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ON CONVERGENCE AND ABSOLUTE CONVERGENCE OF FOURIER SERIES WITH RESPECT TO ORTHOGONAL POLYNOMIALS

(COMMUNICATED BY FRANCISCO MARCELLAN)

JOSEF OBERMAIER

Abstract. Let μ be a probability measure on the Borel σ-algebra of $\mathbb{R}$ with compact and infinite support S, and $\{p_n\}_{n=0}^{\infty}$ be an orthonormal polynomial sequence with respect to μ. A Banach space $B \subset L_1(\mu)$ with norm $\| \ \|$ is called harmonic if the set $\mathcal{P}$ of polynomials is dense in B, and $\|f\|_1 \leq \|f\|$ for all $f \in B$. We are studying Fourier series of $f \in B$ with respect to $\{p_n\}_{n=0}^{\infty}$. Equipped with a proper norm the subspaces $B_D \subset B$ of convergent Fourier series, and $B_A \subset B$ of absolute convergent Fourier series are Banach spaces for its own. We show that in case B is not isomorphic to ℓ_1 it holds $B_A \subsetneq B_D$. For example this result fits for $C(S)$ which is a harmonic Banach space not isomorphic to ℓ_1. In case μ is a Jacobi measures with $\alpha > -1/2$ or $\beta > -1/2$ an explicit function $f \in C([-1,1])$ with convergent but not absolute convergent Fourier series is constructed. For that purpose we prove a modification of Schur's inequality.

1. Introduction

In classical Fourier analysis it is well known that there exists a function $f \in C(\mathbb{T})$ such that the partial sums $\sum_{n=0}^{N} \hat{f}(n)e^{int}$ of its Fourier series are not uniformly converging to f. Also there exist uniformly convergent Fourier series which are not absolutely convergent. For that purpose one can take

$$f(e^{it}) = \int_{-\pi}^{t} g(r)dr \tag{1.1}$$

with

$$g(r) = \sum_{n=1}^{\infty} \frac{\cos(nr)}{\ln(n+1)}. \tag{1.2}$$

Then by simple means the Fourier series of f is not absolutely convergent. Since f is of bounded variation the Dirichlet-Jordan convergence criterion [14] implies that the Fourier series of f is uniformly convergent. Denoting the set of f with uniformly

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convergent Fourier series by $U(\mathbb{T})$ and those f with absolutely convergent Fourier series by $A(\mathbb{T})$, we have

$$A(\mathbb{T}) \subsetneq U(\mathbb{T}) \subsetneq C(\mathbb{T}), \tag{1.3}$$

see [4].

We focus on Fourier series with respect to an orthonormal polynomial sequence $\{p_n\}_{n=0}^{\infty}$, where the support $\mathcal{S} \subset \mathbb{R}$ of the orthogonalization measure μ is assumed to be infinite and compact. It is well-known, that in case $\mathcal{S} = [-1, 1]$ there exists $f \in C([-1, 1])$ such that the Fourier series doesn't converge uniformly, see [2]. However, there also are systems such that every $f \in C(\mathcal{S})$ is represented by its Fourier series, see [6], [7], [8] and [9]. Now the question is, if there also are systems such that every uniformly convergent Fourier series is absolute convergent. In Section 2 we will prove in a more general setting, that this is not the case. Moreover, in case of Jacobi systems we are able to construct functions $f \in C([-1, 1])$ with uniformly but not absolutely convergent Fourier series, see Section 3.

2. Convergent and absolute convergent Fourier series in harmonic Banach spaces

Let μ be a probability Borel measure on $\mathbb{R}$ with compact and infinite support $\mathcal{S}$. As usual, let

$$L_p(\mu) = \{f : \mathcal{S} \to \mathbb{C} : \int |f|^p d\mu < \infty\}, \quad 1 \leq p < \infty, \tag{2.1}$$

with norm $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$, and

$$C(\mathcal{S}) = \{f : \mathcal{S} \to \mathbb{C} : f \text{ continuous}\} \tag{2.2}$$

with norm $\|f\|_\infty = \sup_{x \in S} |f(x)|$. Furthermore, denote by $\mathcal{P}$ the set of algebraic polynomials in one real variable and complex coefficients. $C(\mathcal{S})$ and $L_p(\mu)$ are harmonic Banach spaces in the following sense.

Definition 2.1. *Let $B \subset L_1(\mu)$ be a Banach space with respect to a norm $\| \ \|$ such that $\mathcal{P} \subset B$ is dense in B and*

$$\|f\|_1 \leq \|f\| \quad \text{for all } f \in B. \tag{2.3}$$

Then B is called an harmonic Banach space with respect to μ.

By Gram-Schmidt procedure there exists a unique sequence $\{p_n\}_{n=0}^{\infty} \subset \mathcal{P}$ of orthonormal polynomials with $\int p_n p_m d\mu = \delta_{n,m}$, $\deg p_n = n$ and p_n has positive leading coefficient. We call $\{p_n\}_{n=0}^{\infty}$ the orthonormal polynomial sequence with respect to μ.

The formal Fourier series of $f \in B$ with respect to $\{p_n\}_{n=0}^{\infty}$ is given by

$$f \sim \sum_{n=0}^{\infty} \hat{f}_n p_n, \tag{2.4}$$

where the Fourier coefficients are defined by

$$\hat{f}_n = \int f p_n d\mu. \tag{2.5}$$

If $f \in B$ has a representation $\sum_{n=0}^{\infty} c_n p_n$, then inequality (2.3) implies $c_n = \hat{f}_n$.

The Nth partial sum of the formal Fourier series of $f \in B$ is given by

$$D_N(f) = \sum_{n=0}^{N} \hat{f}_n p_n. \tag{2.6}$$

By simple means D_N is a continuous linear operator from B into B. For to investigate the subspace of convergent Fourier series we rely on the following lemma.

Lemma 2.2. *Let $(X, \| \|)$ denote a Banach space and $\{F_N\}_{N=0}^{\infty}$ a sequence of continuous linear operators from X into X. Set*

$$Y = \{y \in X : \lim_{N\to\infty} F_N(y) = y\}, \tag{2.7}$$

and

$$|||y||| = \sup_{N\in\mathbb{N}_0} \|F_N(y)\|. \tag{2.8}$$

Then $(Y, ||| \, |||)$ is a Banach space, and it holds

$$\|y\| \leq |||y||| \quad \text{for all } y \in Y. \tag{2.9}$$

Since it is standard we omit the proof. Due to Lemma 2.2 we make the following definition.

Definition 2.3. *Let $(B, \| \|)$ be a harmonic Banach space with respect to μ. Then the Banach space*

$$B_D = \{f \in B : \lim_{N\to\infty} \|D_N(f) - f\| = 0\} \tag{2.10}$$

with norm

$$\|f\|_D = \sup_{N\in\mathbb{N}_0} \|D_N(f)\| \tag{2.11}$$

is called space of convergent Fourier series with respect to B.

The absolute convergent Fourier series form a subspace of B_D.

Definition 2.4. *Let $(B, \| \|)$ be a harmonic Banach space with respect to μ. The Banach space*

$$B_A = \{f \in B : \sum_{n=0}^{\infty} \|\hat{f}_n p_n\| < \infty\} \tag{2.12}$$

with norm

$$\|f\|_A = \sum_{n=0}^{\infty} \|\hat{f}_n p_n\| \tag{2.13}$$

is called space of absolute convergent Fourier series with respect to B.

It is easily seen that $(B_A, \| \|_A)$ is isometrically isomorphic to the Banach space $(\ell_1, \| \|_1)$, where

$$\ell_1 = \{\{a_n\}_{n=0}^{\infty} : a_n \in \mathbb{C} \text{ and } \sum_{n=0}^{\infty} |a_n| < \infty\},$$

with norm $\|\{a_n\}_{n=0}^{\infty}\|_1 = \sum_{n=0}^{\infty} |a_n|$.

Our aim is to give sufficient conditions for $B_A \subsetneq B_D$. For instance, if B_D is not isomorphic to ℓ_1, then $B_A \subsetneq B_D$. Assume as a relation of sets that $B_A = B_D$. Then the identity mapping $id : B_A \to B_D$ is continuous, and due to the open mapping theorem [3, (14.16)] it is an isomorphism. Thus, B_D is isomorphic to ℓ_1, which is a

contradiction.
It is exciting that if B is not isomorphic to ℓ_1, then $B_A \subsetneq B_D$, too. For a proof, we need the following lemma.

Lemma 2.5. *Let $(B, \|\ \|)$ be an harmonic Banach space with respect to μ which is not isomorphic to ℓ_1. Then for all $N \in \mathbb{N}_0$ and all $C > 0$ there exists $M > N$ and $a_{N+1}, \ldots, a_M \in \mathbb{C}$ such that*

$$\sum_{n=N+1}^{M} |a_n| > C \| \sum_{n=N+1}^{M} a_n \frac{p_n}{\|p_n\|} \|. \tag{2.14}$$

Proof. Suppose that, contrary to our claim, there exists $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\sum_{n=N+1}^{M} |a_n| \leq C \| \sum_{n=N+1}^{M} a_n \frac{p_n}{\|p_n\|} \| \quad \text{for all } M > N, a_{N+1}, \ldots, a_M \in \mathbb{C}.$$

Let us fix such an N and let $M \in \mathbb{N}_0$ arbitrary. Since norms on finite dimensional spaces are equivalent, there exists $D > 0$ such that

$$\sum_{n=0}^{N} |a_n| \leq D \| \sum_{n=0}^{N} a_n \frac{p_n}{\|p_n\|} \| \quad \text{for all } a_0, \ldots, a_N \in \mathbb{C}.$$

Setting $E = \max(C, D)$ we get

$$\sum_{n=0}^{M} |a_n| \leq E(\| \sum_{n=0}^{\min(M,N)} a_n \frac{p_n}{\|p_n\|} \| + \| \sum_{n=N+1}^{M} a_n \frac{p_n}{\|p_n\|} \|)$$

for all $M \in \mathbb{N}_0$, $a_0, \ldots, a_M \in \mathbb{C}$. If id denotes the identity mapping from $B \to B$, then

$$D_N(\sum_{n=0}^{M} a_n \frac{p_n}{\|p_n\|}) = \sum_{n=0}^{\min(M,N)} a_n \frac{p_n}{\|p_n\|}$$

and

$$(id - D_N)(\sum_{n=0}^{M} a_n \frac{p_n}{\|p_n\|}) = \sum_{n=N+1}^{M} a_n \frac{p_n}{\|p_n\|}.$$

Hence, there exists $F > 0$ such that

$$\sum_{n=0}^{M} |a_n| \leq F \| \sum_{n=0}^{M} a_n \frac{p_n}{\|p_n\|} \| \quad \text{for all } M \in \mathbb{N}_0, a_0, \ldots, a_M \in \mathbb{C}.$$

Therefore, $\{\frac{p_n}{\|p_n\|}\}_{n=0}^{\infty}$ is a basic sequence in B which is equivalent to the standard unit vector basis of ℓ_1, see [5, 4.3.6]. Taking into account that $\mathcal{P}$ is dense in B, [5, 4.3.2] yields that B is isomorphic to ℓ_1. This is a contradiction to our assumption. □

Now, we can state the main result of this section.

Theorem 2.6. *Let B be an harmonic Banach space with respect to μ. If B is not isomorphic to ℓ_1, then $B_A \subsetneq B_D$.*

Proof. Let us assume $B_A = B_D$. Then the identity mapping from B_A onto B_D is continuous. Hence, by the open mapping theorem the norms $\| \|_A$ and $\| \|_D$ are equivalent.
Changing inequality (2.14) of Lemma 2.5 into

$$\| \sum_{n=N+1}^{M} \frac{a_n}{\sum_{k=N+1}^{M} |a_k|} \frac{p_n}{\|p_n\|} \| < \frac{1}{C}$$

shows that we are able to construct polynomials

$$Q_n = \sum_{k=l_n}^{u_n} b_k \frac{p_k}{\|p_k\|},$$

with $u_n < l_{n+1}$ and $\sum_{k=l_n}^{u_n} |b_k| = 1$ for all $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|Q_n\| = 0.$$

Obviously it holds

$$\| \frac{1}{N} \sum_{n=1}^{N} Q_n \|_A = 1$$

for all $N \in \mathbb{N}$. Since

$$\begin{aligned} \| \sum_{n=1}^{N} Q_n \|_D &= \sup_{0\le m\le N-1,\, l_{m+1}\le r\le u_{m+1}} \| \sum_{n=1}^{m} Q_n + \sum_{k=l_{m+1}}^{r} b_k \frac{p_k}{\|p_k\|} \| \\ &\le \sup_{0\le m\le N-1} (\sum_{n=1}^{m} \|Q_n\| + 1) \\ &\le 1 + \sum_{n=1}^{N} \|Q_n\|, \end{aligned}$$

we get by a well-known result on Césaro means

$$\lim_{N\to\infty} \| \frac{1}{N} \sum_{n=1}^{N} Q_n \|_D = 0.$$

This is in contradiction with the equivalence of the norms $\| \|_A$ and $\| \|_D$. □

For classical Banach spaces we get the following corollary.

Corollary 2.7. *In case $B = C(\mathcal{S})$ or $B = L_p(\mu)$, $1 \le p < \infty$, it holds*

$$B_A \subsetneq B_D. \tag{2.15}$$

Proof. Firstly let $B = C(\mathcal{S})$ or $B = L_1(\mu)$.
Assume T is an isomorphism from ℓ_1 onto B. Since the standard vector basis $\{e_n\}_{n=0}^{\infty}$ is an unconditional basis in ℓ_1, we get by [5, 4.2.14] that $\{T(e_n)\}_{n=0}^{\infty}$ is an unconditional basis in B. This is a contradiction to [11, 15.1] and [11, 15.2].
Secondly let $B = L_p(\mu)$, $1 < p < \infty$. By [5, 2.8.12] B doesn't have Schur's property [10]. Since ℓ_1 does have Schur's property, B and ℓ_1 can not be isomorphic. □

Note that a proof of $A(\mathbb{T}) \subsetneq U(\mathbb{T})$ in the classical case could also be given along the lines of above.

3. Jacobi polynomials

Due to Corollary 2.7 there exists $f \in C(\mathcal{S})$ with uniformly but not absolutely converging Fourier series. The proof, however, has not been constructive. Therefore, another goal is to detect such a function. In this chapter we will provide a proper construction for certain Jacobi polynomial systems. The sequence $\{p_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$, $\alpha, \beta > -1$, of orthonormal Jacobi polynomials is defined by the three term recurrence relation

$$x p_n^{(\alpha,\beta)}(x) = \lambda_n p_{n+1}^{(\alpha,\beta)}(x) + \beta_n p_n^{(\alpha,\beta)}(x) + \lambda_{n-1} p_{n-1}^{(\alpha,\beta)}(x), \tag{3.1}$$

where $p_{-1} = 0$, $p_0 = 1$,

$$\lambda_n = \left(\frac{4(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)} \right)^{\frac{1}{2}} \tag{3.2}$$

and

$$\beta_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}. \tag{3.3}$$

They are orthogonal with respect to the measure

$$d\mu^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta} dx \tag{3.4}$$

supported on $[-1, 1]$. It holds the symmetric relation

$$p_n^{(\alpha,\beta)}(x) = (-1)^n p_n^{(\beta,\alpha)}(-x), \tag{3.5}$$

see [13].

For our purpose we need a modification of Schur's inequality [1, Theorem 5.1.9]. Denote by $\mathcal{P}_n \subset \mathcal{P}$ the set of polynomials of degree less or equal to n, and let $\|f(x)\|_\infty = \sup_{x\in[-1,1]} |f(x)|$.

Lemma 3.1. *Let $a, b \geq 0$, $c = \max(a, b)$ and $p \in \mathcal{P}_{n-1}$.*
If $c \geq 1/2$, then

$$\|p(x)\|_\infty \leq 2^{|a-b|} n^{2c} \|(1-x)^a (1+x)^b p(x)\|_\infty, \tag{3.6}$$

and if $c < 1/2$, then

$$\|p(x)\|_\infty \leq 2^{|a-b|} c^{-2c} n^{2c} \|(1-x)^a (1+x)^b p(x)\|_\infty. \tag{3.7}$$

Proof. Let

$$x_k = \cos \frac{(2n-2k+1)\pi}{2n}, \quad k = 1, 2, \ldots, n,$$

denote the zeros of the Chebyshev polynomials of first kind

$$T_n(x) = \cos(n \arccos x).$$

Note that $T_n'(x) = (n \sin n\theta)/(\sin\theta)$, $x = \cos\theta$.
First, let $a = b \geq 1/2$. Without loss of generality we may assume that $p \in \mathcal{P}_{n-1}$ with $\|(1-x^2)^a p(x)\|_\infty = 1$.
If $|y| \leq x_n$, then

$$\begin{aligned} |p(y)| &\leq (1-y^2)^{-a} \leq (1-x_n^2)^{-a} \\ &= \left(\sin\frac{\pi}{2n}\right)^{-2a} \leq \left(\frac{2}{\pi}\frac{\pi}{2n}\right)^{-2a} = n^{2a}. \end{aligned}$$

In case $x_n < y \leq 1$ we get

$$\begin{aligned}
|p(y)| &= \left|\sum_{k=1}^{n} p(x_k)\frac{T_n(y)}{T_n'(x_k)(y-x_k)}\right| \\
&\leq \frac{1}{n}\sum_{k=1}^{n} |p(x_k)(1-x_k^2)^a|(1-x_k^2)^{1/2-a}\frac{T_n(y)}{y-x_k} \\
&\leq \frac{1}{n}\sum_{k=1}^{n}(1-x_k^2)^{1/2-a}\frac{T_n(y)}{y-x_k} \leq \frac{1}{n}(1-x_n^2)^{1/2-a}\sum_{k=1}^{n}\frac{T_n(y)}{y-x_k} \\
&= \frac{1}{n}(1-x_n^2)^{1/2-a}T_n'(y) \leq n(1-x_n^2)^{1/2-a} = n\left(\sin\frac{\pi}{2n}\right)^{1-2a} \\
&\leq nn^{2a-1} = n^{2a},
\end{aligned}$$

where Lagrange interpolation has been applied twice time. For $-1 \leq y < x_1$ we deduce $|p(y)| \leq n^{2a}$ quite similar.
Secondly, let $0 \leq a = b < 1/2$. The case $a = b = 0$ is trivial. Otherwise let k be the least number such that $ka \geq 1/2$. Thus the result of above implies

$$\begin{aligned}
\|p(x)\|_\infty^k = \|p(x)^k\|_\infty &\leq (kn-k+1)^{2ka}\|(1-x^2)^{ka}p(x)^k\|_\infty \\
&< (kn)^{2ka}\|(1-x^2)^a p(x)\|_\infty^k.
\end{aligned}$$

Since $ka < 1$ we get

$$\|p(x)\|_\infty \leq k^{2a}n^{2a}\|(1-x^2)^a p(x)\|_\infty < a^{-2a}n^{2a}\|(1-x^2)^a p(x)\|_\infty.$$

Finally, for the general case we only have to take into account that

$$\|(1-x^2)^c p(x)\|_\infty \leq 2^{|a-b|}\|(1-x)^a(1+x)^b p(x)\|_\infty.$$

□

Next we state a result on orthonormal Jacobi polynomials, which is mainly due to P. K. Suetin [12, Theorem 7.5], who attributes his result to S. N. Bernstein, and to [13, (7.32.1), Theorem 7.32.2].

Lemma 3.2. *Let $\alpha, \beta > -1$, $\alpha' = \max(\alpha+1/2, 0)$ and $\beta' = \max(\beta+1/2, 0)$. Then there exists a constant $D > 0$ with*

$$\sqrt{1-x}^{\alpha'}\sqrt{1+x}^{\beta'}|p_n^{(\alpha,\beta)}(x)| \leq D \tag{3.8}$$

for all $x \in [-1,1]$ and $n \in \mathbb{N}_0$.

Using Lemma 3.1 one gets certain uniform bounds for orthonormal Jacobi polynomials.

Lemma 3.3. *Let $\alpha, \beta > -1$, $\gamma_- = \max(\alpha - 3/2, \beta + 1/2, 0)$ and $\gamma_+ = \max(\beta - 3/2, \alpha + 1/2, 0)$. Then there exists a constant $C > 0$ with*

$$(1 \mp x)|p_n^{(\alpha,\beta)}(x)| \leq Cn^{\gamma_\mp} \tag{3.9}$$

for all $x \in [-1,1]$.

Proof. Let $\alpha' = \max(\alpha + 1/2, 0)$ and $\beta' = \max(\beta + 1/2, 0)$.
In case $\alpha' \leq 2$ Lemma 3.2 implies

$$\sqrt{1+x}^{\beta'}(1-x)|p_n^{(\alpha,\beta)}(x)| \leq D_1$$

for all $x \in [-1, 1]$. Thus by Lemma 3.1 it follows

$$\|(1-x)p_n^{(\alpha,\beta)}(x)\|_\infty \le Cn^{\gamma_-},$$

where $\gamma_- = \max(0, \beta') = \max(\alpha - 3/2, \beta + 1/2, 0)$.
In case $\alpha' > 2$ we get by Lemma 3.2

$$\sqrt{1-x}^{\alpha'-2}\sqrt{1+x}^{\beta'}(1-x)|p_n^{(\alpha,\beta)}(x)| \le D_2$$

for all $x \in [-1, 1]$. Therefore, Lemma 3.1 implies

$$\|(1-x)p_n^{(\alpha,\beta)}(x)\|_\infty \le Cn^{\gamma_-},$$

where $\gamma_- = \max(\alpha' - 2, \beta') = \max(\alpha - 3/2, \beta + 1/2, 0)$.
The remaining assertion with γ_+ holds due to (3.5). □

Now we are able to show the following theorem.

Theorem 3.4. *Let $\alpha > \beta$ and $\alpha > -1/2$ and set $b_n = (-1)^n \left(\frac{1}{n} + \frac{1}{n+1}\right)$. Then*

$$\sum_{n=1}^{\infty} b_n \frac{p_n^{(\alpha,\beta)}}{p_n^{(\alpha,\beta)}(1)} \tag{3.10}$$

is uniformly but not absolutely converging.

Proof. It holds

$$\max_{-1\le x\le 1} |p_n^{(\alpha,\beta)}(x)| = p_n^{(\alpha,\beta)}(1),$$

see [13, (7.32.2)]. Thus

$$\sum_{n=1}^{\infty} \|b_n \frac{p_n^{(\alpha,\beta)}}{p_n^{(\alpha,\beta)}(1)}\|_\infty = \sum_{n=1}^{\infty} |b_n| = \infty.$$

We will show that $\{\sum_{n=1}^{N} b_n p_n^{(\alpha,\beta)}/p_n^{(\alpha,\beta)}(1)\}_{N=1}^{\infty}$ is a Cauchy sequence with respect to the sup-Norm. Set

$$r_n = \sum_{k=n}^{\infty} b_k = \frac{(-1)^n}{n},$$

and take $M \ge N \ge 1$. Then we get

$$\begin{aligned}
\sum_{n=N}^{M+1} b_n \frac{p_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(1)} &= \sum_{n=N}^{M+1} (r_n - r_{n+1}) \frac{p_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(1)} \\
&= r_N \frac{p_N^{(\alpha,\beta)}(x)}{p_N^{(\alpha,\beta)}(1)} - r_{M+2} \frac{p_{M+1}^{(\alpha,\beta)}(x)}{p_{M+1}^{(\alpha,\beta)}(1)} \\
&+ \sum_{n=N}^{M} r_{n+1} \left(\frac{p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(x) - p_n^{(\alpha,\beta)}(x)p_{n+1}^{(\alpha,\beta)}(1)}{p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)} \right).
\end{aligned}$$

It is obvious that

$$\lim_{N\to\infty} \|r_N \frac{p_N^{(\alpha,\beta)}(x)}{p_N^{(\alpha,\beta)}(1)} - r_{M+2} \frac{p_{M+1}^{(\alpha,\beta)}(x)}{p_{M+1}^{(\alpha,\beta)}(1)}\|_\infty = 0,$$

and applying Christoffel-Darboux formula

$$\sum_{n=N}^{M} r_{n+1} \left(\frac{p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(x) - p_n^{(\alpha,\beta)}(x)p_{n+1}^{(\alpha,\beta)}(1)}{p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)} \right)$$

$$
\begin{aligned}
&= (x-1)\sum_{n=N}^{M}\frac{r_{n+1}}{\lambda_n p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)}\sum_{k=0}^{n}p_k^{(\alpha,\beta)}(1)p_k^{(\alpha,\beta)}(x)\\
&= (x-1)\sum_{k=0}^{N}p_k^{(\alpha,\beta)}(1)p_k^{(\alpha,\beta)}(x)\sum_{n=N}^{M}\frac{r_{n+1}}{\lambda_n p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)}\\
&\quad +(x-1)\sum_{k=N+1}^{M}p_k^{(\alpha,\beta)}(1)p_k^{(\alpha,\beta)}(x)\sum_{n=k}^{M}\frac{r_{n+1}}{\lambda_n p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)}.
\end{aligned}
$$

According to [13, (4.1.1) and (4.3.3)] we have

$$
p_n^{(\alpha,\beta)}(1)=\left(\frac{(2n+\alpha+\beta+1)\Gamma(\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(n+1)\Gamma(n+\beta+1)}\right)^{\frac{1}{2}}.
$$

Applying the well-known Stirling's formula with respect to the asymptotic of the Gamma function one gets

$$
p_n^{(\alpha,\beta)}(1)=Cn^{\alpha+1/2}(1+O(n^{-1})),
$$

where $C>0$ is a constant. Moreover, (3.2) implies

$$
\lambda_n=\frac{1}{2}+O(n^{-2}).
$$

Due to the fact that $\{r_n\}_{n=1}^{\infty}$ is alternating it follows

$$
\left|\sum_{n=k}^{M}\frac{r_{n+1}}{\lambda_n p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)}\right|\le Dk^{-2\alpha-2},
$$

where $D>0$ is a constant not depending on M and k. Applying Lemma 3.3 there exists $\gamma<\alpha+\frac{1}{2}$ such that

$$
\begin{aligned}
&|(x-1)\sum_{k=N+1}^{M}p_k^{(\alpha,\beta)}(1)p_k^{(\alpha,\beta)}(x)\sum_{n=k}^{M}\frac{r_{n+1}}{\lambda_n p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)}| \le\\
&\qquad E\sum_{k=N+1}^{M}k^{\alpha+\frac{1}{2}}k^{\gamma}k^{-2\alpha-2}=E\sum_{k=N+1}^{M}k^{\gamma-\alpha-\frac{3}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
&|(x-1)\sum_{k=0}^{N}p_k^{(\alpha,\beta)}(1)p_k^{(\alpha,\beta)}(x)\sum_{n=N}^{M}\frac{r_{n+1}}{\lambda_n p_n^{(\alpha,\beta)}(1)p_{n+1}^{(\alpha,\beta)}(1)}| \le\\
&\qquad F\sum_{k=0}^{N}k^{\alpha+\frac{1}{2}}k^{\gamma}N^{-2\alpha-2}\le GN^{\gamma-\alpha-\frac{1}{2}},
\end{aligned}
$$

with $E,F,G>0$ constants. Therefore, the right hand side of both inequalities above is tending to zero with $N\to\infty$. □

Note that in case $\alpha=\beta>-1/2$ by [13, Theorem 4.1] it holds

$$
\sum_{n=1}^{\infty}b_n\frac{p_{2n}^{(\alpha,\alpha)}(x)}{p_{2n}^{(\alpha,\alpha)}(1)}=\sum_{n=1}^{\infty}b_n\frac{p_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{p_n^{(\alpha,-\frac{1}{2})}(1)}, \tag{3.11}
$$

and in case $\alpha < \beta$ and $\beta > -1/2$ the symmetric relation (3.5) implies

$$\sum_{n=1}^{\infty} b_n \frac{p_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(-1)} = \sum_{n=1}^{\infty} b_n \frac{p_n^{(\beta,\alpha)}(-x)}{p_n^{(\beta,\alpha)}(1)}. \tag{3.12}$$

Hence, due to Theorem 3.4 both series are uniformly but not absolutely convergent.

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References

[1] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, New York, 1995.

[2] G. Faber, *Über die interpolatorische Darstellung stetiger Funktionen*, Jahresber. Deutsch. Math.-Verein. **23** (1914) 192 – 210.

[3] E. Hewitt, K. Stromberg, *Real and Abstract Analysis*, Springer, New York, 1969.

[4] J. P. Kahane, *Séries de Fourier absolument convergentes*, Springer, New York, 1970

[5] Robert E. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 1998.

[6] J. Obermaier, *A continuous function space with a Faber basis*, J. Approx. Theory **125** (2003) 303 – 312.

[7] J. Obermaier, R. Szwarc, *Polynomial bases for continuous function spaces*, in: Trends and Applications in Constructive Approximation, M. G. de Bruin, D. H. Mache and J. Szabados (eds.), International Series of Numerical Mathematics Vol. 151, Birkhäuser Verlag, Basel, 2005, 195 – 205.

[8] J. Obermaier, R. Szwarc, *Nonnegative linearization for little q-Laguerre polynomials and Faber basis*, J. Comp. Appl. Math. **199** (2007) 89 – 94.

[9] J. Obermaier, R. Szwarc, *Orthogonal polynomials of discrete variable and boundedness of Dirichlet kernel*, Constr. Approx. **27** (2008) 1 – 13.

[10] J. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, J. Reine Angew. Math. **151** (1920) 79 – 111.

[11] I. Singer, *Bases in Banach Spaces I*, Springer, New York, 1970.

[12] P. K. Suetin, *Classical orthogonal polynomials*, Nauka, Moscow, 1976.

[13] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Volume **23**, Amer. Math. Soc. Providence, RI, 1975 (Fourth Edition).

[14] A. Zygmund, *Trigonometric series*, Vol. I, Third edition, With a foreword by Robert A. Fefferman, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.

Josef Obermaier

Institute of Computational Biology, Helmholtz Zentrum München, Ingolstädter Landstrasse 1, D-85764 Neuherberg, Germany

E-mail address: josef.obermaier@helmholtz-muenchen.de