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COMPATIBILITY AND WEAK COMPATIBILITY FOR FOUR SELF MAPS IN A CONE METRIC SPACE

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Abstract. The object of this paper is to introduce the concept of compatibility of pair of self maps in a cone metric space without assuming its normality. Using this concept we establish a unique common fixed point theorem for four self maps satisfying a generalized contractive condition in a cone metric space which generalizes and synthesizes the results of L. G. Huang and X. Zhang [3](J. Math. Anal. Appl 332(2007) 1468-1476). All the results presented in this paper are new.

1. Introduction

There has been a number of generalizations of metric space. One such generalization is a Cone metric space initiated by Huang and Zhang [3]. In this space they replaced the set of real numbers of a metric space by an ordered Banach Space and gave some fundamental results for a self map satisfying a contractive condition. In [1] Abbas and Jungck generalized the result of [3] for two self maps through weak compatibility in a normal cone metric space. On the same line Vetro [7] proved some fixed point theorem for two self maps satisfying a contractive condition through weak compatibility.

Recently, Rezapour and Hamlbarani [5] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space.

In section 2, of this paper we introduce the concept of compatibility of pair of self maps and prove some propositions using it in a cone metric space. Also we prove a unique common fixed point theorem for four self maps through compatibility satisfying a more generalized contractive condition than one adopted in [1, 2, 3, 7] for a non- normal cone metric space. Our results generalize, extend and unify several well-known fixed point results in cone metric spaces. Example 2, illustrates the main result of this paper.

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2. Preliminaries

Definition 2.1. [3] *:Let E be a real Banach space and P be a subset of $E.P$ is called a cone if.*
$(i)P$ is a closed, non-empty and $P \neq \{0\}$;
$(ii)a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;
$(iii)x \in Pand - x \in P$ imply $x = 0$.
Given a cone $P \subseteq E$, we define a partial ordering "$\leq$" in E by $x \leq y$ if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x << y$ to denote $y - x \in P^0$, where P^0 stands for the interior of P.

Proposition 2.2. [4]*: Let P be a cone in a real Banach space E . If $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then, $a = 0$.*

Proof: For $a \in P, k \in [0, 1)$ and $a \leq ka$ give $(k-1)a \in P$ implies $-(1-k)a \in P$. Therefore by (ii) we have $-a \in P$, as $1/(1 - k) > 0$. Hence $a = 0$, by (iii).

Proposition 2.3. [4]*: Let P be a cone in a real Banach space E. If for $a \in E$ and $a << c$, for all $c \in P^0$, then $a = 0$.*

Remark 2.4. [5]*: $\lambda P^0 \subseteq P^0$, for $\lambda > 0$ and $P^0 + P^0 \subseteq P^0$.*

Definition 2.5. [3]*: Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:*
(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$;
(b) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(c)$d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

For examples of cone metric spaces we refer Huang et al. [3].

Definition 2.6. [3]*: Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 << c$ there is a positive integer N_c such that for all $n > N_c, d(x_n, x) << c$, then the sequence $\{x_n\}$ is said to converges to x, and x is called limit of $\{x_n\}$. We write $lim_{n\to\infty} x_n = x$ or $x_n \to x$, as $n \to \infty$.*

Definition 2.7. [3]*: Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 << c$ there is a N such that for all $n, m > N, d(x_n, x_m) << c$, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in X.*

Definition 2.8. [3]*: Let (X, d) be a cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.*

Proposition 2.9. *: Let (X, d) be a cone metric space and P be a cone in a real Banach space E. If $u \leq v, v << w$ then $u << w$.*

Lemma 2.10. *: Let (X, d) be a cone metric space and P be a cone in a real Banach space E and $k_1, k_2, k_3, k_4, k > 0$. If $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ in X and*

$$ka \leq k_1 d(x_n, x) + k_2 d(y_n, y) + k_3 d(z_n, z) + k_4 d(p_n, p), \tag{1.1}$$

then $a = 0$.

Proof: As $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ for $c \in P^0$ there exists a positive integer N_c such that

$\frac{c}{(k_1+k_2+k_3+k_4)} - d(x_n, x), \frac{c}{(k_1+k_2+k_3+k_4)} - d(y_n, y),$
$\frac{c}{(k_1+k_2+k_3+k_4)} - d(z_n, z), \frac{c}{(k_1+k_2+k_3+k_4)} - d(p_n, p) \in P^0$, for all $n > N_c$.
Therefore by Remark 2.4, we have
$\frac{k_1 c}{(k_1+k_2+k_3+k_4)} - k_1 d(x_n, x), \frac{k_2 c}{(k_1+k_2+k_3+k_4)} - k_2 d(y_n, y),$
$\frac{k_3 c}{(k_1+k_2+k_3+k_4)} - k_3 d(z_n, z), \frac{k_4 c}{(k_1+k_2+k_3+k_4)} - k_4 d(p_n, p) \in P^0$, for all $n > N_c$.
Again by adding and Remark 2.4, we have
$c - k_1 d(x_n, x) - k_2 d(y_n, y) - k_3 d(z_n, z) - k_4 d(p_n, p) \in P^0$ for all $n > N_c$.
From (1.1) and Proposition 2.9 we have i. e. $ka << c$, for each $c \in P^0$. By Proposition 2.3 , we have $a = 0$, as $k > 0$.

Definition 2.11. [1]*: Let A and S be self maps of a set X. If $w = Ax = Sx$, for some $x \in X$, then w is called a coincidence point of A and S.*

Definition 2.12. [5] *:Let X be any set. A pair of self maps (A, S) in X is said to be weakly compatible if $u \in X, Au = Su$ imply $SAu = ASu$.*

Compatibility in a Cone Metric Space

Here we will define compatibility of self maps in a cone metric space and prove some Propositions to be used in the main result of this manuscript.

Definition 2.13. *: Let (X, d) be a cone metric space. A pair of self maps (A, S) in X is said to be compatible if for $\{x_n\}$ in X, $Ax_n \to u$ and $Sx_n \to u$, for some $u \in X$, then for every $c \in P^0$, there is a positive integer N_c such that $d(ASx_n, SAx_n) << c$, for all $n > N_c$.*

Proposition 2.14. *: In a cone metric space every commuting pair of self maps is compatible but the converse is not true, as observed in the following example.*

Example 2.15. *: Let $E = R^2, P = \{(x, y) : x, y \geq 0\} \subseteq R^2$ be a cone in E. Taking $X = R$. Fix a real number $\alpha > 0$ and define $d : X \times X \to E$ by $d(x, y) = |x-y|(1, \alpha)$. Then (X, d) is a complete cone metric space. Define self maps A and S on X as follows:*
$A(x) = \begin{cases} 0, & if \quad x \quad is \quad rational \\ 1, & if \quad x \quad is \quad irrational \end{cases} \qquad S(x) = \begin{cases} x/2, & x \in [0, 2] \\ 2, & otherwise. \end{cases}$
If $\{r_n\}$ is a sequence of rationals such that $A(r_n) \to u$ and $S(r_n) \to u$ then $u = 0$ and $SAr_n = S(0) = 0$ and $S(r_n) = r_n/2$ gives $AS(r_n) = 0$. Thus $d(ASr_n, SAr_n) = 0$. Hence the pair of the self maps (A, S) is compatible. It is observed that the pair of self maps (A, S) is non-commuting at $\sqrt{8}$.

Proposition 2.16. *: In a cone metric space every compatible pair of self maps is weakly compatible.*

Proposition 2.17. *:Let (A, S) be a compatible pair of self maps in a cone metric space (X, d). If $Ax_n \to u$ and $Sx_n \to u$, for some $u \in X$ and $ASx_n \to Au$ then $SAx_n \to Au$.*

Proof :We have
$d(SAx_n, Au) \leq d(SAx_n, ASx_n) + d(ASx_n, Au). \qquad (*)$
As the pair (A, S) is compatible and $ASx_n \to Au$, for $c \in P^0$ there exists a positive integer N_c such that
$\frac{c}{2} - d(ASx_n, SAx_n), \frac{c}{2} - d(ASx_n, Au) \in P^0$, for all $n > N_c$.

Therefore by Remark 2.4 , we have
$c - d(SAx_n, ASx_n) - d(ASx_n, Au) \in P^0$ for all $n > N_c$.
From $(*)$ we have,
$d(SAx_n, ASx_n) + d(ASx_n, Au) - d(SAx_n, Au) \in P$ for all $n > N_c$.
Now adding and using Proposition 2.9, we have $c - d(ASx_n, Au) \in P^0$ i. e.
$d(SAx_n, Au) << c$, for all $n > N_c$.
Hence $SAx_n \to Au$.
Note : In above Proposition, if $SAx_n \to Su$ then it will follow that $ASx_n \to Su$.

3. MAIN RESULTS

Theorem 3.1. *:Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let A, B, S and T be self mappings on X satisfying:*
(3.1.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;
(3.1.2) *pair (A, S) is compatible and the pair (B, T) is weakly compatible;*
(3.1.3) *one of A or S is continuous;*
(3.1.4) *for some $\lambda, \mu, \delta, \gamma \in [0, 1)$ with $\lambda + \mu + \delta + 2\gamma < 1$ such that for all $x, y \in X$*
$d(Ax, By) \leq \lambda d(Ax, Sx) + \mu d(By, Ty) + \delta d(Sx, Ty) + \gamma[d(Ax, Ty) + d(Sx, By)]$.
Then A, B, S and T have a unique common fixed point in X.

Proof. : Let $x_0 \in X$ be any point in X. Using (3.1.4) construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n}, Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}, foralln. \tag{3.1}$$

We show that $\{y_n\}$ is a Cauchy sequence in X.
Step I: Taking $x = x_{2n}, y = x_{2n+1}$ in (3.1.4) we get,

$$\begin{aligned} d(Ax_{2n}, Bx_{2n+1}) \leq \quad & \lambda d(Ax_{2n}, Sx_{2n}) + \mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Sx_{2n}, Tx_{2n+1}) \\ & +\gamma[d(Ax_{2n}, Tx_{2n+1}) + d(Sx_{2n}, Bx_{2n+1})]. \end{aligned}$$

Using (3.1) we get,

$$\begin{aligned} d(y_{2n}, y_{2n+1}) \quad & \leq \lambda d(y_{2n}, y_{2n-1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n-1}, y_{2n}) + \gamma[d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})] \\ & \leq \lambda d(y_{2n}, y_{2n-1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n-1}, y_{2n}) + \gamma[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \end{aligned}$$

Writing $d(y_n, y_{n+1}) = d_n$, we have
$d_{2n} \leq \lambda d_{2n-1} + \mu d_{2n} + \delta d_{2n-1} + \gamma[d_{2n} + d_{2n-1}]$,
i.e.
$(1 - \mu - \gamma)d_{2n} \leq (\lambda + \delta + \gamma)d_{2n-1}$,
which implies

$$d_{2n} \leq hd_{2n-1}, \tag{3.2}$$

where $h = \frac{(\lambda+\delta+\gamma)}{1-\mu-\gamma}$.
Inview of (3.1.4), $h < 1$.
Taking $x = x_{2n+2}, y = x_{2n+1}$ in (3.1.4) we get,

$$\begin{aligned} d(Ax_{2n+2}, Bx_{2n+1}) \leq \quad & \lambda d(Ax_{2n+2}, Sx_{2n+2}) + \mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Sx_{2n+2}, Tx_{2n+1}) \\ & +\gamma[d(Ax_{2n+2}, Tx_{2n+1}) + d(Sx_{2n+2}, Bx_{2n+1})]. \end{aligned}$$

Using (3.1) we get,

$$\begin{aligned} d(y_{2n+2}, y_{2n+1}) &\leq \lambda d(y_{2n+2}, y_{2n+1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n+1}, y_{2n}) \\ &+\gamma[d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})] \\ &\leq \lambda d(y_{2n+2}, y_{2n+1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n+1}, y_{2n}) \\ &+\gamma[d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})]. \end{aligned}$$

So we have
$d_{2n+1} \leq \lambda d_{2n+1} + \mu d_{2n} + \delta d_{2n} + \gamma[d_{2n+1} + d_{2n}]$,
i.e.
$(1 - \lambda - \gamma)d_{2n+1} \leq (\mu + \delta + \gamma)d_{2n}$,
which implies

$$d_{2n+1} \leq k d_{2n}, \tag{3.3}$$

where $k = \frac{(\mu+\delta+\gamma)}{(1-\lambda-\gamma)}$.
By condition (3.1.4), we have $k < 1$.
Inview of (3.2) and (3.3) we have
$d_{2n+1} \leq k d_{2n} \leq hk d_{2n-1} \leq k^2 h d_{2n-2} \leq \ldots \leq k^{n+1} h^n d_0$, where $d_0 = d(y_0, y_1)$, and
$d_{2n} \leq h d_{2n-1} \leq hk d_{2n-2} \leq h^2 k d_{2n-3} \leq \ldots \leq h^n k^n d_0$, where $d_0 = d(y_0, y_1)$.
Therefore,
$d_{2n+1} \leq k^{n+1} h^n d_0$ and $d_{2n} \leq h^n k^n d_0$.
Also
$d(y_{n+p}, y_n) \leq d(y_{n+p}, y_{n+p-1}) + d(y_{n+p-1}, y_{n+p-2}) + \ldots + d(y_{n+1}, y_n)$,
i. e.

$$d(y_{n+p}, y_n) \leq d_{n+p-1} + d_{n+p-2} + \ldots + d_n. \tag{3.4}$$

If $n + p - 1$ is even, then by (3.4) we have

$$\begin{aligned} d(y_{n+p}, y_n) &\leq (h^{(n+p-1)/2} k^{(n+p-1)/2} + h^{(n+p-1)/2} k^{(n+p)/2} + \cdots +)d_0. \\ &= h^{(n+p-1)/2} k^{(n+p-1)/2}[1 + k + hk + hk^2 + h^2k^2 + \ldots]d_0, \\ &= h^{(n+p-1)/2} k^{(n+p-1)/2}[(1 + hk + h^2k^2 + \ldots) + (k + hk^2 + h^2k^3 + \ldots)]d_0, \\ &= h^{(n+p-1)/2} k^{(n+p-1)/2}[(1 + hk + h^2k^2 +) + k(1 + hk + h^2k^2 + \ldots)]d_0, \\ &= h^{(n+p-1)/2} k^{(n+p-1)/2}(1 + k)(1 + hk + h^2k^2 + \ldots)d_0, \\ &\leq h^{(n+p-1)/2} k^{(n+p-1)/2}(1 + k)d_0/(1 - hk), \end{aligned}$$

as $hk < 1$ and P is closed.
Thus

$$d(y_{n+p}, y_n) \leq h^{(n+p-1)/2} k^{(n+p-1)/2}(1 + k)d_0/(1 - hk). \tag{3.5}$$

Now for $c \in P^0$, there exists $r > 0$ such that $c - y \in P^0$ if $||y|| < r$. Choose a positive integer N_c such that for all $n \geq N_c$, $||h^{(n+p-1)/2} k^{(n+p-1)/2}(1 + k)d_0/(1 - hk)|| < r$, which implies $c - h^{(n+p-1)/2} k^{(n+p-1)/2}(1 + k)d_0/(1 - hk) \in P^0$ and $h^{(n+p-1)/2} k^{(n+p-1)/2}(1 + k)d_0/(1 - hk) - d(y_{n+p}, y_n) \in P$, using (3.5).
So we have $c - d(y_{n+p}, y_n) \in P^0$, for all $n > N_c$ and for all p by Proposition 2.9 . The same thing is true if $n + p - 1$ is odd. This implies $d(y_{n+p}, y_n) << c$, for all $n > N_c$, for all p. Hence $\{y_n\}$ is a Cauchy sequence in X, which is complete. So $\{y_n\} \to u \in X$. Hence its subsequences

$$\{Ax_{2n}\} \to u \quad \text{and} \quad \{Bx_{2n+1}\} \to u \tag{3.6}$$

$$\{Sx_{2n}\} \to u \quad \text{and} \quad \{Tx_{2n+1}\} \to u \tag{3.7}$$

Case I: Map S is continuous.
As S is continuous we have

$$S^2x_{2n} \to Su, SAx_{2n} \to Su \tag{3.8}$$

Step II: As the pair (A, S) is compatible by Proposition 2.17, we have, $ASx_{2n} \to Su$.
Now,

$$\begin{aligned} d(Su, u) &\leq d(Su, ASx_{2n}) + d(ASx_{2n}, Bx_{2n+1}) + d(Bx_{2n+1}, u) \\ &= d(Su, ASx_{2n}) + d(y_{2n+1}, u) + d(ASx_{2n}, Bx_{2n+1}). \end{aligned}$$

Using (3.1.4) with $x = Sx_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned} d(Su, u) \leq\ & d(Su, ASx_{2n}) + d(y_{2n+1}, u) + \lambda d(ASx_{2n}, S^2x_{2n}) + \mu d(Bx_{2n+1}, Tx_{2n+1}) \\ &+\delta d(S^2x_{2n}, Tx_{2n+1}) + \gamma[d(ASx_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, S^2x_{2n})] \\ &= d(Su, ASx_{2n}) + d(y_{2n+1}, u) + \lambda d(ASx_{2n}, S^2x_{2n}) + \mu d(y_{2n+1}, y_{2n}) \\ &+\delta d(S^2x_{2n}, y_{2n}) + \gamma[d(ASx_{2n}, y_{2n}) + d(y_{2n+1}, S^2x_{2n})] \\ &\leq d(Su, ASx_{2n}) + d(y_{2n+1}, u) + \lambda[d(ASx_{2n}, Su) + d(Su, S^2x_{2n})] \\ &+\mu[d(y_{2n+1}, u) + d(u, y_{2n})] + \delta[d(S^2x_{2n}, Su) + d(Su, u) + d(u, y_{2n})] \\ &+\gamma[d(ASx_{2n}, Su) + d(Su, u) + d(u, y_{2n}) + d(y_{2n+1}, u) + d(u, Su) + d(Su, S^2x_{2n})], \end{aligned}$$

implies

$$\begin{aligned} [1 - \delta - 2\gamma]d(Su, u) \leq\ & [1 + \lambda + \gamma]d(Su, ASx_{2n}) + [\lambda + \delta + \gamma]d(Su, S^2x_{2n})] \\ &+[1 + \mu + \gamma]d(y_{2n+1}, u) + [\mu + \delta + \gamma]d(u, y_{2n}). \end{aligned}$$

As $ASx_{2n} \to Su, S^2x_{2n} \to Su, \{y_{2n}\} \to u$ and $\{y_{2n+1}\} \to u$, using Lemma 2.10, we have $d(Su, u) = 0$, and we get $Su = u$.
Now,

$$\begin{aligned} d(Au, Su) &\leq d(Au, Bx_{2n+1}) + d(Bx_{2n+1}, Su) \\ &= d(y_{2n+1}, Su) + d(Au, Bx_{2n+1}). \end{aligned}$$

Using (3.1.4) with $x = u$ and $y = x_{2n+1}$ we have

$$\begin{aligned} d(Au, Su) \leq\ & d(y_{2n+1}, Su) + \lambda d(Au, Su) + \mu d(Bx_{2n+1}, Tx_{2n+1}) \\ &+\delta d(Su, Tx_{2n+1}) + \gamma[d(Au, Tx_{2n+1}) + d(Bx_{2n+1}, Su)] \\ &= d(y_{2n+1}, Su) + \lambda d(Au, Su) + \mu d(y_{2n+1}, y_{2n}) \\ &+\delta d(Su, y_{2n}) + \gamma[d(Au, y_{2n}) + d(y_{2n+1}, Su)] \\ &\leq d(y_{2n+1}, Su) + \lambda d(Au, Su) + \mu[d(y_{2n+1}, Su) + d(Su, y_{2n})] \\ &+\delta d(Su, y_{2n}) + \gamma[d(Au, Su) + d(Su, y_{2n}) + d(y_{2n+1}, Su)]. \end{aligned}$$

So
$(1 - \lambda - \gamma)d(Au, Su) \leq (\mu + \delta + \gamma)d(y_{2n}, Su) + (1 + \mu + \gamma)d(y_{2n+1}, Su)$.
Using $Su = u$ we have
$(1 - \lambda - \gamma)d(Au, u) \leq (\mu + \delta + \gamma)d(y_{2n}, u) + (1 + \mu + \gamma)d(y_{2n+1}, u)$.
As $\{y_{2n}\} \to u$ and $\{y_{2n+1}\} \to u$, using Lemma 2.10, we have $d(Au, u) = 0$, and we get $Au = Su = u$. Thus u is a point of coincidence of the pair of maps (A, S).
Step III: As $A(X) \subseteq T(X)$, there exists $v \in X$ such that $u = Au = Tv$. So

$$u = Au = Su = Tv. \tag{3.9}$$

Taking $x = u$ and $y = v$ in (3.1.4) we have
$d(Au, Bv) \leq \lambda d(Au, Su) + \mu d(Bv, Tv) + \delta d(Su, Tv) + \gamma[d(Au, Tv) + d(Bv, Su)]$.
Using (3.9) we have
$d(u, Bv) \leq [\mu + \gamma]d(u, Bv)$.
As $\mu+\gamma < 1$, using Proposition 2.2, it follows that $d(Bv, u) = 0$ and we get $Bv = u$.
Thus $Bv = Tv = u$. As the pair (B, T) is weak compatible we get $Bu = Tu$.
Taking $x = u, y = u$ in (3.1.4) and using $Au = Su, Bu = Tu$ we get
$d(Au, Bu) \leq (\delta + 2\gamma)d(Au, Bu)$.
Hence $Au = Bu$, by Proposition 2.2, as $\delta + 2\gamma < 1$, and we have $u = Au = Su =$

$Bu = Tu$. Thus u is a point of coincidence of the four self maps A, B, S and T in this case.

Case II: Map A is continuous.

As A is continuous we have
$A^2x_{2n} \to Au, ASx_{2n} \to Au$.
As the pair (A, S) is compatible by Proposition 2.17, we have, $SAx_{2n} \to Au$.
Now,

$$\begin{aligned} d(Au, u) &\leq d(Au, A^2x_{2n}) + d(A^2x_{2n}, Bx_{2n+1}) + d(Bx_{2n+1}, u) \\ &= d(Au, A^2x_{2n}) + d(y_{2n+1}, u) + d(A^2x_{2n}, Bx_{2n+1}). \end{aligned}$$

Using (3.1.4) with $x = Ax_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned} d(Au, u) \leq\ & d(Au, A^2x_{2n}) + d(y_{2n+1}, u) + \lambda d(A^2x_{2n}, SAx_{2n}) + \mu d(Bx_{2n+1}, Tx_{2n+1}) \\ & + \delta d(SAx_{2n}, Tx_{2n+1}) + \gamma[d(A^2x_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, SAx_{2n})] \\ & \leq d(Au, A^2x_{2n}) + d(y_{2n+1}, u) + \lambda[d(A^2x_{2n}, Au) + d(Au, SAx_{2n})] \\ & + \mu[d(y_{2n+1}, u) + d(u, y_{2n})] + \delta[d(SAx_{2n}, Au) + d(Au, u) + d(u, Tx_{2n+1})] \\ & + \gamma[d(A^2x_{2n}, Au) + d(Au, u) + d(u, Tx_{2n+1}) + d(Bx_{2n+1}, u) + d(u, Au) + d(Au, SAx_{2n})]. \end{aligned}$$

So

$$\begin{aligned} (1 - \delta - 2\gamma)d(Au, u) \leq\ & (1 + \lambda + \gamma)d(Au, A^2x_{2n}) + (\lambda + \delta + \gamma)d(Au, SAx_{2n}) \\ & + (\mu + \delta + \gamma)d(y_{2n}, u) + (1 + \mu + \gamma)d(y_{2n+1}, u). \end{aligned}$$

As $SAx_{2n} \to Au, A^2x_{2n} \to Au, \{y_{2n}\} \to u$ and $\{y_{2n+1}\} \to u$, using Lemma 2.10, we have $d(Au, u) = 0$, and we get $Au = u$.
As $A(X) \subseteq T(X)$ there exists $v_1 \in X$ such that $u = Au = Tv_1$.

Step IV: Now
$d(u, Bv_1) \leq d(Ax_{2n}, Bv_1) + d(Ax_{2n}, u)$
i. e.
$d(u, Bv_1) \leq d(Ax_{2n}, Bv_1) + d(y_{2n}, u)$
Taking $x = x_{2n}, y = v_1$ in (3.1.4) and using $u = Tv_1$

$$\begin{aligned} d(u, Bv_1) \leq\ & \lambda d(Ax_{2n}, Sx_{2n}) + \mu d(Bv_1, Tv_1) + \delta d(Sx_{2n}, Tv_1) \\ & + \gamma[d(Ax_{2n}, Tv_1) + d(Sx_{2n}, Bv_1)] + d(y_{2n}, u) \\ & = \lambda d(y_{2n}, y_{2n-1}) + \mu d(Bv_1, u) + \delta d(y_{2n-1}, u) \\ & + \gamma[d(y_{2n}, u) + d(y_{2n-1}, Bv_1)] + d(y_{2n}, u) \\ & \leq \lambda[d(y_{2n}, u) + d(u, y_{2n-1})] + \mu d(Bv_1, u) + \delta d(y_{2n-1}, u) \\ & + \gamma[d(y_{2n}, u) + d(y_{2n-1}, u) + d(u, Bv_1)] + d(y_{2n}, u). \end{aligned}$$

So
$(1 - \mu - \gamma)d(Bv_1, u) \leq (1 + \lambda + \gamma)d(u, y_{2n}) + (\gamma + \lambda + \delta)d(u, y_{2n-1})$,
As $\{y_{2n}\} \to u$ and $\{y_{2n-1}\} \to u$, and using Lemma 2.10, we have $d(u, Bv_1) = 0$, and we get $Bv_1 = u$. Thus $u = Bv_1 = Tv_1$. As (B, T) is weak compatible we have $Bu = Tu$. Again
$d(u, Bu) \leq d(Ax_{2n}, u) + d(Ax_{2n}, Bu)$
i. e. $d(u, Bu) \leq d(y_{2n}, u) + d(Ax_{2n}, Bu)$
Taking $x = x_{2n}$ and $y = u$ in (3.1.4) and using $Tu = Bu$ we have

$$\begin{aligned} d(u, Bu) \leq\ & d(y_{2n}, u) + \lambda d(Ax_{2n}, Sx_{2n}) + \mu d(Bu, Tu) + \delta d(Sx_{2n}, Tu) \\ & + \gamma[d(Ax_{2n}, Tu) + d(Sx_{2n}, Bu)] \\ & = d(y_{2n}, u) + \lambda d(y_{2n}, y_{2n-1}) + \mu d(Bu, Bu) + \delta d(y_{2n-1}, Bu) \\ & + \gamma[d(y_{2n}, Bu) + d(y_{2n-1}, Bu)] \\ & \leq d(y_{2n}, u) + \lambda[d(y_{2n}, u) + d(u, y_{2n-1})] + +\delta[d(y_{2n-1}, u) + d(u, Bu)] \\ & + \gamma[d(y_{2n}, u) + d(y_{2n-1}, u) + 2d(u, Bu)]. \end{aligned}$$

So
$(1 - \delta - 2\gamma)d(Bu, u) \leq (1 + \lambda + \gamma)d(u, y_{2n}) + (\lambda + \gamma + \delta)d(u, y_{2n-1})$.
As $\{y_{2n}\} \to u$ and $\{y_{2n-1}\} \to u$, using Lemma 2.10, we have $d(u, Bu) = 0$, and we

get $Bu = u$. Thus $u = Bu = Tu = Au$.
Now as $B(X) \subseteq S(X)$, there exists $w_1 \in X$ such that $u = Bu = Sw_1$. Also $d(Aw_1, u) = d(Aw_1, Bu)$.
Using (3.1.4) with $x = w_1$ and $y = u$ with $u = Tu = Bu = Sw_1$ we have

$$\begin{aligned} d(Aw_1, Bu) \leq \quad & \lambda d(Aw_1, Sw_1) + \mu d(Bu, Tu) + \delta d(Sw_1, Tu) + \gamma[d(Aw_1, Tu) + d(Bu, Sw_1)] \\ & = \lambda d(Aw_1, u) + \mu d(u, u) + \delta d(u, u) + \gamma[d(Aw_1, u) + d(u, u)] \\ & = \lambda d(Aw_1, u) + \gamma d(Aw_1, u). \end{aligned}$$

So
$d(Aw_1, u) \leq [\lambda + \gamma] d(Aw_1, u)$.
Hence $Aw_1 = u$, by Proposition 2.2, as $\lambda + \gamma < 1$. Thus $Aw_1 = Sw_1 = u$. As (A, S) is compatible so by Proposition 2.16, (A, S) is weakly compatible.Therefore $Au = Su$. Thus $u = Au = Bu = Su = Tu$. Hence u is a common fixed point of the four self maps in both the cases.
Step V (Uniqueness):Let $w = Aw = Bw = Sw = Tw$ be another common fixed point of the four self maps. Taking $x = u$ and $y = w$ in (3.1.4) we get
$d(Au, Bw) \leq \lambda d(Au, Su) + \mu d(Bw, Tw) + \delta d(Su, Tw) + \gamma[d(Au, Tw) + d(Bw, Su)]$
implies
$d(u, w) \leq [\delta + 2\gamma] d(u, w)$.
Hence $u = w$, by Proposition 2.2, as $\delta + 2\gamma < 1$. Thus the four self maps A, B, S and T have a unique common fixed point. □

On the lines of B. Singh, Shishir Jain [6] our Theorem 3.1 can be extended for six self maps as follows:

Theorem 3.2. *:Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let A, B, S, T, L and M be self mappings on X satisfying:*
(3.2.1) *$L(X) \subseteq ST(X), M(X) \subset AB(X)$;*
(3.2.2) *$AB = BA, ST = TS, LB = BL, MT = TM$;*
(3.2.3) *pair (L, AB) is compatible and the pair (M, ST) is weakly compatible;*
(3.2.4) *for some $\lambda, \mu, \delta, \gamma \in [0, 1)$ with $\lambda + \mu + \delta + 2\gamma < 1$,*

$$\begin{aligned} d(Lx, My) \leq \quad & \lambda d(Lx, ABx) + \mu d(My, STy) + \delta d(ABx, STy) \\ & + \gamma[d(Lx, STy) + d(ABx, My)]. \end{aligned}$$

for all $x, y \in X$.
Then A, B, S, T, L and M have a unique common fixed point in X.

Taking $B = A$ and $T = S$ in Theorem 3.1, we get

Corollary 3.3. *:Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let A and S be self mappings on X satisfying:*
(3.3.1) *$A(X) \subseteq S(X)$;*
(3.3.2) *pair (A, S) is compatible ;*
(3.3.3) *one of A or S is continuous;*
(3.3.4) *for some $\lambda, \mu, \delta, \gamma \in [0, 1)$ with $\lambda + \mu + \delta + 2\gamma < 1$ such that for all $x, y \in X$*
$d(Ax, Ay) \leq \lambda d(Ax, Sx) + \mu d(Ay, Sy) + \delta(Sx, Sy) + \gamma[d(Ax, Sy) + d(Sx, Ay)]$.
Then A and S have a unique common fixed point in X.

Taking $S = I$, the identity map on X, in above Corollary we get

Corollary 3.4. *: Let (X, d) be a complete cone metric space. Let A be self mapping on X satisfying:*
(3.4.1) *for some $\lambda, \mu, \delta, \gamma \in [0, 1)$ with $\lambda + \mu + \delta + 2\gamma < 1$*
$d(Ax, Ay) \leq \lambda d(Ax, x) + \mu d(Ay, y) + \delta d(x, y) + \gamma[d(Ax, y) + d(Ay, x)]$,
for all $x, y \in X$.
Then the map A has the unique fixed point in X and for any $x \in X$, the iterative sequence $\{A^n x\}$ converges to the fixed point.

Proof. : Existence and uniqueness of the fixed point follow from Corollary 3.3, by taking $S = I$ there. Taking $T = S = I, B = A$ and $x_0 = x$ in Theorem 3.1, we have $y_0 = Ax, y_1 = A^2 x, \ldots, y_{n+1} = A^n x$ etc. Thus for each x, the sequence $\{A^n x\}$ converges to the fixed point z. □

Taking $\gamma = 0$ in Corollary 3.4 we have

Corollary 3.5. *:Let (X, d) be a complete cone metric space. Let A be self mapping on X satisfying:*
(3.5.1) *for some $\lambda, \mu, \delta, \gamma \in [0, 1)$ with $\lambda + \mu + \delta + \gamma < 1$,*
$d(Ax, Ay) \leq \lambda d(Ax, x) + \mu d(Ay, y) + \delta d(x, y)$, *for all $x, y \in X$.*
Then the map A has the unique fixed point in X and for any $x \in X$, the iterative sequence $\{A^n x\}$ converges to the fixed point.

Remark 3.6. *: Taking $\lambda = k$ and $\mu = \delta = 0$ in Corollary 3.5, we get Theorem 1 of Huang et. al* [3] *even for a non-normal cone metric space.*

Remark 3.7. *: Taking $\lambda = \mu = k$ and $\delta = 0$ in Corollary 3.5, $k \in [0, 1/2)$ and we get Theorem 3 of Huang et. al* [3] *even for a non-normal cone metric space.*

Remark 3.8. *: Taking $\lambda = \mu = \delta = 0$ and $\gamma = k$ in Corollary 3.4, $k \in [0, 1/2)$ and we get Theorem 4 of Huang et. al* [3] *even for a non-normal cone metric space.*

Example 3.9 (of Theorem 3.1). *: Let $X = R^+, E = R^2, P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\} \subseteq R^2$ be a cone in E. Fix a real number $\alpha > 0$ and define $d : X \times X \to E$ by $d(x, y) = |x - y|(1, \alpha)$. Then (X, d) is a complete cone metric space. Define self maps A, B, S and T on X as follows:*
$A(x) = B(x) = \frac{3x}{4} \qquad S(x) = T(x) = 2x, forallx.$
Conditions $(3.1.1), (3.1.2) and (3.1.3)$ *of Theorem* 3.1 *hold trivially. If we take* $\lambda = \frac{2}{5}, \mu = \frac{1}{110}, \gamma = \frac{1}{11}$ *and* $\delta = \frac{1}{4}$ *the contractive condition* (3.1.4) *of above said Theorem holds good and* 0 *is the unique common fixed point of the maps A, B, S and T.*

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