 3, \ldots, k − 1\} \cup \{v(0,i)v(1,i)\cdots v(m−1,i) \mid i = k + 1, k + 3, \ldots, 2k + 1\}$$

are two linear forests.

Case 3. $k = 2$.
One can partition the vertex set $V(G)$ into two subsets
\[
\{v(0, 0), v(0, 2), v(0, 4), v(1, 0), v(1, 2)\} \quad \text{and} \quad \{v(0, 1), v(0, 3), v(0, 5), v(1, 1)\}.
\]
It is easy to verify that every subset induces a linear forest.

**Corollary 3.1.** The vertex arboricity for the Schrijver graph $SG(2k+2,k)$, $(k \geq 2)$, is 2.

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**References**