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COMMON FIXED POINT OF AN ARBITRARY FAMILY OF NONEXPANSIVE MAPPINGS IN A LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE

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Abstract. In this paper the existence of a common fixed point for an arbitrary family of nonexpansive mappings is proved in a locally convex topological vector space. Our theorem is a (partial) generalization of the theorem of Lim [8] to this more general setting. Our main tool is the gauge function in locally convex spaces.

1. Introduction

Let X be a vector space. A point $x \in X$ is said to be a fixed point of the mapping T, with domain $D(T)$ and range $R(T)$ in X if and only if $x = Tx$.

Fixed point theorems have been studied by many authors. The oldest fixed point theorem is probably the contraction mapping principle.

Let (X, ρ) be a complete metric space and $f : X \to X$ be a contraction map, that is, there exists $\alpha \in (0, 1)$ such that

$$\rho(fx, fy) \le \alpha\rho(x, y)\ \forall x, y \in X.$$

Then f has a unique fixed point in X. Moreover the sequence $\{x_n\}$, defined for any arbitrary $x_o \in X$ by

$$x_n = f^n x_o$$

converges to the fixed point.

From this theorem arose the study of fixed point theorems in different settings. Thousands of fixed point theorems have been studied and proved by various authors. Notable of the early fixed point theorems are the Brouwer's, the Schauder Tychonov's and so many others. Although many fixed point theorems have been flourishing for both single maps and groups of maps in metric and Banach spaces, only very few have been reported in the general topological vector spaces. The Caristi fixed point theorem provides a link to some of the current known fixed point theorems, see ([4]). Existence and approximation results for common fixed

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points of families of mappings have also been studied by various authors (see, for example [1, 2, 5, 7, 8]).

R. De Marr [5] studied the existence of a common fixed point of an arbitrary family of commuting nonexpansive mappings of a Banach space into itself and proved the following theorem:

Theorem 1.1. *Let E be a Banach space and K a nonempty compact convex subset of E. If $\mathcal{F}$ is a nonempty commuting family of nonexpansive mappings of K into itself, then the family $\mathcal{F}$ has a common fixed point in K.*

Browther [3] proved the result of De Marr in a uniformly convex Banach space, requiring that K be only nonempty closed bounded and convex. He proved the following theorem:

Theorem 1.2. *Let E be a uniformly convex Banach space, K a nonempty closed convex and bounded subset of E, $\{T_\lambda\}$ a commuting family of nonexpansive self mappings of K. Then the family $\{T_\lambda\}$ has a common fixed point in K.*

Belluce and Kirk [2] proved the existence of a common fixed point of a finite family of nonexpansive mappings in a reflexive Banach space with normal structure.

Lim [8] proved the existence of a common fixed point for an arbitrary family of nonexpansive mappings in a Banach space. His theorem is as follows:

Theorem 1.3. *Let K be a nonempty weakly compact convex subset of a Banach space and assume that K has normal structure. Let $\mathcal{F}$ be an arbitrary family of commuting nonexpansive maps from K into itself. Then $\mathcal{F}$ has a common fixed point.*

The above theorem of Lim is a generalization of the results of Belluce and Kirk [2]. It is our purpose in this paper to generalize the result of Lim to an arbitrary locally convex topological vector space.

2. Main Result

Let K be a convex subset of a locally convex topological vector space E. Then there is a coarsest topology compatible with the algebraic structure of E in which K is a neighborhood. Under this topology E is a locally convex space and a base of neighborhoods is formed by the sets

$$\{\epsilon K :\ \epsilon > 0\}.$$

Now let p be the gauge of K, and let $\{x_\alpha\}_{\alpha\in\Delta}$ be a bounded net in K. For each $x \in K$ and every $\beta \in \Delta$, let

$$\begin{aligned} r_\beta(x) &= \sup\{p(x - x_\alpha) :\ \alpha \geq \beta\} \\ r(x) &= \inf\{r_\delta(x) :\ \delta \in \Delta\} = \limsup p(x - x_\alpha) \end{aligned}$$

and let $r = \inf\{r(x) :\ x \in K\}$.
The set $\{x \in K :\ r(x) = r\}$ and the number r will be called the asymptotic center

and asymptotic radius respectively of $\{x_\alpha\}_{\alpha\in\Delta}$.
It is easy to see that

(1) For each $x \in K$, $\{r_\beta(x)\}_{\beta\in\Delta}$ is a decreasing net with limit $r(x)$.
(2) $r(x) = 0$ iff $x_\alpha \to x$
(3) $|r(x) - r(y)| \le p(x - y)\ \forall x, y \in K$.

A convex subset K of a locally convex space will be said to have normal structure if every bounded convex subset D of C with $|D| > 1$ contains a point x such that $\sup\{p(x - y) :\ y \in D\} < d(D)$.

By mimicking Lim's theorem (Theorem 1.3) by replacing $||$ with p, where applicable and with the definitions and conditions above, we have established the following theorem:

Theorem 2.1. *Let K be a nonempty weakly compact convex absorbent and balanced subset of a locally convex topological vector space and assume that K has normal structure. Let $\mathcal{F}$ be an arbitrary family of commuting nonexpansive maps from K into itself. Then $\mathcal{F}$ has a common fixed point.*

Thus this theorem extends Lim's theorem from a Banach space to a locally convex space but with the additional condition that the subset K need be balanced and absorbent.

References

[1] H. H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 202 (1996), 150-159.
[2] L. P. Belluce and W. A. Kirk, Fixed point theorem for families of contraction mappings, Pacific J. Math, 18 (1966), 213-217.
[3] F. E. Browder, Nonexpansive nonlinear operators in Banach space, Proc. Nat. Acad. Sci. USA 54 (1965), 1041-1044.
[4] J. V. Caristi, Ph. D Thesis, University of Iowa, 1975.
[5] R. De Marr, Common fixed points for commuting contraction mappings, Pacific J. Math. 13 (1963), 1139-1141.
[6] M. Edelstein, Fixed point theorems in uniformly convex Banach spaces, Proc. Amer. Math. Soc. 44 (1974), 369-374.
[7] J. S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 302 (2005), 509-520.
[8] T. C. Lim, A fixed point theorem for families of nonexpansive mappings, Pacific J. Math. 53 (1974), 487-493.
[9] A. P. Robertson and W. J. Robertson, Topological vector spaces, Cambridge University Press, 1973.
[10] H. H. Schaefer, Topological vector spaces, Springer-Verlag New York, 1971.

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