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SOME PROPERTIES OF CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS INVOLVING THE DZIOK-SRIVASTAVA OPERATOR

ZHI-GANG WANG, HAI-PING SHI

Abstract. The main purpose of the present paper is to derive such results as inclusion relationships and convolution properties for certain new subclasses of multivalent analytic functions involving the Dziok-Srivastava operator. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

1. Introduction and Preliminaries

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For simplicity, we write

$$\mathcal{A}_1 =: \mathcal{A}.$$

Let $f,\ g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} =: (g * f)(z).$$

For parameters

$$\alpha_j \in \mathbb{C} \ \ (j = 1, \ldots, l) \quad \text{and} \quad \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \ (\mathbb{Z}_0^- := \{0, -1, -2, \ldots\};\ j = 1, \ldots, m),$$

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the generalized hypergeometric function

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) $$

is defined by the following infinite series:

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} $$

$$ (l \leqq m+1;\ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}), $$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$ (\lambda)_n := \begin{cases} 1 & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases} $$

Recently, Dziok and Srivastava [8] introduced a linear operator

$$ H_p(\alpha_1, \ldots, \alpha_l; \beta_, \ldots, \beta_m) :\ \mathcal{A}_p \longrightarrow \mathcal{A}_p $$

defined by the Hadamard product

$$ H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := [z^p\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)] * f(z) \tag{1.2} $$

$$ (l \leqq m+1;\ l, m \in \mathbb{N}_0;\ z \in \mathbb{U}). $$

If $f \in \mathcal{A}_p$ is given by (1.1), then we have

$$ H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n}\ a_{n+p} \frac{z^{n+p}}{n!} \qquad (n \in \mathbb{N}). $$

In order to make the notation simple, we write

$$ H_p^{l,m}(\alpha_1) := H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \qquad (l \leqq m+1;\ l, m \in \mathbb{N}_0). $$

It is easily verified from the definition (1.2) that

$$ z\left(H_p^{l,m}(\alpha_1) f\right)'(z) = \alpha_1 H_p^{l,m}(\alpha_1 + 1) f(z) - (\alpha_1 - p) H_p^{l,m}(\alpha_1) f(z). \tag{1.3} $$

Let $\mathcal{P}$ denote the class of functions of the form

$$ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, $$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$ \Re(p(z)) > 0 \qquad (z \in \mathbb{U}). $$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$ f(z) \prec g(z) \qquad (z \in \mathbb{U}), $$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$ \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}) $$

such that

$$ f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}). $$

Indeed, it is known that

$$ f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}). $$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Throughout this paper, we assume that

$$p, k \in \mathbb{N}, \quad l, m \in \mathbb{N}_0, \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right),$$

$$f_{p,k}^{l,m}(\alpha_1; z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{-jp} \left(H_p^{l,m}(\alpha_1) f\right)(\varepsilon_k^j z) = z^p + \cdots \qquad (f \in \mathcal{A}_p), \tag{1.4}$$

$$g_p^{l,m}(\alpha_1; z) = \frac{1}{2}\left[H_p^{l,m}(\alpha_1) f(z) + \overline{H_p^{l,m}(\alpha_1) f(\overline{z})}\right] = z^p + \cdots \qquad (f \in \mathcal{A}_p), \tag{1.5}$$

and

$$h_p^{l,m}(\alpha_1; z) = \frac{1}{2}\left[H_p^{l,m}(\alpha_1) f(z) - \overline{H_p^{l,m}(\alpha_1) f(-\overline{z})}\right] = z^p + \cdots \qquad (f \in \mathcal{A}_p). \tag{1.6}$$

Clearly, for $k = 1$, we have

$$f_{p,1}^{l,m}(\alpha_1; z) = H_p^{l,m}(\alpha_1) f(z).$$

In recent years, several authors obtained many interesting results involving the Dziok-Srivastava operator $H_p^{l,m}(\alpha_1)$ (see, for details, [1, 2, 3, 4, 6, 7, 8, 9, 11, 13, 14, 15, 17, 18, 19, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 34]). In the present paper, by making use of the Dziok-Srivastava operator $H_p^{l,m}(\alpha_1)$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclasses of the class $\mathcal{A}_p$ of p-valent analytic functions.

Definition 1.1. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$ if it satisfies the subordination condition

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1) f\right)'(z) + \alpha\left(H_p^{l,m}(\alpha_1+1) f\right)'(z)\right]}{p\left[(1-\alpha) f_{p,k}^{l,m}(\alpha_1; z) + \alpha f_{p,k}^{l,m}(\alpha_1+1; z)\right]} \prec \phi(z) \qquad (z \in \mathbb{U}), \tag{1.7}$$

where $\phi \in \mathcal{P}$, $f_{p,k}^{l,m}(\alpha_1; z)$ is defined by (1.4) and

$$f_{p,k}^{l,m}(\alpha_1 + 1; z) \neq 0 \qquad (z \in \mathbb{U}).$$

For simplicity, we write

$$\mathcal{F}_{p,k}^{l,m}(0; \alpha_1; \phi) =: \mathcal{F}_{p,k}^{l,m}(\alpha_1; \phi).$$

Remark 1.1. If we set

$$p = 1, \ l = 2, \ m = 1, \quad \text{and} \quad \alpha_1 = \alpha_2 = \beta_1 = 1$$

in the class $\mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$, then it reduces to the known class $\mathcal{S}^{(k)}(\alpha; \phi)$ of functions α-starlike with respect to k-symmetric points, which was studied earlier by Parvatham and Radha [21]. If we set

$$p = 1, \ l = 2, \ m = 1, \ \alpha_1 = \alpha_2 = \beta_1 = 1, \quad \text{and} \quad \alpha = 0$$

in the class $\mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$, then it reduces to the class $\mathcal{S}^{(k)}(\phi)$ of functions starlike with respect to k-symmetric points, which was considered by Wang *et al.* [33]. Furthermore, we note that the class $\mathcal{F}_{p,k}^{l,m}(\alpha_1; \phi)$ was introduced and investigated recently by Wang *et al.* [34] (see also Huang and Liu [12] and Xu and Yang [35]).

Definition 1.2. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)$ if it satisfies the subordination condition

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)g_p^{l,m}(\alpha_1;z)+\alpha g_p^{l,m}(\alpha_1+1;z)\right]} \prec \phi(z) \qquad (z\in\mathbb{U}),$$

where $\phi \in \mathcal{P}$, $g_p^{l,m}(\alpha_1;z)$ is defined by (1.5) and

$$g_p^{l,m}(\alpha_1+1;z) \neq 0 \qquad (z\in\mathbb{U}).$$

For simplicity, we write

$$\mathcal{G}_p^{l,m}(0;\alpha_1;\phi) =: \mathcal{G}_p^{l,m}(\alpha_1;\phi).$$

Definition 1.3. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)$ if it satisfies the subordination condition

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)h_p^{l,m}(\alpha_1;z)+\alpha h_p^{l,m}(\alpha_1+1;z)\right]} \prec \phi(z) \qquad (z\in\mathbb{U}),$$

where $\phi \in \mathcal{P}$, $h_p^{l,m}(\alpha_1;z)$ is defined by (1.6) and

$$h_p^{l,m}(\alpha_1+1;z) \neq 0 \qquad (z\in\mathbb{U}).$$

For simplicity, we write

$$\mathcal{H}_p^{l,m}(0;\alpha_1;\phi) =: \mathcal{H}_p^{l,m}(\alpha_1;\phi).$$

Remark 1.2. In 1996, Chen *et al.* [5] introduced and investigated a subclass $\mathcal{S}_{sc}^*(\alpha)$ of $\mathcal{A}$ consisting of functions which are α-starlike with respect to symmetric conjugate points and satisfy the inequality

$$\Re\left(\frac{z\left[(1-\alpha)f'(z)+\alpha(zf'(z))'\right]}{(1-\alpha)T_{sc}f(z)+\alpha z(T_{sc}f(z))'}\right) > 0 \qquad (\, z\in\mathbb{U}),$$

where

$$T_{sc}f(z) = \frac{1}{2}\left[f(z)-\overline{f(-\overline{z})}\right].$$

It is easy to see that, if we set

$$p=1,\ l=2,\ m=1,\ \alpha_1=\alpha_2=\beta_1=1, \ \text{ and } \ \phi(z)=\frac{1+z}{1-z}$$

in the class $\mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)$, then it reduces to the class $\mathcal{S}_{sc}^*(\alpha)$.

Definition 1.4. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathfrak{F}_{p,k}^{l,m}(\alpha;\alpha_1;\phi)$ if it satisfies the subordination condition

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)\mathfrak{f}_{p,k}^{l,m}(\alpha_1;z)+\alpha\mathfrak{f}_{p,k}^{l,m}(\alpha_1+1;z)\right]} \prec \phi(z) \qquad (z\in\mathbb{U}),$$

where $\phi \in \mathcal{P}$, $\mathfrak{f}_{p,k}^{l,m}(\alpha_1;z)$ is defined as in (1.4) and

$$\mathfrak{f}_{p,k}^{l,m}(\alpha_1+1;z) \neq 0 \qquad (z\in\mathbb{U}).$$

For simplicity, we write

$$\mathfrak{F}_{p,k}^{l,m}(0;\alpha_1;\phi) =: \mathfrak{F}_{p,k}^{l,m}(\alpha_1;\phi).$$

Remark 1.3. If we set

$$p = 1,\ l = 2,\ m = 1, \quad \text{and} \quad \alpha_1 = \alpha_2 = \beta_1 = 1$$

in the class $\mathfrak{F}_{p,k}^{l,m}(\alpha;\alpha_1;\phi)$, then it reduces to the known class $\mathcal{C}^{(k)}(\alpha,\phi)$ of functions α-close-to-convex with respect to k-symmetric points, which was also studied earlier by Parvatham and Radha [21]. Furthermore, we also note that the class $\mathfrak{F}_{p,k}^{l,m}(\alpha_1;\phi)$ was introduced and investigated recently by Wang *et al.* [34].

Definition 1.5. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathfrak{G}_p^{l,m}(\alpha;\alpha_1;\phi)$ if it satisfies the subordination condition

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z) + \alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)\mathfrak{g}_p^{l,m}(\alpha_1;z) + \alpha\mathfrak{g}_p^{l,m}(\alpha_1+1;z)\right]} \prec \phi(z) \qquad (z \in \mathbb{U}),$$

where $\phi \in \mathcal{P}$, $\mathfrak{g}_p^{l,m}(\alpha_1;z)$ is defined as in (1.5) and

$$\mathfrak{g}_p^{l,m}(\alpha_1+1;z) \neq 0 \qquad (z \in \mathbb{U}).$$

For simplicity, we write

$$\mathfrak{G}_p^{l,m}(0;\alpha_1;\phi) =: \mathfrak{G}_p^{l,m}(\alpha_1;\phi).$$

Definition 1.6. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathfrak{H}_p^{l,m}(\alpha;\alpha_1;\phi)$ if it satisfies the subordination condition

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z) + \alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)\mathfrak{h}_p^{l,m}(\alpha_1;z) + \alpha\mathfrak{h}_p^{l,m}(\alpha_1+1;z)\right]} \prec \phi(z) \qquad (z \in \mathbb{U}),$$

where $\phi \in \mathcal{P}$, $\mathfrak{h}_p^{l,m}(\alpha_1;z)$ is defined as in (1.6) and

$$\mathfrak{h}_p^{l,m}(\alpha_1+1;z) \neq 0 \qquad (z \in \mathbb{U}).$$

For simplicity, we write

$$\mathfrak{H}_p^{l,m}(0;\alpha_1;\phi) =: \mathfrak{H}_p^{l,m}(\alpha_1;\phi).$$

In order to establish our main results, we need the following lemmas.

Lemma 1.1. (See [10, 16]) *Let $\beta, \gamma \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \text{and} \quad \Re(\beta\phi(z) + \gamma) > 0 \qquad (z \in \mathbb{U}).$$

If $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the subordination

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{p}(z) + \gamma} \prec \phi(z) \qquad (z \in \mathbb{U})$$

implies that

$$\mathfrak{p}(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Lemma 1.2. (See [20]) *Let $\beta, \gamma \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \textit{and} \quad \Re(\beta\phi(z) + \gamma) > 0 \qquad (z \in \mathbb{U}).$$

Also let

$$\mathfrak{q}(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

If $\mathfrak{p} \in \mathcal{P}$ and satisfies the subordination

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{q}(z) + \gamma} \prec \phi(z) \qquad (z \in \mathbb{U}),$$

then

$$\mathfrak{p}(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Lemma 1.3. *Let $f \in \mathcal{F}_{p,k}^{l,m}(\alpha;\alpha_1;\phi)$. Then*

$$\frac{z\left[(1-\alpha)\left(f_{p,k}^{l,m}(\alpha_1)f\right)'(z) + \alpha\left(f_{p,k}^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z) + \alpha f_{p,k}^{l,m}(\alpha_1+1;z)\right]} \prec \phi(z) \qquad (z \in \mathbb{U}). \quad (1.8)$$

Furthermore, if $\phi \in \mathcal{P}$ with

$$\Re\left(p\phi(z) + \frac{\alpha_1}{\alpha} - p\right) > 0 \qquad (\alpha > 0;\ z \in \mathbb{U}).$$

Then

$$\frac{z\left(f_{p,k}^{l,m}(\alpha_1;z)\right)'}{pf_{p,k}^{l,m}(\alpha_1;z)} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Proof. Making use of (1.4), we have

$$\begin{aligned} f_{p,k}^{l,m}(\alpha_1;\varepsilon_k^j z) &= \frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-np}\left(H_p^{l,m}(\alpha_1)f\right)(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{jp}\cdot\frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-(n+j)p}\left(H_p^{l,m}(\alpha_1)f\right)(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{jp}f_{p,k}^{l,m}(\alpha_1;z) \qquad (j \in \{0,1,\ldots,k-1\}), \end{aligned} \quad (1.9)$$

and

$$\left(f_{p,k}^{l,m}(\alpha_1;z)\right)' = \frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-j(p-1)}\left(H_p^{l,m}(\alpha_1)f\right)(\varepsilon_k^j z). \quad (1.10)$$

Replacing α_1 by $\alpha_1 + 1$ in (1.9) and (1.10), respectively, we get

$$f_{p,k}^{l,m}(\alpha_1+1;\varepsilon_k^j z) = \varepsilon_k^{jp}f_{p,k}^{l,m}(\alpha_1+1;z) \qquad (j \in \{0,1,\ldots,k-1\}), \quad (1.11)$$

and

$$\left(f_{p,k}^{l,m}(\alpha_1+1;z)\right)' = \frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-j(p-1)}\left(H_p^{l,m}(\alpha_1+1)f\right)(\varepsilon_k^j z). \quad (1.12)$$

From (1.9) to (1.12), we get

$$\begin{aligned}
&\frac{z\left[(1-\alpha)\left(f_{p,k}^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(f_{p,k}^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;z)\right]}\\
&\quad=\frac{1}{k}\sum_{j=0}^{k-1}\frac{\varepsilon_k^{-j(p-1)}z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(\varepsilon_k^j z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(\varepsilon_k^j z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;z)\right]}\\
&\quad=\frac{1}{k}\sum_{j=0}^{k-1}\frac{\varepsilon_k^{j}z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(\varepsilon_k^j z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(\varepsilon_k^j z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;\varepsilon_k^j z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;\varepsilon_k^j z)\right]}.
\end{aligned} \tag{1.13}$$

Moreover, since $f\in\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$, it follows that

$$\frac{\varepsilon_k^{j}z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(\varepsilon_k^j z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(\varepsilon_k^j z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;\varepsilon_k^j z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;\varepsilon_k^j z)\right]}\prec\phi(z) \tag{1.14}$$

$$(z\in\mathbb{U};\ j\in\{0,1,\dots,k-1\}).$$

By noting that $\phi(z)$ is convex and univalent in $\mathbb{U}$, from (1.13) and (1.14), we conclude that the assertion (1.8) of Lemma 1.3 holds true.

Next, making use of the relationships (1.3) and (1.4), we know that

$$\begin{aligned}
z\left(f_{p,k}^{l,m}(\alpha_1;z)\right)'+(\alpha_1-p)f_{p,k}^{l,m}(\alpha_1;z)&=\frac{\alpha_1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-jp}\left(H_p^{l,m}(\alpha_1+1)f\right)(\varepsilon_k^j z)\\
&=\alpha_1 f_{p,k}^{l,m}(\alpha_1+1;z).
\end{aligned} \tag{1.15}$$

Let $f\in\mathcal{F}_{p,k}^{l,m}(\alpha;\alpha_1;\phi)$ and suppose that

$$\psi(z)=\frac{z\left(f_{p,k}^{l,m}(\alpha_1;z)\right)'}{pf_{p,k}^{l,m}(\alpha_1;z)}\qquad(z\in\mathbb{U}). \tag{1.16}$$

Then ψ is analytic in $\mathbb{U}$ and $\psi(0)=1$. It follows from (1.15) and (1.16) that

$$\alpha_1-p+p\psi(z)=\alpha_1\frac{f_{p,k}^{l,m}(\alpha_1+1;z)}{f_{p,k}^{l,m}(\alpha_1;z)}\qquad(z\in\mathbb{U}). \tag{1.17}$$

From (1.16) and (1.17), we have

$$z\left(f_{p,k}^{l,m}(\alpha_1+1;z)\right)'=\frac{p}{\alpha_1}[z\psi'(z)+(\alpha_1-p+p\psi(z))\psi(z)]f_{p,k}^{l,m}(\alpha_1;z). \tag{1.18}$$

It now follows from (1.8), (1.16), (1.17) and (1.18) that

$$\begin{aligned}
&\frac{z\left[(1-\alpha)\left(f_{p,k}^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(f_{p,k}^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;z)\right]}\\
&\quad=\frac{p(1-\alpha)\psi(z)f_{p,k}^{l,m}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p[z\psi'(z)+(\alpha_1-p+p\psi(z))\psi(z)]f_{p,k}^{l,m}(\alpha_1;z)}{p(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p(\alpha_1-p+p\psi(z))f_{p,k}^{l,m}(\alpha_1;z)}\\
&\quad=\frac{(1-\alpha)\psi(z)+\frac{\alpha}{\alpha_1}\left[z\psi'(z)+(\alpha_1-p+p\psi(z))\psi(z)\right]}{(1-\alpha)+\frac{\alpha}{\alpha_1}(\alpha_1-p+p\psi(z))}\\
&\quad=\frac{\frac{\alpha}{\alpha_1}z\psi'(z)+\left[(1-\alpha)+\frac{\alpha}{\alpha_1}(\alpha_1-p+p\psi(z))\right]\psi(z)}{(1-\alpha)+\frac{\alpha}{\alpha_1}(\alpha_1-p+p\psi(z))}\\
&\quad=\psi(z)+\frac{z\psi'(z)}{\frac{\alpha_1}{\alpha}-p+p\psi(z)}\prec\phi(z).
\end{aligned} \tag{1.19}$$

Since

$$\Re\left(p\phi(z)+\frac{\alpha_1}{\alpha}-p\right)>0 \qquad (\alpha>0;\ z\in\mathbb{U}).$$

Thus, by (1.19) and Lemma 1.1, we know that

$$\psi(z)=\frac{z\left(f_{p,k}^{l,m}(\alpha_1;z)\right)'}{pf_{p,k}^{l,m}(\alpha_1;z)}\prec\phi(z).$$

This completes the proof of Lemma 1.3. □

By similarly applying the method of proof of Lemma 1.3 for the classes $\mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)$ and $\mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)$, we get the following results.

Lemma 1.4. *Let* $f\in\mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)$. *Then*

$$\frac{z\left[(1-\alpha)\left(g_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(g_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)g_p^{l,m}(\alpha_1;z)+\alpha g_p^{l,m}(\alpha_1+1;z)\right]}\prec\phi(z) \qquad (z\in\mathbb{U}).$$

Furthermore, if $\phi\in\mathcal{P}$ *with*

$$\Re\left(p\phi(z)+\frac{\alpha_1}{\alpha}-p\right)>0 \qquad (\alpha>0;\ z\in\mathbb{U}).$$

Then

$$\frac{z\left(g_p^{l,m}(\alpha_1;z)\right)'}{pg_p^{l,m}(\alpha_1;z)}\prec\phi(z) \qquad (z\in\mathbb{U}).$$

Lemma 1.5. *Let* $f\in\mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)$. *Then*

$$\frac{z\left[(1-\alpha)\left(h_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(h_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)h_p^{l,m}(\alpha_1;z)+\alpha h_p^{l,m}(\alpha_1+1;z)\right]}\prec\phi(z) \qquad (z\in\mathbb{U}).$$

Furthermore, if $\phi \in \mathcal{P}$ *with*

$$\Re\left(p\phi(z) + \frac{\alpha_1}{\alpha} - p\right) > 0 \qquad (\alpha > 0;\ z \in \mathbb{U}).$$

Then

$$\frac{z\left(h_p^{l,m}(\alpha_1; z)\right)'}{p h_p^{l,m}(\alpha_1; z)} \prec \phi(z) \qquad (z \in \mathbb{U}).$$

In the present paper, we aim at proving such results as inclusion relationships and convolution properties for the function classes $\mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$, $\mathcal{G}_p^{l,m}(\alpha; \alpha_1; \phi)$, $\mathcal{H}_p^{l,m}(\alpha; \alpha_1; \phi)$, $\mathfrak{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$, $\mathfrak{G}_p^{l,m}(\alpha; \alpha_1; \phi)$, and $\mathfrak{H}_p^{l,m}(\alpha; \alpha_1; \phi)$. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

2. A Set of Inclusion Relationships

At first, we provide some inclusion relationships for the classes $\mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$, $\mathcal{G}_p^{l,m}(\alpha; \alpha_1; \phi)$, $\mathcal{H}_p^{l,m}(\alpha; \alpha_1; \phi)$, $\mathfrak{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$, $\mathfrak{G}_p^{l,m}(\alpha; \alpha_1; \phi)$, and $\mathfrak{H}_p^{l,m}(\alpha; \alpha_1; \phi)$, which were defined in the preceding section.

Theorem 2.1. *Let* $\phi \in \mathcal{P}$ *with*

$$\Re\left(p\phi(z) + \frac{\alpha_1}{\alpha} - p\right) > 0 \qquad (\alpha > 0;\ z \in \mathbb{U}).$$

Then

$$\mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi) \subset \mathcal{F}_{p,k}^{l,m}(\alpha_1; \phi).$$

Proof. Let $f \in \mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$ and suppose that

$$q(z) = \frac{z\left(H_p^{l,m}(\alpha_1; z)f\right)'(z)}{p f_{p,k}^{l,m}(\alpha_1; z)} \qquad (z \in \mathbb{U}). \tag{2.1}$$

Then q is analytic in $\mathbb{U}$ and $q(0) = 1$. It follows from (1.3) and (2.1) that

$$q(z) f_{p,k}^{l,m}(\alpha_1; z) = \frac{\alpha_1}{p} H_p^{l,m}(\alpha_1 + 1) f(z) - \frac{\alpha_1 - p}{p} H_p^{l,m}(\alpha_1) f(z). \tag{2.2}$$

Differentiating both sides of (2.2) with respect to z and using (2.1), we have

$$zq'(z) + \left(\alpha_1 - p + \frac{z\left(f_{p,k}^{l,m}(\alpha_1; z)\right)'}{f_{p,k}^{l,m}(\alpha_1; z)}\right) q(z) = \frac{\alpha_1}{p} \frac{z\left(H_p^{l,m}(\alpha_1 + 1)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_1; z)}. \tag{2.3}$$

It now follows from (1.7), (1.17), (2.1) and (2.3) that

$$\begin{aligned}
&\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;z)\right]}\\
&=\frac{p(1-\alpha)q(z)f_{p,k}^{l,m}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p\left[zq'(z)+(\alpha_1-p+p\psi(z))q(z)\right]f_{p,k}^{l,m}(\alpha_1;z)}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\frac{\alpha}{\alpha_1}(\alpha_1-p+p\psi(z))f_{p,k}^{l,m}(\alpha_1;z)\right]}\\
&=\frac{(1-\alpha)q(z)+\frac{\alpha}{\alpha_1}\left[zq'(z)+(\alpha_1-p+p\psi(z))q(z)\right]}{(1-\alpha)+\frac{\alpha}{\alpha_1}(\alpha_1-p+p\psi(z))}\\
&=\frac{\frac{\alpha}{\alpha_1}zq'(z)+\left[(1-\alpha)+\frac{\alpha}{\alpha_1}(\alpha_1-p+p\psi(z))\right]q(z)}{(1-\alpha)+\frac{\alpha}{\alpha_1}(\alpha_1-p+p\psi(z))}\\
&=q(z)+\frac{zq'(z)}{\frac{\alpha_1}{\alpha}-p+p\psi(z)}\prec\phi(z)\qquad(z\in\mathbb{U}).
\end{aligned}\tag{2.4}$$

Moreover, since

$$\Re\left(p\phi(z)+\frac{\alpha_1}{\alpha}-p\right)>0\qquad(\alpha>0;\ z\in\mathbb{U}),$$

by Lemma 1.1, we know that

$$\psi(z)=\frac{z\left(f_{p,k}^{l,m}(\alpha_1;z)\right)'}{pf_{p,k}^{l,m}(\alpha_1;z)}\prec\phi(z)\qquad(z\in\mathbb{U}).$$

Thus, an application of Lemma 1.2 to (2.4), we have

$$q(z)\prec\phi(z)\qquad(z\in\mathbb{U}),$$

that is $f\in\mathcal{F}_{p,k}^{l,m}(\alpha_1;\phi)$. This implies that

$$\mathcal{F}_{p,k}^{l,m}(\alpha;\alpha_1;\phi)\subset\mathcal{F}_{p,k}^{l,m}(\alpha_1;\phi).$$

Hence the proof of Theorem 2.1 is complete. □

In view of Lemmas 1.4 and 1.5, by similarly applying the method of proof of Theorem 2.1 for the classes $\mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)$ and $\mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)$, we easily get the following inclusion relationships.

Corollary 2.1. *Let $\phi\in\mathcal{P}$ with*

$$\Re\left(p\phi(z)+\frac{\alpha_1}{\alpha}-p\right)>0\qquad(\alpha>0;\ z\in\mathbb{U}).$$

Then

$$\mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)\subset\mathcal{G}_p^{l,m}(\alpha_1;\phi).$$

Corollary 2.2. *Let $\phi\in\mathcal{P}$ with*

$$\Re\left(p\phi(z)+\frac{\alpha_1}{\alpha}-p\right)>0\qquad(\alpha>0;\ z\in\mathbb{U}).$$

Then

$$\mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)\subset\mathcal{H}_p^{l,m}(\alpha_1;\phi).$$

Theorem 2.2. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(p\phi(z) + \frac{\alpha_1}{\alpha} - p\right) > 0 \qquad (\alpha > 0;\ z \in \mathbb{U}).$$

Then

$$\mathfrak{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi) \subset \mathfrak{F}_{p,k}^{l,m}(\alpha_1; \phi). \tag{2.5}$$

Proof. Let $f \in \mathfrak{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$ and suppose that

$$p(z) = \frac{z\left(H_p^{l,m}(\alpha_1; z)f\right)'(z)}{p\mathfrak{f}_{p,k}^{l,m}(\alpha_1; z)} \qquad (z \in \mathbb{U}). \tag{2.6}$$

Then p is analytic in $\mathbb{U}$ and $p(0) = 1$. It follows from (1.3) and (2.6) that

$$p(z)\mathfrak{f}_{p,k}^{l,m}(\alpha_1; z) = \frac{\alpha_1}{p} H_p^{l,m}(\alpha_1 + 1)f(z) - \frac{\alpha_1 - p}{p} H_p^{l,m}(\alpha_1)f(z). \tag{2.7}$$

Differentiating both sides of (2.7) with respect to z and using (2.6), we have

$$zp'(z) + \left(\alpha_1 - p + \frac{z\left(\mathfrak{f}_{p,k}^{l,m}(\alpha_1; z)\right)'}{\mathfrak{f}_{p,k}^{l,m}(\alpha_1; z)}\right) p(z) = \frac{\alpha_1}{p} \frac{z\left(H_p^{l,m}(\alpha_1 + 1)f\right)'(z)}{\mathfrak{f}_{p,k}^{l,m}(\alpha_1; z)}.$$

Furthermore, we suppose that

$$\varphi(z) = \frac{z\left(\mathfrak{f}_{p,k}^{l,m}(\alpha_1; z)\right)'}{p\mathfrak{f}_{p,k}^{l,m}(\alpha_1; z)} \qquad (z \in \mathbb{U}).$$

The remainder of the proof of Theorem 2.2 is similar to that of Theorem 2.1. We, therefore, choose to omit the analogous details involved. We thus find that

$$p(z) \prec \phi(z) \qquad (z \in \mathbb{U}),$$

which implies that $f \in \mathfrak{F}_{p,k}^{l,m}(\alpha_1; \phi)$. The proof of Theorem 2.2 is evidently completed. □

By similarly applying the method of proof of Theorem 2.1 for the classes $\mathfrak{G}_p^{l,m}(\alpha; \alpha_1; \phi)$ and $\mathfrak{H}_p^{l,m}(\alpha; \alpha_1; \phi)$, we easily get the following inclusion relationships.

Corollary 2.3. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(p\phi(z) + \frac{\alpha_1}{\alpha} - p\right) > 0 \qquad (\alpha > 0;\ z \in \mathbb{U}).$$

Then

$$\mathfrak{G}_p^{l,m}(\alpha; \alpha_1; \phi) \subset \mathfrak{G}_p^{l,m}(\alpha_1; \phi).$$

Corollary 2.4. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(p\phi(z) + \frac{\alpha_1}{\alpha} - p\right) > 0 \qquad (\alpha > 0;\ z \in \mathbb{U}).$$

Then

$$\mathfrak{H}_p^{l,m}(\alpha; \alpha_1; \phi) \subset \mathfrak{H}_p^{l,m}(\alpha_1; \phi).$$

By similarly applying the method of proof of Theorems 1 and 2 obtained by Wang *et al.* [34] for the function classes $\mathcal{G}_p^{l,m}(\alpha; \alpha_1; \phi)$, $\mathfrak{G}_p^{l,m}(\alpha; \alpha_1; \phi)$, $\mathcal{H}_p^{l,m}(\alpha; \alpha_1; \phi)$ and $\mathfrak{H}_p^{l,m}(\alpha; \alpha_1; \phi)$, we also easily get the following inclusion relationships.

Corollary 2.5. *Let* $\phi \in \mathcal{P}$ *with*

$$\Re(p\phi(z) + \alpha_1 - p) > 0 \qquad (z \in \mathbb{U}).$$

Then

$$\mathcal{G}_p^{l,m}(\alpha_1 + 1; \phi) \subset \mathcal{G}_p^{l,m}(\alpha_1; \phi).$$

Corollary 2.6. *Let* $\phi \in \mathcal{P}$ *with*

$$\Re(p\phi(z) + \alpha_1 - p) > 0 \qquad (z \in \mathbb{U}).$$

Then

$$\mathcal{H}_p^{l,m}(\alpha_1 + 1; \phi) \subset \mathcal{H}_p^{l,m}(\alpha_1; \phi).$$

Corollary 2.7. *Let* $\phi \in \mathcal{P}$ *with*

$$\Re(p\phi(z) + \alpha_1 - p) > 0 \qquad (z \in \mathbb{U}).$$

Then

$$\mathfrak{G}_p^{l,m}(\alpha_1 + 1; \phi) \subset \mathfrak{G}_p^{l,m}(\alpha_1; \phi).$$

Corollary 2.8. *Let* $\phi \in \mathcal{P}$ *with*

$$\Re(p\phi(z) + \alpha_1 - p) > 0 \qquad (z \in \mathbb{U}).$$

Then

$$\mathfrak{H}_p^{l,m}(\alpha_1 + 1; \phi) \subset \mathfrak{H}_p^{l,m}(\alpha_1; \phi).$$

3. Convolution Properties

In this section, we provide some convolution properties for the function classes $\mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$, $\mathcal{G}_p^{l,m}(\alpha; \alpha_1; \phi)$, and $\mathcal{H}_p^{l,m}(\alpha; \alpha_1; \phi)$.

Theorem 3.1. *Let* $f \in \mathcal{A}_p$ *and* $\phi \in \mathcal{P}$. *Then* $f \in \mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$ *if and only if*

$$\begin{aligned}
\frac{1}{z}\Bigg\{ f * \Bigg[& (1-\alpha)\left(pz^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{n+p}{n!} z^{n+p} \right) \\
& - p\,(1-\alpha)\phi(e^{i\theta}) \left(z^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{1}{n!} z^{n+p} \right) * \left(\frac{1}{k} \sum_{\upsilon=0}^{k-1} \frac{z^p}{1-\varepsilon^{\upsilon} z} \right) \\
& + \alpha \left(pz^p + \sum_{n=1}^{\infty} \frac{(\alpha_1+1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{n+p}{n!} z^{n+p} \right) \\
& - p\,\alpha\phi(e^{i\theta}) \left(z^p + \sum_{n=1}^{\infty} \frac{(\alpha_1+1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{1}{n!} z^{n+p} \right) * \left(\frac{1}{k} \sum_{\upsilon=0}^{k-1} \frac{z^p}{1-\varepsilon^{\upsilon} z} \right) \Bigg] \Bigg\} \neq 0
\end{aligned}$$

$$(z \in \mathbb{U};\ \ 0 \leqq \theta < 2\pi).$$

Proof. Suppose that $f \in \mathcal{F}_{p,k}^{l,m}(\alpha; \alpha_1; \phi)$. Since

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z) + \alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1; z) + \alpha f_{p,k}^{l,m}(\alpha_1+1; z)\right]} \prec \phi(z) \qquad (z \in \mathbb{U})$$

is equivalent to

$$\frac{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;z)\right]}\neq\phi(e^{i\theta})\qquad(0\leqq\theta<2\pi),\tag{3.1}$$

it is easy to see that the condition (3.1) can be written as follows:

$$\frac{1}{z}\Bigg\{z\left[(1-\alpha)\left(H_p^{l,m}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)\right] - p\left[(1-\alpha)f_{p,k}^{l,m}(\alpha_1;z)+\alpha f_{p,k}^{l,m}(\alpha_1+1;z)\right]\phi(e^{i\theta})\Bigg\}\neq 0\qquad(0\leqq\theta<2\pi).\tag{3.2}$$

On the other hand, we know from (1.2) that

$$z\left(H_p^{l,m}(\alpha_1)f\right)'(z)=\left(pz^p+\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{n+p}{n!}z^{n+p}\right)*f(z).\tag{3.3}$$

Moreover, from the definition of $f_{p,k}^{l,m}(\alpha_1;z)$, we have

$$\begin{aligned}f_{p,k}^{l,m}(\alpha_1;z)&=H_p^{l,m}(\alpha_1)f(z)*\left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^{\upsilon}z}\right)\\&=\left(z^p+\sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{1}{n!}z^{n+p}\right)*\left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^{\upsilon}z}\right)*f(z).\end{aligned}\tag{3.4}$$

Replacing α_1 by α_1+1 in (3.3) and (3.4), we know that (3.3) and (3.4) also hold true, that is,

$$z\left(H_p^{l,m}(\alpha_1+1)f\right)'(z)=\left(pz^p+\sum_{n=1}^{\infty}\frac{(\alpha_1+1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{n+p}{n!}z^{n+p}\right)*f(z),\tag{3.5}$$

and

$$\begin{aligned}f_{p,k}^{l,m}(\alpha_1+1;z)=&H_p^{l,m}(\alpha_1+1)f(z)*\left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^{\upsilon}z}\right)\\&=\left(z^p+\sum_{n=1}^{\infty}\frac{(\alpha_1+1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{1}{n!}z^{n+p}\right)*\left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^{\upsilon}z}\right)*f(z).\end{aligned}\tag{3.6}$$

Upon substituting from (3.3) to (3.6) into (3.2), we easily deduce the convolution property asserted by Theorem 3.1.

By similarly applying the method of proof of Theorem 3.1 for the classes $\mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)$ and $\mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)$, we easily get the following convolution properties. □

Corollary 3.1. *Let $f \in \mathcal{A}_p$ and $\phi \in \mathcal{P}$. Then $f \in \mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)$ if and only if*

$$\frac{1}{z}\Bigg\{f * \Bigg[(1-\alpha)\left(pz^p + \sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{n+p}{n!}z^{n+p}\right) - \frac{p(1-\alpha)\phi(e^{i\theta})}{2}h_1$$
$$+\alpha\left(pz^p + \sum_{n=1}^{\infty}\frac{(\alpha_1+1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{n+p}{n!}z^{n+p}\right) - \frac{p\,\alpha\phi(e^{i\theta})}{2}h_2\Bigg]$$
$$-\frac{p\,(1-\alpha)\phi(e^{i\theta})}{2}\overline{(h_1 * f)(\overline{z})} - \frac{p\,\alpha\phi(e^{i\theta})}{2}\overline{(h_2 * f)(\overline{z})}\Bigg\} \neq 0 \qquad (0 \leqq \theta < 2\pi),$$

where

$$h_1(z) = z^p + \sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{1}{n!}z^{n+p}, \tag{3.7}$$

and

$$h_2(z) = z^p + \sum_{n=1}^{\infty}\frac{(\alpha_1+1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{1}{n!}z^{n+p}. \tag{3.8}$$

Corollary 3.2. *Let $f \in \mathcal{A}_p$ and $\phi \in \mathcal{P}$. Then $f \in \mathcal{H}_p^{l,m}(\alpha;\alpha_1;\phi)$ if and only if*

$$\frac{1}{z}\Bigg\{f * \Bigg[(1-\alpha)\left(pz^p + \sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{n+p}{n!}z^{n+p}\right) - \frac{p(1-\alpha)\phi(e^{i\theta})}{2}h_1$$
$$+\alpha\left(pz^p + \sum_{n=1}^{\infty}\frac{(\alpha_1+1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{n+p}{n!}z^{n+p}\right) - \frac{p\alpha\phi(e^{i\theta})}{2}h_2\Bigg]$$
$$+\frac{p(1-\alpha)\phi(e^{i\theta})}{2}\overline{(h_1 * f)(-\overline{z})} + \frac{p\alpha\phi(e^{i\theta})}{2}\overline{(h_2 * f)(-\overline{z})}\Bigg\} \neq 0 \qquad (0 \leqq \theta < 2\pi),$$

where h_1 and h_2 are given by (3.7) *and* (3.8)*, respectively.*

Theorem 3.2. *Let $f \in \mathcal{H}_p^{l,m}(\alpha_1;\phi)$. Then*

$$\begin{aligned} f(z) = &\left[p\int_0^z \zeta^{p-1}\phi(\omega(\zeta))\cdot\exp\left(\frac{p}{2}\int_0^{\zeta}\frac{\phi(\omega(\xi)) - \overline{\phi(\omega(-\overline{\xi}))} - 2}{\xi}d\xi\right)d\zeta\right] \\ &* \left(\sum_{n=0}^{\infty}\frac{n!(\beta_1)_n\cdots(\beta_m)_n}{(\alpha_1)_n\cdots(\alpha_l)_n}z^{n+p}\right), \end{aligned} \tag{3.9}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Proof. From the definition of $\mathcal{H}_p^{l,m}(\alpha_1;\phi)$, we know that

$$\frac{z\left(H_p^{l,m}(\alpha_1)f\right)'(z)}{p h_p^{l,m}(\alpha_1;z)} = \frac{2z\left(H_p^{l,m}(\alpha_1)f\right)'(z)}{p\left[(H_p^{l,m}(\alpha_1)f)(z) - \overline{(H_p^{l,m}(\alpha_1)f)(-\overline{z})}\right]} = \phi(\omega(z)), \tag{3.10}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

From (3.10), we get

$$\frac{2\overline{z\left(H_p^{l,m}(\alpha_1)f\right)'(-\overline{z})}}{p\left[(H_p^{l,m}(\alpha_1)f)(z)-\overline{(H_p^{l,m}(\alpha_1)f)(-\overline{z})}\right]}=\overline{\phi(\omega(-\overline{z}))}. \tag{3.11}$$

It now follows from (3.10) and (3.11) that

$$\frac{z\left(h_p^{l,m}(\alpha_1;z)\right)'}{ph_p^{l,m}(\alpha_1;z)}=\frac{1}{2}\left[\phi(\omega(z))-\overline{\phi(\omega(-\overline{z}))}\right]. \tag{3.12}$$

We next find from (3.12) that

$$\frac{\left(h_p^{l,m}(\alpha_1;z)\right)'}{h_p^{l,m}(\alpha_1;z)}-\frac{p}{z}=\frac{p}{2}\cdot\frac{\phi(\omega(z))-\overline{\phi(\omega(-\overline{z}))}-2}{z}. \tag{3.13}$$

Upon integrating (3.13), we have

$$\log\left(\frac{h_p^{l,m}(\alpha_1;z)}{z^p}\right)=\frac{p}{2}\int_0^z\frac{\phi(\omega(\xi))-\overline{\phi(\omega(-\overline{\xi}))}-2}{\xi}d\xi, \tag{3.14}$$

which implies that

$$h_p^{l,m}(\alpha_1;z)=z^p\cdot\exp\left(\frac{p}{2}\int_0^z\frac{\phi(\omega(\xi))-\overline{\phi(\omega(-\overline{\xi}))}-2}{\xi}d\xi\right). \tag{3.15}$$

It now follows from (3.10) and (3.15) that

$$\begin{aligned}\left(H_p^{l,m}(\alpha_1)f\right)'(z)=&\frac{ph_p^{l,m}(\alpha_1;z)}{z}\cdot\phi(\omega(z))\\ =&pz^{p-1}\phi(\omega(z))\cdot\exp\left(\frac{p}{2}\int_0^z\frac{\phi(\omega(\xi))-\overline{\phi(\omega(-\overline{\xi}))}-2}{\xi}d\xi\right).\end{aligned} \tag{3.16}$$

Upon integrating (3.16), we get

$$H_p^{l,m}(\alpha_1)f(z)=p\int_0^z\zeta^{p-1}\phi(\omega(\zeta))\cdot\exp\left(\frac{p}{2}\int_0^\zeta\frac{\phi(\omega(\xi))-\overline{\phi(\omega(-\overline{\xi}))}-2}{\xi}d\xi\right)d\zeta. \tag{3.17}$$

Combining (1.2) and (3.17), we find that

$$\begin{aligned}&p\int_0^z\zeta^{p-1}\phi(\omega(\zeta))\cdot\exp\left(\frac{p}{2}\int_0^\zeta\frac{\phi(\omega(\xi))-\overline{\phi(\omega(-\overline{\xi}))}-2}{\xi}d\xi\right)d\zeta\\ &=[z^p\ {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)]*f(z).\end{aligned} \tag{3.18}$$

Thus, from (3.18), we easily get the convolution property (3.9). □

Remark 3.1. Putting

$$p=1,\ l=2,\ m=1\quad\text{and}\quad\alpha_1=\alpha_2=\beta_1=1$$

in Theorem 3.2, we get the corresponding result obtained by Ravichandran [25].

By similarly applying the method of proof of Theorem 3.2 for the class $\mathcal{G}_p^{l,m}(\alpha;\alpha_1;\phi)$, we easily get the following convolution property.

Corollary 3.3. *Let* $f \in \mathcal{G}_p^{l,m}(\alpha_1;\phi)$. *Then*

$$f(z) = \left[p \int_0^z \zeta^{p-1} \phi(\omega(\zeta)) \cdot \exp\left(\frac{p}{2} \int_0^{\zeta} \frac{\phi(\omega(\xi)) + \overline{\phi(\omega(\bar{\xi}))} - 2}{\xi} d\xi \right) d\zeta \right] * \left(\sum_{n=0}^{\infty} \frac{n!(\beta_1)_n \cdots (\beta_m)_n}{(\alpha_1)_n \cdots (\alpha_l)_n} z^{n+p} \right),$$

where ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

References

[1] R. Aghalary and G. Azadi, *The Dziok-Srivastava operator and k-uniformly starlike functions,* J. Inequal. Pure Appl. Math. **6** (2005), Article 52, pp. 1–7 (electronic).

[2] R. Aghalary, S. B. Joshi, R. N. Mohapatra and V. Ravichandran, *Subordinations for analytic functions defined by the Dziok-Srivastava linear operator,* Appl. Math. Comput. **187** (2007), 13–19.

[3] R. M. Ali, V. Ravichandran, M. H. Khan and K. G. Subramanian, *Applications of first order differential superordination to certain linear operators,* Southeast Asian Bull. Math. **30** (2006), 799–810.

[4] M. K. Aouf and G. Murugusundaramoorthy, *On a subclass of uniformly convex functions defined by the Dziok-Srivastava operator,* Aust. J. Math. Anal. Appl. **5** (2008), Article 3, (electronic).

[5] M.-P. Chen, Z.-R. Wu and Z.-Z. Zou, *On functions α-starlike with respect to symmetric conjugate points,* J. Math. Anal. Appl. **201** (1996), 25–34.

[6] J. Dziok, *On the convex combination of the Dziok-Srivastava operator,* Appl. Math. Comput. **188** (2007), 1214–1220.

[7] J. Dziok, *On some applications of the Briot-Bouquet differential subordination,* J. Math. Anal. Appl. **328** (2007), 295–301.

[8] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generzlized hypergeometric function,* Appl. Math. Comput. **103** (1999), 1–13.

[9] J. Dziok and H. M. Srivastava, *Certain sbuclasses of analytic functions associated with the generzlized hypergeometric function,* Integral Transform. Spec. Funct. **14** (2003), 7–18.

[10] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination,* in *General Mathematics* **3**, International Series of Numerical Mathematics, Vol. **64**, Birkhäuser Verlag, Basel, 1983, pp. 339–348; see also Rev. Roumaine Math. Pures Appl. **29** (1984), 567–573.

[11] A. Gangadharan, T. N. Shanmugam and H. M. Srivastava, *Generalized hypergeometric function associated with k-uniformly convex functions,* Comput. Math. Appl. **44** (2002), 1515–1526.

[12] Y.-Y. Huang and M.-S. Liu, *Properties of certain subclasses of multivalent analytic functions involving the Dziok-Srivastava operator,* Appl. Math. Comput. **204** (2008), 137–149.

[13] S. S. Kumar, H. C. Taneja and V. Ravichandran, *Classes of multivalent functions defined by Dziok-Srivastava linear operator and multitiplier transformation,* Kyungpook Math. J. **46** (2006), 97–109.

[14] J.-L. Liu, *Strongly starlike functions associated with the Dziok-Srivastava operator,* Tamkang J. Math. **35** (2004), 37–42.

[15] J.-L. Liu and H. M. Srivastava, *Certain properties of the Dziok-Srivastava operator,* Appl. Math. Comput. **159** (2004), 485–493.

[16] S. S. Miller and P. T. Mocanu, *On some classes of first order differential subordinations,* Michigan Math. J. **32** (1985), 185–195.

[17] G. Murugusundaramoorthy and N. Magesh, *Starlike and convex functions of complex order involving the Dziok-Srivastava operator,* Integral Transform. Spec. Funct. **18** (2007), 419–425.

[18] G. Murugusundaramoorthy, T. Rosy and S. Sivasubramanian, *On certain classes of analytic functions of complex order defined by Dziok-Srivastava operator,* J. Math. Inequal. **1** (2007), 553–562.

[19] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergermetric functions,* Canad. J. Math. **39** (1987), 1057–1077.
[20] K. S. Padmanabhan and R. Parvatham, Some applications of differential subordination, *Bull. Austral. Math. Soc.* **32** (1985), 321–330.
[21] R. Parvatham and S. Radha, On α-starlike and α-close-to-convex functions with respect to n-symmetric points, *Indian J. Pure Appl. Math.* **17** (1986), 1114–1122.
[22] J. Patel, A. K. Mishra and H. M. Srivastava, Classes of multivalent analytic functions involving the Dziok-Srivastava operator, *Comput. Math. Appl.* **54** (2007), 599–616.
[23] K. Piejko and J. Sokól, *On the Dziok-Srivastava operator under multivalent analytic functions,* Appl. Math. Comput. **177** (2006), 839–843.
[24] C. Ramachandran, T. N. Shanmugam, H. M. Srivastava and A. Swaminathan, *A unified class of k-uniformly convex functions defined by the Dziok-Srivastava linear operator,* Appl. Math. Comput. **190** (2007), 1627–1636.
[25] V. Ravichandran, *Starlike and convex functions with respect to conjugate points,* Acta Math. Acad. Paedagog. Nyházi. (N.S.) **20** (2004), 31–37 (electronic).
[26] T. N. Shanmugam and M. P. Jeyaraman, *Strongly Mocanu convex functions associated with the Dziok-Srivastava operator,* Far East J. Math. Sci. **20** (2006), 49–57.
[27] T. N. Shanmugam, M. P. Jeyarama and A. Singaravelu, *On sandwich theorems for certain subclass of analytic functions involving Dziok-Srivastava operator,* Aust. J. Math. Anal. Appl. **4** (2007), Article 17, (electronic).
[28] T. N. Shanmugam, C. Ramachandran, A. Singaravelu and M. P. Jeyaraman, *On differential Sanwich theorems for certain subclass of analytic functions using Dziok-Srivastava operator,* Far East J. Math. Sci. **25** (2007), 621–632.
[29] T. N. Shanmugam, S. Sivasubramanian, and G. Murugusundaramoorthy, *On the unified class of functions of complex order involving Dziok-Srivastava operator,* Gen. Math. **15** (2007), 149–160.
[30] H. M. Srivastava, *Generalized hypergeometric functions and associated families of k-uniformly convex and k-starlike functions,* Gen. Math. **15** (2007), 201–226.
[31] H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, *Hypergeometric functions in the parabolic starlike and uniformly convex domains,* Integral Transform. Spec. Funct. **18** (2007), 511–520.
[32] K. Suchithra, A. Gangadharan and S. Owa, *Neighborhood properties of certain classes of analytic functions using the Dziok-Srivastava operator,* Tamsui Oxf. J. Math. Sci. **24** (2008), 185–200.
[33] Z.-G. Wang, C.-Y. Gao and S.-M. Yuan, *On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points,* J. Math. Anal. Appl. **322** (2006), 97–106.
[34] Z.-G. Wang, Y.-P. Jiang and H. M. Srivastava, *Some subclasses of multivalent analytic functions involving the Dziok-Srivastava operator,* Integral Transforms Spec. Funct. **19** (2008), 129–146.
[35] N. Xu and D.-G. Yang, *Some classes of analytic and multivalent functions involving a linear operator,* Math. Comput. Modelling **49** (2009), 955–965.

Zhi-Gang Wang
Department of Mathematics and Computing Science, Hengyang Normal University, Hengyang 421008, Hunan, P. R. China
E-mail address: zhigangwang@foxmail.com

Hai-Ping Shi
Department of Basic Courses, Guangdong Baiyun Institute, Guangzhou 510450, Guangdong, P. R. China
E-mail address: shp7971@163.com