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A CLASS OF MULTIVALENT ANALYTIC FUNCTIONS WITH FIXED ARGUMENT OF COEFFICIENTS INVOLVING WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, a class of analytic functions with fixed argument of its coefficients involving Wright's generalized hypergeometric function is defined with the help of subordination. The coefficient inequalities have been derived. Growth, distortion bounds and extreme points for functions belonging to the defined class have been investigated with consequent results.

1. Introduction and Preliminaries

Let A_m denote a class of functions $f(z)$ of the form:

$$f(z) = z^m + \sum_{k=1}^{\infty} a_{m+k} z^{m+k} \ (a_{m+k} \in \mathrm{C},\, m \in \mathrm{N} = \{1, 2, 3...\}), \tag{1.1}$$

which are analytic in an open unit disk $\mathrm{U} := \{z : z \in \mathrm{C} \text{ and } |z| < 1\}$ and its subclass is denoted by A_m^{θ} whose members are of the form:

$$f(z) = z^m + e^{i\theta} \sum_{k=1}^{\infty} |a_{m+k}| z^{m+k} \ (m \in \mathrm{N} = \{1, 2, 3...\}), \tag{1.2}$$

where θ is the fixed argument of $a_{m+k} \neq 0\,(k \geq 1)$. Note that $\mathrm{A}_1 \equiv \mathrm{A}$.

A function $f(z) \in \mathrm{A}_m$ is said to be in the class $\mathrm{S}_m^*(\alpha)$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ (0 \leq \alpha < 1,\ z \in \mathrm{U})$$

and functions therein are called starlike of order α.

The Wright's (psi) function ${}_p\Psi_q(z)$ is a generalized hypergeometric function [12] whose series representation is given by

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$$ {}_p\Psi_q\left((a_1,A_1),...,(a_p,A_p);(b_1,B_1),...,(b_q,B_q);z\right)=\sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{p}\Gamma(a_i+A_ik)z^n}{\prod\limits_{i=1}^{q}\Gamma(b_i+B_ik)k!}, \quad (1.3) $$

where $a_i\,(i=1,2,...,p), b_i\,(i=1,2,...,q)$ are positive real numbers and $A_i\,(i=1,2,...,p)$, $B_i(i=1,2,...,q)$ are positive integers such that $1+\sum\limits_{i=1}^{q}B_i-\sum\limits_{i=1}^{p}A_i\geq 0$. Taking $A_i=1(i=1,2,...,p)$, $B_i=1(i=1,2,...,q)$, we see that $\frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(a_i)}\;{}_p\Psi_q(z)$ reduces to a familiar generalized hypergeometric function:

$$ {}_pF_q(z)\equiv{}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k....(a_p)_kz^k}{(b_1)_k...(b_q)_kk!}, \quad (1.4) $$

where the Pochhammer symbol is defined by $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$ for non-negative integrs k.

Similar to the operator defined earlier by Dziok and Raina [4] (see also [1, 2], [6] and [5]), involving psi function ${}_{q+1}\Psi_q(z)$ for positive real numbers $a_i, b_i\,(i=1,2,...,q)$ and for positive integers A_i, $B_i(i=1,2,...,q)$ such that $\sum\limits_{i=1}^{q}(B_i-A_i)\geq 0$, an operator $I_m^q\left([a_1]\right)f\equiv I_m^q\left((a_i,A_i)_{1,q},(b_i,B_i)_{1,q}\right)f\,:\,\mathrm{A}_m\to\mathrm{A}_m$ is defined as:

$$ I_m^q\left([a_1]\right)f=z^m\prod_{i=1}^{q}\frac{\Gamma(b_i)}{\Gamma(a_i)}\,{}_{q+1}\Psi_q(z)*f, \quad (1.5) $$

where

$$ {}_{q+1}\Psi_q(z)\equiv{}_{q+1}\Psi_q\left((a_1,A_1),...,(a_q,A_q),(1,1);(b_1,B_1),...,(b_q,B_q);z\right). $$

Let $f(z)\in\mathrm{A}_m$ be of the form (1.1), then the series expansion of $I_m^q\left([a_1]\right)f$ is given by

$$ I_m^q\left([a_1]\right)f=z^m+\sum_{k=1}^{\infty}\theta_{m+k}a_{m+k}z^{m+k}, \quad (1.6) $$

where

$$ \theta_{m+k}=\prod_{i=1}^{q}\frac{\Gamma(a_i+A_ik)\Gamma(b_i)}{\Gamma(b_i+B_ik)\Gamma(a_i)},\ k\geq 1. \quad (1.7) $$

The series expansion of $I_m^q\left([a_1+1]\right)f\equiv I_m\left[(a_1+1,A_1),(a_i,A_i)_{2,q},(b_i,B_i)_{1,q}\right]f\,:\,\mathrm{A}_m\to\mathrm{A}_m$ is given by

$$ I_m^q\left([a_1+1]\right)f=z^m+\sum_{k=1}^{\infty}\theta^{+}_{m+k}a_{m+k}z^{m+k} \quad (1.8) $$

with

$$ \theta^{+}_{m+k}=\frac{(a_1+A_1k)}{a_1}\theta_{m+k}. $$

From (1.6) and (1.8), we get an identity:

$$A_1 z\left(I_m^q([a_1])f\right)' = a_1 I_m^q([a_1+1])f - (a_1 - mA_1)I_m^q([a_1])f \tag{1.9}$$

which shows that for $a_i = mA_i$, $I_m^q([a_1+1])f = \frac{z(I_m^q([a_1])f)'}{m}$. Note that $I_m^q f \equiv f$ if $A_i = B_i$, $a_i = b_i$ $(i = 1, 2, ..., q)$.

With the use of subordination, classes in which the linear operator $I_1^q([a_1])f$ is involved, are defined and studied in [2] and [6]. Involving $I_m^q([a_1])f$ in [9], a class of functions $f \in \mathrm{A}_m^\theta$ with the help of subordination is defined and studied. Motivated with the work of Raina [9], involving $I_m^q([a_1])f$ and $I_m^q([a_1+1])f$, we define a class for functions $f \in \mathrm{A}_m$ as follows:

Definition 1.1. *Let $A_m(p, [a_1], A, B, \lambda)$ denote a class of functions $f \in A_m$ satisfying*

$$(1-\lambda)\frac{I_m^q([a_1])f}{z^m} + \lambda\frac{I_m^q([a_1+1])f}{z^m} \prec \frac{1+Az}{1+Bz}, \tag{1.10}$$

where $0 \le \lambda \le 1$, and $-1 \le A < B \le 1$, $0 \le B$. Denote $A_m^\theta(p, [a_1], A, B, \lambda) \equiv A_m(p, [a_1], A, B, \lambda) \cap A_m^\theta$.

For positive real a and for positive integer A, we have [[11], 240, Eq. (1.26)]:

$$\Gamma(a+kA) = \Gamma(a)\left(\frac{a}{A}\right)_k\left(\frac{a+1}{A}\right)_k \cdots \left(\frac{a+A-1}{A}\right)_k (A)^{kA},\ k = 0, 1, 2, ...$$

when it is used together with the result [[3], p.57]:

$$\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)} = k^{a+b-c-d}\left[1 + O\left(\frac{1}{k}\right)\right],\ k = 1, 2, 3, ...$$

we obtain that for positive real numbers $a_i, b_i\,(i = 1, 2, ..., q)$ and positive integers A_i, $B_i(i = 1, 2, ..., q)$ such that $\sum_{i=1}^{q}(B_i - A_i) \ge 0$, the series $\sum_{k=1}^{\infty}\theta_{m+k}$, where θ_{m+k} is given by (1.7), converges absolutly if

$$\sum_{i=1}^{q}(b_i - a_i) > 1 + \frac{1}{2}\sum_{i=1}^{q}(A_i - B_i). \tag{1.11}$$

For details one may refer to [8] and [10].

The purpose of this paper is to investigate the coefficient inequalities, growth and distortion bounds with certain conditions on parameters and extreme points for functions belonging to the class $\mathrm{A}_m^\theta(p, [a_1], A, B, \lambda)$. Some consequent results are also mentioned.

2. Coefficient Inequalities

Theorem 2.1. *Let $f(z) \in A_m^\theta$ of the form (1.2) belong to $A_m^\theta(p, [a_1], A, B, \lambda)$, then*

$$\sum_{k=1}^{\infty}(a_1 + \lambda A_1 k)\theta_{m+k}|a_{m+k}| < \frac{a_1(B-A)}{\left[\sqrt{1 - B^2 sin^2\theta} - Bcos\theta\right]}, \tag{2.1}$$

where θ_{m+k} is given by (1.7) and $0 \le \lambda \le 1$, $-1 \le A < B \le 1$, $0 \le B$, θ is the argument of $a_{m+k} \ne 0\,(k \ge 1)$.

Proof. Let a function f belong to the class $\mathrm{A}_m^\theta(p,[a_1],A,B,\lambda)$, then from Definition 1.1, we have

$$(1-\lambda)\frac{I_m^q([a_1])f}{z^m}+\lambda\frac{I_m^q([a_1+1])f}{z^m}=\frac{1+Aw(z)}{1+Bw(z)},$$

where $w(0)=0$ and $|w(z)|<1$ for $z\in\mathrm{U}$. Hence we have

$$|w(z)|=\left|\frac{z^m-(1-\lambda)I_m^q([a_1])f-\lambda I_m^q([a_1+1])f}{B\left[(1-\lambda)I_m^q([a_1])f+\lambda I_m^q([a_1+1])f\right]-Az^m}\right|<1,\ z\in\mathrm{U}. \quad (2.2)$$

On using (1.6) and (1.8), inequality (2.2) yields

$$\left|\sum_{k=1}^{\infty}(a_1+\lambda A_1k)\theta_{m+k}\left|a_{m+k}\right|z^k\right|<\left|a_1(B-A)+Be^{i\theta}\sum_{k=1}^{\infty}(a_1+\lambda A_1k)\theta_{m+k}\left|a_{m+k}\right|z^k\right|. \quad (2.3)$$

For $0<z=r<1$, $\sum\limits_{k=1}^{\infty}(a_1+\lambda A_1k)\theta_{m+k}\left|a_{m+k}\right|r^k=:\eta$ is real, we get

$$|\eta\,|<\left|a_1(B-A)+Be^{i\theta}\eta\right|. \quad (2.4)$$

On solving (2.4) for η, we get

$$\eta<\frac{a_1\,(B-A)}{\left[\sqrt{1-B^2sin^2\theta}-Bcos\theta\right]},$$

which proves the inequality (2.1). □

The inequality (2.1) is sharp and the extremal function is given by

$$f_k(z)=z^m+e^{i\theta}\frac{a_1\,(B-A)}{(a_1+\lambda A_1k)\theta_{m+k}\left[\sqrt{1-B^2sin^2\theta}-Bcos\theta\right]}z^{m+k},\ k\geq1. \quad (2.5)$$

Corollary 2.2. *Let $f(z)\in A_m^\theta$ of the form (1.2) belong to $A_m^\theta(p,[a_1],A,B,\lambda)$ then for $\sum\limits_{i=1}^{q}(b_i-a_i)>1+\frac{1}{2}\sum\limits_{i=1}^{q}(A_i-B_i)$,*

$$\sum_{k=1}^{\infty}|a_{m+k}|<\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\phi\left[\sqrt{1-B^2sin^2\theta}-Bcos\theta\right]},$$

where $\phi=\min\limits_{k\geq1}\{\theta_{m+k}\}$, θ_{m+k} is given by (1.7).

Proof. Under the given hypothesis, convergence of the series $\sum\limits_{k=1}^{\infty}\theta_{m+k}$ implies that θ_{m+k}, given in (1.7) is bounded for $k\geq1$. Let $\phi:=\min\limits_{k\geq1}\{\theta_{m+k}\}$, then by Theorem 2.1 we get the result. □

Corollary 2.3. *Let $f(z)\in A_m^\theta$ of the form (1.2) belong to $A_m^\theta(p,[a_1],A,B,\lambda)$ then for $\sum\limits_{i=1}^{q}(b_i-a_i)>\frac{1}{2}\sum\limits_{i=1}^{q}(A_i-B_i)$,*

$$\sum_{k=1}^{\infty}(m+k)|a_{m+k}|<\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\varphi\left[\sqrt{1-B^2sin^2\theta}-Bcos\theta\right]},$$

where $\varphi=\min\limits_{k\geq1}\left\{\frac{\theta_{m+k}}{m+k}\right\}$, θ_{m+k} is given by (1.7).

Proof. Under the given hypothesis, convergence of the series $\sum\limits_{k=1}^{\infty}\frac{\theta_{m+k}}{m+k}$ implies that $\frac{\theta_{m+k}}{m+k}$ is bounded for $k \geq 1$, where θ_{m+k} is given by (1.7) and hence let $\varphi := \min\limits_{k\geq 1}\left\{\frac{\theta_{m+k}}{m+k}\right\}$, then by Theorem 2.1 we get the result. □

Taking $\theta = (2n-1)\pi$, $n = 1, 2, 3, ..$ in Corollary 2.2 we get following result.

Corollary 2.4. *Let $f(z) \in A_m^{(2n-1)\pi}$ of the form* (1.2) *belong to $A_m^{(2n-1)\pi}(p, [a_1], A, B, \lambda)$, then for $\sum\limits_{i=1}^{q}(b_i - a_i) > 1 + \frac{1}{2}\sum\limits_{i=1}^{q}(A_i - B_i)$,*

$$\sum_{k=1}^{\infty}|a_{m+k}| < \frac{a_1\,(B-A)}{(a_1 + \lambda A_1)\phi(1+B)},$$

where $\phi = \min\limits_{k\geq 1}\{\theta_{m+k}\}$, θ_{m+k} is given by (1.7).

Remark. *Raina* [9] *studied the class of functions $f(z) \in A_m^{\theta}$ and obtained results assuming θ_{m+k} and $\frac{\theta_{m+k}}{m+k}$ to be increasing functions.*

Theorem 2.5. *A function $f(z) \in A_m^{(2n-1)\pi}$ of the form* (1.2) *belong to $A_m^{(2n-1)\pi}(p, [a_1], A, B, \lambda)$ if and only if*

$$\sum_{k=1}^{\infty}(a_1 + \lambda A_1 k)\theta_{m+k}|a_{m+k}| < \frac{a_1\,(B-A)}{(1+B)}, \tag{2.6}$$

where θ_{m+k} is given by (1.7) *and $0 \leq \lambda \leq 1$, $-1 \leq A < B \leq 1$, $0 \leq B$.*

Proof. We need to show only sufficient condition for $f \in \mathrm{A}_m^{(2n-1)\pi}(p, [a_1], A, B, \lambda)$. Consider

$$\left|\sum_{k=1}^{\infty}(a_1 + \lambda A_1 k)\theta_{m+k}\,|a_{m+k}|\,z^k\right| - \left|a_1(B-A) - B\sum_{k=1}^{\infty}(a_1 + \lambda A_1 k)\theta_{m+k}\,|a_{m+k}|\,z^k\right|$$

$$\leq \sum_{k=1}^{\infty}(a_1 + \lambda A_1 k)\theta_{m+k}\,|a_{m+k}| - \left[a_1\,(B-A) - B\sum_{k=1}^{\infty}(a_1 + \lambda A_1 k)\theta_{m+k}\,|a_{m+k}|\right]$$

$$\leq \sum_{k=1}^{\infty}(1+B)(a_1+\lambda A_1 k)\theta_{m+k}\,|a_{m+k}| - a_1\,(B-A) < 0,$$

if (2.6) holds, which proves (2.3) for $\theta = (2n-1)\pi$ and hence the result. □

The result is sharp for the function:

$$f_k(z) = z^m - \frac{a_1\,(B-A)}{(a_1 + \lambda A_1 k)\theta_{m+k}(1+B)}z^{m+k},\ k \geq 1. \tag{2.7}$$

Corollary 2.6. *Let a function $f(z) \in A_m^{(2n-1)\pi}$ of the form (1.2) belong to $A_m^{(2n-1)\pi}(p,[a_1],A,B,\lambda)$ then*

$$|a_{m+k}| < \frac{a_1\,(B-A)}{(a_1+\lambda A_1 k)\theta_{m+k}(1+B)},\ k \geq 1. \tag{2.8}$$

3. Growth and Distortion Bounds

In this section we find growth and distortion bounds for functions belonging to the class $\mathrm{A}_m^\theta(p,[a_1],A,B,\lambda)$ with the use of Corollaries 2.2 and 2.3.

Theorem 3.1. *Let $f(z) \in A_m^\theta$ of the form (1.2) belong to $A_m^\theta(p,[a_1],A,B,\lambda)$, then for $0<|z|=r<1$ and for $\sum\limits_{i=1}^{q}(b_i-a_i) > 1+\frac{1}{2}\sum\limits_{i=1}^{q}(A_i-B_i)$*

$$r^m - r^{m+1}\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\phi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]} \leq |f(z)|$$

$$\leq r^m + r^{m+1}\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\phi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]}, \tag{3.1}$$

where $\phi = \min\limits_{k\geq 1}\{\theta_{m+k}\}$, θ_{m+k} is given by (1.7).

Proof. From (1.2), we have

$$|f(z)| \leq r^m + \sum_{k=1}^{\infty}|a_{m+k}|\,r^{m+k},\ 0<|z|=r<1.$$

Using Corollary 2.2, we get

$$|f(z)| \leq r^m + r^{m+1}\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\phi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]}\ ,0<|z|=r<1.$$

Similary we get

$$|f(z)| \geq r^m - r^{m+1}\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\phi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]},$$

which completes the proof.

□

Theorem 3.2. *Let $f(z) \in A_m^\theta$ of the form (1.2) belong to $A_m^\theta(p,[a_1],A,B,\lambda)$, then for $0<|z|=r<1$ and for $\sum\limits_{i=1}^{q}(b_i-a_i) > \frac{1}{2}\sum\limits_{i=1}^{q}(A_i-B_i)$,*

$$mr^{m-1} - r^m\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\varphi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]} \leq |f'(z)|$$

$$\leq mr^{m-1} + r^m\frac{a_1\,(B-A)}{(a_1+\lambda A_1)\varphi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]}, \tag{3.2}$$

where $\varphi = \min\limits_{k\geq 1}\left\{\frac{\theta_{m+k}}{m+k}\right\}$, θ_{m+k} is given by (1.7).

Proof. From (1.2), we have

$$|f'(z)| \le mr^{m-1} + \sum_{k=1}^{\infty}(m+k)\,|a_{m+k}|\,r^{m+k-1},\ 0<|z|=r<1.$$

Using Corollary 2.3, we get

$$|f'(z)| \le mr^{m-1} + r^m \frac{a_1\,(B-A)}{(a_1+\lambda A_1)\varphi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]}.$$

Similary we get

$$|f'(z)| \ge mr^{m-1} - r^m \frac{a_1\,(B-A)}{(a_1+\lambda A_1)\varphi\left[\sqrt{1-B^2 sin^2\theta}-Bcos\theta\right]},$$

which completes the proof. □

4. Extreme Points

In this section we find extreme points for the class $\mathrm{A}_m^{(2n-1)\pi}(p,[a_1],A,B,\lambda)$.

Theorem 4.1. *Let* $f(z) \in A_m^{(2n-1)\pi}$ *and*

$$f_0(z)=z^m,\ f_k(z)=z^m-\frac{a_1(B-A)}{(a_1+\lambda A_1 k)\theta_{m+k}(1+B)}z^{m+k},\ k\ge 1. \tag{4.1}$$

Then $f(z)\in A_m^{(2n-1)\pi}(p,[a_1],A,B,\lambda)$ *if only if it can be expressed in the form*

$$f(z)=\sum_{k=0}^{\infty}\lambda_k f_k(z), \tag{4.2}$$

where $\lambda_k \ge 0$ *and* $\sum\limits_{k=0}^{\infty}\lambda_k = 1$ *and* $f_k's$ *for* $k\ge 0$ *are the extreme points for* $A_m^{(2n-1)\pi}(p,[a_1],A,B,\lambda)$ *class.*

Proof. Let

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}\lambda_k f_k(z) \\ &= (1-\sum_{k=1}^{\infty}\lambda_k)z^m+\sum_{k=1}^{\infty}\lambda_k\left(z^m-\frac{a_1(B-A)}{(a_1+\lambda A_1 k)\theta_{m+k}(1+B)}z^{m+k}\right) \\ &= z^m-\sum_{k=1}^{\infty}\lambda_k\frac{a_1(B-A)}{(a_1+\lambda A_1 k)\theta_{m+k}(1+B)}z^{m+k}, \end{aligned}$$

which proves that $f(z)\in \mathrm{A}_m^{(2n-1)\pi}(p,[a_1],A,B,\lambda)$. Since by Theorem 2.5

$$\sum_{k=1}^{\infty}\frac{(1+B)(a_1+\lambda A_1 k)\theta_{m+k}}{a_1\,(B-A)}\left[\frac{a_1(B-A)\lambda_k}{(a_1+\lambda A_1 k)\theta_{m+k}(1+B)}\right]=\sum_{k=1}^{\infty}\lambda_k<1-\lambda_0\le 1.$$

Conversely, suppose $f(z)\in \mathrm{A}_m^{(2n-1)\pi}(p,[a_1],A,B,\lambda)$ then using Corollary 2.6, we set $\lambda_k=\frac{(a_1+\lambda A_1 k)\theta_{m+k}(1+B)}{a_1(B-A)}a_{m+k}$ for $k\ge 1$ and $\lambda_0 = 1-\sum\limits_{k=1}^{\infty}\lambda_k$.

Thus

$$\begin{aligned} f(z) &= z^m - \sum_{k=1}^{\infty} |a_{m+k}| z^{m+k} \\ &= f_0(z) - \sum_{k=1}^{\infty} \frac{a_1(B-A)\lambda_k}{(a_1+\lambda A_1 k)\theta_{m+k}(1+B)} z^{m+k} \\ &= f_0(z) - \sum_{k=1}^{\infty} \left[z^m - f_k(z)\right] \lambda_k \\ &= \sum_{k=0}^{\infty} \lambda_k f_k(z). \end{aligned}$$

This completes the proof. □

5. Some Consequences of Main Results

Taking $a_1 = mA_1$ in Theorm 2.1, we get following result.

Corollary 5.1. *A function $f(z) \in A_m^{(2n-1)\pi}$ of the form (1.2) satisfies*

$$(1-\lambda)\frac{I_m^q([a_1])f}{z^m} + \lambda \frac{z\left(I_m^q([a_1])f\right)'}{z^m} \prec \frac{1+Az}{1+Bz}$$

if and only if

$$\sum_{k=1}^{\infty} (m+\lambda k)\theta_{m+k}|a_{m+k}| < \frac{m(B-A)}{(1+B)},$$

where θ_{m+k} is defined in (1.7) and $0 \le \lambda \le 1$, $-1 \le A < B \le 1$, $0 \le B$.

In Corolary 5.1, on replacing θ_{m+k} by b_{m+k}, following result can be obtained.

Corollary 5.2. *Let $g(z) \in A_m$ of the form:*

$$g(z) = z^m + \sum_{k=1}^{\infty} b_{m+k} z^{m+k} \ (b_{m+k} \ge 0,\ m \in N = \{1, 2, 3...\})$$

and $0 \le \lambda \le 1$, $-1 \le A < B \le 1$, $0 \le B$. A function $f(z) \in A_m^{(2n-1)\pi}$ of the form (1.2) satisfies

$$(1-\lambda)\frac{(f*g)}{z^m} + \lambda \frac{z(f*g)'}{z^m} \prec \frac{1+Az}{1+Bz},\ z \in U$$

if and only if

$$\sum_{k=1}^{\infty} (m+\lambda k) b_{m+k} |a_{m+k}| < \frac{m(B-A)}{(1+B)}.$$

Corollary 5.3. *Let $0 < \lambda \le 1, -1 < A < 0$, if a function $f(z) \in A_m$ satisfies*

$$(1-\lambda)\frac{I_m^q([a_1])f}{z^m} + \lambda \frac{I_m^q([a_1+1])f}{z^m} \prec 1 + Az,$$

then $I_m^q([a_1])f \in S_m^*(\alpha)$, $\alpha := \left(m - \frac{2a_1|A|}{\lambda A_1(1-|A|)}\right)$.

Proof. Let $p(z) := \frac{I_m^q([a_1])f}{z^m}$ then with the use of identity (1.9) we have

$$p(z) + \frac{\lambda A_1}{a_1} z p'(z) \prec 1 + Az.$$

By a well known Lemma of Hallenbeck and Ruscheweyh [7] we get that $p(z) \prec 1+Az$ and hence:

$$\left| \frac{I_m^q\left([a_1]\right) f}{z^m} - 1 \right| < |A| \text{ and } \left| \frac{I_m^q\left([a_1]\right) f}{z^m} \right| > 1 - |A|,$$

which with the use of hypothesis and the identity(1.9), derives:

$$\left| \frac{z\left(I_m^q\left([a_1]\right) f\right)'}{z^m} - m \frac{I_m^q\left([a_1]\right) f}{z^m} \right| < \frac{2a_1 |A|}{\lambda A_1 \left(1 - |A|\right)} \left| \frac{I_m^q\left([a_1]\right) f}{z^m} \right|.$$

That evidently yields: $\left| \frac{z(I_m^q([a_1])f)'}{I_m^q([a_1])f} - m \right| < \frac{2a_1|A|}{\lambda A_1(1-|A|)}$.

Therefore $\mathrm{Re}\left\{ \frac{z(I_m^q([a_1])f)'}{I_m^q([a_1])f} \right\} > \alpha$, where $\alpha = \left(m - \frac{2a_1|A|}{\lambda A_1(1-|A|)} \right)$ which proves the result. □

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