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ON THE GENERALIZED ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. In this paper, we generalize a known result concerning the absolute Riesz summability factors of infinite series.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α we denote the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [4]),

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \text{ and } t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,\ ({t_n}^1 = t_n) \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = O(n^\alpha),\ A_{-n}^\alpha = 0 \ for\ n > 0. \tag{1.2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [5, 7])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n}|t_n^\alpha|^k < \infty. \tag{1.3}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.4}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.5}$$

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defines the sequence (w_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid w_n - w_{n-1} \mid^k < \infty. \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ (resp.$\mid \bar{N}, p_n \mid$) summability. Given a normal matrix $A = (a_{nv})$, we associate two lower semi-matrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \ n, v = 0, 1, \ldots \tag{1.7}$$

and

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}, \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \ n = 1, 2, \ldots \tag{1.8}$$

It may be noted that $\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \ n = 0, 1, \ldots \tag{1.9}$$

and

$$A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \widehat{a}_{nv} a_v. \tag{1.10}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \overline{\Delta} A_n(s) \mid^k < \infty, \tag{1.11}$$

where

$$\overline{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

In the special case, for $a_{nv} = p_v/P_n$, $|A, p_n|_k$ summability is the same as $\mid \bar{N}, p_n \mid_k$ summability. For any sequence (λ_n) we write that $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

2. Known results

The following theorems are known dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series.

Theorem A ([2])**.** *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \ as \ n \to \infty. \tag{2.1}$$

If the conditions

$$\lambda_n = o(1) \ as \ n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} nX_n \mid \Delta^2 \lambda_n \mid = O(1) \ as \ m \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} \mid t_n \mid^k = O(X_m) \ as \ m \to \infty, \tag{2.4}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

If we take $p_n = 1$ for all values of n, then we get the known result of Mazhar dealing with $\mid C, 1 \mid_k$ summability factors of infinite series (see [8]).

Theorem B ([3])**.** *If the sequences* (X_n), (λ_n), *and* (p_n) *satisfy the conditions (12)-(14) and*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{\mid t_n \mid^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.5}$$

then the series $\sum a_n \lambda_n$ *is summable* $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

Remark. *It should be noted that condition (2.5) is reduced to the condition (2.4), when* $k = 1$. *When* $k > 1$, *condition (2.5) is weaker than condition (2.4) but the converse is not true (see* [3]*).*

3. Main result

The aim of this paper is to prove the following theorem.

Theorem 3.1. *Let* $A = (a_{nv})$ *be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \ n = 0, 1, 2, ..., \tag{3.1}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1 \tag{3.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right) \tag{3.3}$$

$$\hat{a}_{n,v+1} = O\left(v|\Delta_v \hat{a}_{nv}|\right). \tag{3.4}$$

If the sequences (X_n) *be a positive non-decreasing sequence. If the sequences* (X_n), (λ_n), *and* (p_n) *satisfy the conditions (2.1)-(2.3) and (2.5), then the series* $\sum a_n \lambda_n$ *is summable* $\mid A, p_n \mid_k$, $k \geq 1$.

We need the following lemma for the proof of our theorem.

Lemma 3.2 ([2])**.** *Under the conditions of the theorem, we get*

$$nX_n \mid \Delta\lambda_n \mid = O(1), \ as \ n \to \infty, \tag{3.5}$$

$$\sum_{n=1}^{\infty} X_n \mid \Delta\lambda_n \mid < \infty, \tag{3.6}$$

$$X_n \mid \lambda_n \mid = O(1), \ as \ n \to \infty. \tag{3.7}$$

4. Proof of the theorem

Let (T_n) be the A-transform of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$\begin{aligned} T_n - T_{n-1} &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{v=1}^{n} v a_v \\ &= \sum_{v=1}^{n-1} \Delta_v \frac{\hat{a}_{nv}\lambda_v}{v}(v+1)t_v + \frac{a_{nn}\lambda_n}{n}(n+1)t_n \end{aligned}$$

Now, since

$$\begin{aligned} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) &= \frac{(v+1)\hat{a}_{nv}\lambda_v - v\hat{a}_{n,v+1}\lambda_{v+1}}{v(v+1)} \\ &= \frac{(v+1)\Delta_v(\hat{a}_{nv})\lambda_v + (v+1)\hat{a}_{n,v+1}\Delta\lambda_v + \hat{a}_{n,v+1}\lambda_{v+1}}{v(v+1)}, \end{aligned}$$

we have that

$$\begin{aligned} T_n - T_{n-1} &= \frac{(n+1)a_{nn}t_n\lambda_n}{n} - \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) t_v \lambda_v \frac{v+1}{v} \\ &+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v t_v \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1} t_v \frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1,2,3,4. \tag{4.1}$$

Firstly, we have that

$$\begin{aligned} \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,1} \mid^k &= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid a_{nn} \mid t_n \mid^k \\ &= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \frac{p_n}{P_n} \frac{\mid t_n \mid^k}{X_n^{k-1}} \\ &= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \frac{p_v}{P_v} \frac{\mid t_v \mid^k}{{X_v}^{k-1}} + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \frac{p_n}{P_n} \frac{\mid t_n \mid^k}{{X_n}^{k-1}} \\ &= O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m = O(1), \quad as \quad m \to \infty, \end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma. Also, as in $T_{n,1}$ we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,2}\mid^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(a_{nn})^{k-1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{p_v}{P_v}\frac{|t_v|^k}{X_v^{k-1}} = O(1) \quad as \quad m\to\infty.
\end{aligned}$$

Again, by using (2.1), we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\mid\Delta\lambda_v\mid\mid t_v\mid\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}(v\mid\Delta\lambda_v\mid)^k|t_v|^k|\Delta_v(\hat{a}_{nv})|\right)\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}(v\mid\Delta\lambda_v\mid)^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}v\mid\Delta\lambda_v\mid\frac{p_v|t_v|^k}{P_vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\mid\Delta\lambda_v\mid)\sum_{r=1}^{v}\frac{p_r|t_r|^k}{rX_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{p_v|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\mid\Delta\lambda_v\mid)X_v + O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v\mid\Delta^2\lambda_v\mid) + O(1)\sum_{v=1}^{m-1}X_v\mid\Delta\lambda_v\mid) + O(1)m|\Delta\lambda_m|X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma. Finally by using (2.1), as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,4}\mid^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}}{v}|\lambda_{v+1}||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{p_v|t_v|^k}{P_vX_v^{k-1}} = O(1), \quad as \quad m\to\infty.
\end{aligned}$$

This completes the proof of the theorem.

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