

SZÁSZ-SCHURER OPERATORS VIA GENERAL-APPELL POLYNOMIALS

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ABSTRACT. This manuscript associates with a study of general-Appell Polynomials. In this research work, we construct a new sequence of Szász-Schurer type operators via general-Appell Polynomials to discuss approximation properties for the Lebesgue integrable functions $(L_1[0, \infty))$. Further, estimates in view of test functions and central moments are studied. Next, rate of convergence is discussed with the aid of Korovkin theorem and Voronovskaja type theorem. Moreover, direct approximation results in terms of modulus of continuity of first and second order, Peetre's K-functional, Lipschitz type space, and the r^{th} order Lipschitz type maximal functions are investigated. In subsequent section, we present weighted approximation results, and statistical approximation theorems are discussed.

1. INTRODUCTION AND PRELIMINARIES

The development of operator theory gained momentum in the late 19th century. A crucial aspect of approximation in this field is finding simple, computationally efficient approximations that capture the essential properties of complex functions. These approximations have numerous applications in fields like quantum mechanics, signal processing, and control theory, providing powerful tools for problem-solving and system analysis.

In recent years, research in approximation theory within operator theory has continued to evolve, with a focus on advanced techniques applicable to data science and machine learning. The Weierstrass approximation theorem, formulated by Weierstrass [1] in 1885, is a foundational result in approximation theory. This theorem has been the subject of interest for many prominent mathematicians, who have sought to provide more straightforward and comprehensible proofs.

One notable contribution is the work of Bernstein [2], who developed a sequence of polynomials known as Bernstein polynomials in 1912. These polynomials provide a concise demonstration of the Weierstrass approximation theorem using the

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binomial distribution as follows:

$$B_\kappa(\tilde{g}; u) = \sum_{\nu=0}^{\kappa} \tilde{g}\left(\frac{\nu}{\kappa}\right) \binom{\kappa}{\nu} u^\nu (1-u)^{\kappa-\nu}, \quad u \in [0, 1], \quad (1.1)$$

where $\tilde{g}$ is a continuous and bounded function on $[0, 1]$. The sequences of operators in (1.1) restrict the approximation for continuous functions on bounded interval $[0, 1]$. In order to discuss approximation properties on unbounded interval $[0, \infty)$, Szász [3] provided the modifications to the sequences in (1.1) which has a significant role to the evolution of operator theory as follows:

$$S_\kappa(\tilde{g}; u) = e^{-(\kappa+s)u} \sum_{\nu=0}^{\infty} \frac{((\kappa+s)u)^\nu}{\nu!} \tilde{g}\left(\frac{\nu}{\kappa}\right), \quad \kappa \in \mathbb{N}, \quad (1.2)$$

where real valued function $\tilde{g} \in C[0, \infty)$. As given in (1.2), linear positive operators are limited solely to a continuous functional space. Many integral variants of these sequences of operators are obtained in order to approximate the longer class of functions, i.e., the space of Lebesgue measurable functions. Moreover, in order to discuss better approximation properties, Schurer [4] in 1962 constructed a new sequence of Bernstein operators [2] is given as $B_{m,p} : C([0, 1+s]) \rightarrow C([0, 1])$ and defined by:

$$B_{\kappa+s}(\tilde{g}; u) = \sum_{\nu=0}^{\kappa+s} \tilde{g}\left(\frac{\nu}{\kappa}\right) \binom{\kappa+s}{\nu} u^\nu (1-u)^{\kappa+s-\nu}, \quad u \in [0, 1+p], \quad (1.3)$$

where $s \in \mathbb{N} \cup \{0\}$ and $\tilde{g} \in C[0, 1+p]$.

Further, many mathematicians, e.g., Braha et al. ([5], [6]), Aslan ([7], [8]), Mohiuddine et al. ([9], [10]), Mursaleen et al. ([11], [12]), Özger et al. ([13], [14]), Acu et al. ([15], [16]), Ayman Mursaleen et al. ([17], [18]), Ansari et al. ([19], [20]), Khan et al. [21], Nasiruzzaman [22], Rao et al. ([23], [24]), and Jha et al. [25] provided a number of generalizations for these kinds of sequences to investigate flexibility in approximation properties across several functional spaces. Recently, Raza et al. [26] provided a class of sequence of operators $G_{\kappa,A}(\cdot; \cdot)$, $\kappa \in \mathbb{N}$, given by the formula

$$G_{\kappa,A}(\tilde{g}; u) = \frac{e^{-\kappa u}}{\tilde{\Lambda}(1)\tilde{\xi}(h, 1)} \sum_{\nu=0}^{\infty} \frac{A_{p,\nu}(\kappa u, h)}{\nu!} \tilde{g}\left(\frac{\nu}{\kappa}\right), \quad u \in \mathbb{R}_0^+, \quad (1.4)$$

where $A_{p,\nu}$ is the two variable Appell polynomials (see [26]).

The operators given by (1.4) are positive and linear. The basic information about positive linear operators, including their generalizations and applications can be observed in [29].

As the operators described in (1.4) are limited for continuous function only, we present a sequence of positive linear operators to provide approximations in larger class of functions. To achieve better approximation properties in various functional spaces we introduced the Szász-Schurer operators in context of general Appell Polynomials as:

$$\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) = \sum_{\nu=0}^{\infty} A_{\nu}^p((\kappa+s)u, h) \tilde{g}\left(\frac{\nu}{\kappa+s}\right), \quad \text{for } u \in \mathbb{R}_0^+, \quad (1.5)$$

where

$$A_\nu^p((\kappa + s)u, h) = \frac{e^{-(\kappa+s)u}}{\tilde{\Lambda}(1)\tilde{\xi}(h, 1)} \frac{A_{p,\nu}((\kappa + s)u, h)}{\nu!}.$$

Lemma 1.1. *The sequence of operators introduced in (1.5) are linear.*

Proof. In view of (1.5), we have

$$\begin{aligned} \mathcal{R}_{\kappa+s}^A(\lambda_1 \tilde{g}_1 + \lambda_2 \tilde{g}_2; u) &= \sum_{\nu=0}^{\infty} A_\nu^p((\kappa + s)u, h) \left(\lambda_1 g_1 \left(\frac{\nu}{\kappa + s} \right) + \lambda_2 g_2 \left(\frac{\nu}{\kappa + s} \right) \right) \\ &= \lambda_1 \sum_{\nu=0}^{\infty} A_\nu^p((\kappa + s)u, h) g_1 \left(\frac{\nu}{\kappa + s} \right) + \lambda_2 \sum_{\nu=0}^{\infty} A_\nu^p((\kappa + s)u, h) g_2 \left(\frac{\nu}{\kappa + s} \right) \\ &= \lambda_1 \mathcal{R}_{\kappa+s}^A(\tilde{g}_1; u) + \lambda_2 \mathcal{R}_{\kappa+s}^A(\tilde{g}_2; u). \end{aligned}$$

□

Lemma 1.2. *As discussed by Raza et al. in [26], we can have the following equalities:*

$$\begin{aligned} \sum_{\nu=0}^{\infty} \frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \tilde{\Lambda}(1) e^{\kappa u} \tilde{\xi}(h, 1); \\ \sum_{\nu=0}^{\infty} \nu \frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \left[\kappa u \tilde{\Lambda}(1) \tilde{\xi}(h, 1) + \tilde{\Lambda}(1) \tilde{\xi}'(h, 1) + \tilde{\Lambda}'(1) \tilde{\xi}(h, 1) \right] e^{\kappa u}; \\ \sum_{\nu=0}^{\infty} \nu^2 \frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \left[(\kappa^2 u^2 + \kappa u) \tilde{\Lambda}(1) \tilde{\xi}(h, 1) + (2\kappa u + 1) [\tilde{\Lambda}(1) \tilde{\xi}'(h, 1) \right. \\ &\quad \left. + \tilde{\Lambda}'(1) \tilde{\xi}(h, 1)] + 2\tilde{\xi}'(h, 1) \tilde{\Lambda}'(1) + \tilde{\xi}''(h, 1) \tilde{\Lambda}(1) \right. \\ &\quad \left. + \tilde{\xi}(h, 1) \tilde{\Lambda}''(1) \right] e^{\kappa u}. \end{aligned}$$

Lemma 1.3. *Let $\tilde{g}_s(y) = y^s$, $s \in \{0, 1, 2\}$ be the test functions by (1.4). Then, we have*

$$\begin{aligned} \mathcal{R}_{\kappa+s}^A(1; u) &= 1; \\ \mathcal{R}_{\kappa+s}^A(y; u) &= u + \frac{1}{\kappa + s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right); \\ \mathcal{R}_{\kappa+s}^A(y^2; u) &= u^2 + \frac{1}{\kappa + s} u \left(2 \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + 2 \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} + 1 \right) \\ &\quad + \frac{1}{(\kappa + s)^2} \left(\frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} + 2 \frac{\tilde{\xi}'(h, 1) \tilde{\Lambda}'(1)}{\tilde{\xi}(h, 1) \tilde{\Lambda}(1)} + \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} \right); \end{aligned}$$

for each $u \in \mathbb{R}_0^+$. In the direction of (1.5), we have

Proof. With the aid of linearity property and Lemma 1.2, we can easily prove Lemma 1.3. Hence, the details can be omitted. □

Lemma 1.4. *Let $\gamma_s^u(y) = (y - u)^s$, $s = 0, 1, 2$. Then, for the operators (1.5), we have central moments $\mathcal{R}_{\kappa+s}^A(\gamma_s^u(y), u)$ as:*

$$\begin{aligned}\mathcal{R}_{\kappa+s}^A(\gamma_0^u(y); u) &= 1; \\ \mathcal{R}_{\kappa+s}^A(\gamma_1^u(y); u) &= \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right); \\ \mathcal{R}_{\kappa+s}^A(\gamma_2^u(y); u) &= \frac{1}{\kappa+s} u + \frac{1}{(\kappa+s)^2} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 2 \frac{\tilde{\xi}'(h, 1) \tilde{\Lambda}'(1)}{\tilde{\xi}(h, 1) \tilde{\Lambda}(1)} \right. \\ &\quad \left. + \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} \right); \end{aligned}$$

for each $u \in \mathbb{R}_0^+$.

Proof. With the aid of linearity property and Lemma 1.3, we can easily prove Lemma 1.4. $\square$

In the following sections, we examine the rate of convergence of operators and their approximation order. Specifically, we discuss direct results both locally and globally across several spaces. In the final section, we explore some results of A-Statistical approximation in various functional spaces.

2. APPROXIMATION PROPERTIES: UNIFORM RATE OF CONVERGENCE AND ORDER OF APPROXIMATION

Definition 2.1. [29] *The modulus of smoothness for $\tilde{g} \in C[0, \infty)$ is given by*

$$\omega(\tilde{g}; \delta) = \sup_{|u_1 - u_2| \leq \delta} |\tilde{g}(u_1) - \tilde{g}(u_2)|, \quad u_1, u_2 \in [0, \infty).$$

Theorem 2.1. *Let $\mathcal{R}_{\kappa+s}^A(., .)$ be a sequence of operators described in Eq. (1.5). Then, on each closed and bounded subset of $[0, \infty)$, $\mathcal{R}_{\kappa+s}^A(\tilde{g}; .) \rightrightarrows \tilde{g}$, for all $\tilde{g} \in C_B[0, \infty)$, where $\rightrightarrows$ denotes uniform convergence.*

Proof. Considering the classical Korovkin type theorem [27], which characterizes the uniform convergence for the sequence of positive linear operators, it is enough to note that

$$\lim_{\kappa \rightarrow \infty} \mathcal{R}_{\kappa+s}^A(\tilde{g}_s; u) = u^s, \quad s = 0, 1, 2,$$

uniformly on all closed and bounded subsets of $[0, \infty)$. We can easily establish this result with the help of Lemma 1.3. $\square$

Now, we show that Voronovskaja type asymptotic approximation theorem for the $\mathcal{R}_{\kappa+s}^A(., .)$ given in (1.5).

Theorem 2.2. *Let $\tilde{g} \in C_B[0, \infty)$ and $\tilde{g}', \tilde{g}''$ exist at a fixed point $u \in [0, \infty)$. Then, we get*

$$\lim_{\kappa \rightarrow \infty} (\kappa + s) (\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)) = \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \tilde{g}'(u) + \frac{u}{2} \tilde{g}''(u).$$

Proof. In accordance with Taylor's formula for the function $\tilde{g}$, we have

$$\tilde{g}(y) = \tilde{g}(u) + (y - u)\tilde{g}'(u) + \frac{1}{2}(y - u)^2\tilde{g}''(u) + t(y, u)(y - u)^2, \quad (2.1)$$

where $t(y, u)$ is the Peano remainder and

$$\lim_{y \rightarrow u} t(y, u) = 0.$$

Applying operators on both the sides in (2.1), we yield

$$\begin{aligned} (\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)) &= \tilde{g}'(u)\mathcal{R}_{\kappa+s}^A((y - u); u) + \frac{1}{2}\tilde{g}''(u)\mathcal{R}_{\kappa+s}^A((y - u)^2; u) \\ &+ \mathcal{R}_{\kappa+s}^A(t(y, u)(y - u)^2; u). \end{aligned}$$

In view of Lemma 1.4

$$\begin{aligned} (\kappa + s)(\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)) &= \tilde{g}'(u) \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) + \frac{u}{2}\tilde{g}''(u) \\ &+ \frac{1}{2} \frac{\tilde{g}''(u)}{(\kappa + s)} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 2 \frac{\tilde{\xi}'(h, 1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h, 1)\tilde{\Lambda}(1)} \right. \\ &\left. + \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} \right) + (\kappa + s)\mathcal{R}_{\kappa+s}^A(t(y, u)(y - u)^2; u). \end{aligned}$$

Operate the limits on both the sides of the above expression, we get

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} (\kappa + s)(\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)) &= \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \tilde{g}'(u) + \frac{u}{2}\tilde{g}''(u) \\ &+ \lim_{\kappa \rightarrow \infty} (\kappa + s)\mathcal{R}_{\kappa+s}^A(t(y, u)(y - u)^2; u). \end{aligned}$$

Now, we need to show that

$$\lim_{\kappa \rightarrow \infty} (\kappa + s)\mathcal{R}_{\kappa+s}^A(t(y, u)(y - u)^2; u) = 0.$$

In view of Cauchy-Schwarz inequality, we calculate the last term of the above expression as:

$$(\kappa + s)\mathcal{R}_{\kappa+s}^A(t(y, u)(y - u)^2; u) \leq \sqrt{\mathcal{R}_{\kappa+s}^A(t^2(y, u); u)} \sqrt{(\kappa + s)^2 \mathcal{R}_{\kappa+s}^A((y - u)^4; u)}. \quad (2.2)$$

We see that $t^2(u, u) = 0$ and $t^2(y, u) \in C_B[0, \infty)$. Thus we have

$$\lim_{\kappa \rightarrow \infty} \mathcal{R}_{\kappa+s}^A(t^2(y, u); u) = t^2(u, u) = 0. \quad (2.3)$$

From (2.2) and (2.3) it follows that

$$\lim_{\kappa \rightarrow \infty} (\kappa + s)\mathcal{R}_{\kappa+s}^A(t(y, u)(y - u)^2; u) = 0.$$

Hence, the proof is completed. $\square$

In accordance with Shisha et al. [28], order of the convergence relative to Ditzian-Totik modulus of continuity can easily be proved.

Theorem 2.3. Consider $\tilde{g} \in C_B[0, \infty)$ and for the operators $\mathcal{R}_{\kappa+s}^A(\cdot; \cdot)$ presented in Eq. (1.5), we acquire

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq 2\omega(\tilde{g}; \delta),$$

where $\delta = \sqrt{\mathcal{R}_{\kappa+s}^A((y-u)^2; u)}$.

Proof. In accordance with Lemma 1.3, 1.4 and Cauchy-Schwartz inequality, we have

$$\begin{aligned} |\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq \mathcal{R}_{\kappa+s}^A(|\tilde{g}(y) - \tilde{g}(u)|; u) \\ &\leq \mathcal{R}_{\kappa+s}^A\left(\left(1 + \frac{|y-u|}{\delta}\right)\omega(\tilde{g}, \delta); u\right) \\ &\leq \omega(\tilde{g}, \delta) \left[1 + \frac{1}{\delta} \mathcal{R}_{\kappa+s}^A(|y-u|; u)\right] \\ &\leq \omega(\tilde{g}, \delta) \left[1 + \frac{1}{\delta} \sqrt{\mathcal{R}_{\kappa+s}^A((y-u)^2; u)}\right] \end{aligned}$$

By selecting $\delta = \sqrt{\mathcal{R}_{\kappa+s}^A((y-u)^2; u)}$, we obtained the desired result. $\square$

3. LOCALLY APPROXIMATION RESULTS

We recall a few functional spaces and functional relations in this part as: $C_B[0, \infty)$: Denotes a real valued functional space which acquires bounded and continuous functions. Now, Peetre's K-functional [29] is given by

$$K_2(\tilde{g}, \delta) = \inf_{\tilde{h} \in C_B^2[0, \infty)} \left\{ \|\tilde{g} - \tilde{h}\|_{C_B[0, \infty)} + \delta \|\tilde{h}''\|_{C_B^2[0, \infty)} \right\},$$

where $C_B^2[0, \infty) = \{\tilde{h} \in C_B[0, \infty) : \tilde{h}', \tilde{h}'' \in C_B[0, \infty)\}$ associated with the norm $\|\tilde{g}\| = \sup_{0 \leq y < \infty} |\tilde{g}(y)|$ and second order Ditzian-Totik modulus of smoothness is presented by

$$\omega_2(\tilde{g}; \sqrt{\delta}) = \sup_{0 < k \leq \sqrt{\delta}} \sup_{y \in [0, \infty)} |\tilde{g}(y+2k) - 2\tilde{g}(y+k) + \tilde{g}(y)|.$$

We revisit a result from DeVore and Lorentz ([29] page no. 177, Theorem 2.4) as:

$$K_2(\tilde{g}; \delta) \leq C\omega_2(\tilde{g}; \sqrt{\delta}), \quad (3.1)$$

where C is an absolute constant. To establish the next result, we consider the auxiliary operator defined as:

$$\hat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) = \mathcal{R}_{\kappa+s}^A(\tilde{g}; u) + \tilde{g}(u) - \tilde{g}\left(u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)}\right)\right). \quad (3.2)$$

where $\tilde{g} \in C_B[0, \infty)$, $u \geq 0$. From Eq. (3.2), one can yield

$$\hat{\mathcal{R}}_{\kappa+s}^A(1; u) = 1, \quad \hat{\mathcal{R}}_{\kappa+s}^A(\gamma_1^u(y); u) = 0 \text{ and } |\hat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u)| \leq 3\|\tilde{g}\|. \quad (3.3)$$

Lemma 3.1. If $u \geq 0$, we have

$$|\hat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \theta(u)\|\tilde{g}''\|,$$

where $\tilde{g} \in C_B^2[0, \infty)$ and $\theta(u) = \hat{\mathcal{R}}_{\kappa+s}^A(\gamma_1^u(y); u) + (\hat{\mathcal{R}}_{\kappa+s}^A(\gamma_1^u(y); u))^2$.

Proof. For $\tilde{g} \in C_B^2[0, \infty)$ and by Taylor expansion, we have

$$\tilde{g}(y) = \tilde{g}(u) + (y - u)\tilde{g}'(u) + \int_u^y (y - v)\tilde{g}''(v)dv. \quad (3.4)$$

Implementing the auxiliary operators $\widehat{\mathcal{R}}_{\kappa+s}^A(.,.)$ introduced in Eq.(3.2) to both sides of Eq. (3.4), we get

$$\widehat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u) = \tilde{g}'(u)\widehat{\mathcal{R}}_{\kappa+s}^A(\gamma_1^u(y); u) + \widehat{\mathcal{R}}_{\kappa+s}^A\left(\int_u^y (y - v)\tilde{g}''(v)dv; u\right).$$

Using the Eqs. (3.3) and (3.4), one yield

$$\begin{aligned} \widehat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u) &= \widehat{\mathcal{R}}_{\kappa+s}^A\left(\int_u^y (y - v)\tilde{g}''(v)dv; u\right) \\ &= \mathcal{R}_{\kappa+s}^A\left(\int_u^y (y - v)\tilde{g}''(v)dv; u\right) \\ &\quad - \int_u^{u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right)} \left(u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) - v \right) \tilde{g}''(v)dv, \\ |\widehat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq \left| \widehat{\mathcal{R}}_{\kappa+s}^A\left(\int_u^y (y - v)\tilde{g}''(v)dv; u\right) \right| \\ &\quad + \left| \int_u^{u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right)} \left(u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) - v \right) \tilde{g}''(v)dv \right|. \end{aligned} \quad (3.5)$$

Since,

$$\left| \int_u^y (y - v)\tilde{g}''(v)dv \right| \leq (y - u)^2 \|\tilde{g}''\|, \quad (3.6)$$

then

$$\left| \int_u^{u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right)} \left(u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) - v \right) \tilde{g}''(v)dv \right| \quad (3.7)$$

$$\leq \left(\frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h,1)}{\tilde{\xi}(h,1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \right)^2 \|\tilde{g}''\|. \quad (3.8)$$

In accordance with (3.5), (3.6) and (3.7), we accquire

$$\begin{aligned} |\widehat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq \left\{ \widehat{\mathcal{R}}_{\kappa+s}^A(\gamma_2^u(y); u) + \left(\frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \right)^2 \right\} \|\tilde{g}''\| \\ &= \theta(u) \|\tilde{g}''\|. \end{aligned}$$

Which proves the required result. $\square$

Theorem 3.2. *Let $\tilde{g} \in C_B^2[0, \infty)$. Then, there corresponds a non-negative constant $\tilde{C} > 0$ such that*

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \tilde{C} \omega_2(\tilde{g}; \sqrt{\theta(u)}) + \omega(\tilde{g}; \mathcal{R}_{\kappa+s}^A(\gamma_1^u(y); u),$$

where $\theta(u)$ is given by in Lemma 3.1.

Proof. For $\tilde{h} \in C_B^2[0, \infty)$ and $\tilde{g} \in C_B[0, \infty)$ and with the definition of $\widehat{\mathcal{R}}_{\kappa+s}^A(\cdot; \cdot)$ given in (3.2), we get

$$\begin{aligned} |\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq |\widehat{\mathcal{R}}_{\kappa+s}^A(\tilde{g} - \tilde{h}; u)| + |(\tilde{g} - \tilde{h})(u)| + |\widehat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \\ &\quad + \left| \tilde{g} \left(u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \right) - \tilde{g}(u) \right|. \end{aligned}$$

In accordance with Lemma 3.1 and inequalities mentioned in Eq. (3.3), we acquire

$$\begin{aligned} |\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq 4\|\tilde{g} - \tilde{h}\| + |\widehat{\mathcal{R}}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \\ &\quad + \left| \tilde{g} \left(u + \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \right) - \tilde{g}(u) \right| \\ &\leq 4\|\tilde{g} - \tilde{h}\| + \theta(u) \|\tilde{h}''\| + \omega(\tilde{g}; \mathcal{R}_{\kappa+s}^A(y - u); u). \end{aligned}$$

By employing Eq. (3.1), we established the desired result. $\square$

Now, we address the next result in Lipschitz type space presented by [31] as:

$$Lip_M^{\zeta_1, \zeta_2}(\eta) := \left\{ \tilde{g} \in C_B[0, \infty) : |\tilde{g}(y) - \tilde{g}(u)| \leq \tilde{M} \frac{|y - u|^\eta}{(y + \zeta_1 u + \zeta_2 u^2)^{\frac{\eta}{2}}} : u, y \in (0, \infty) \right\},$$

where $\tilde{M} > 0$, $0 < \eta \leq 1$ and $\zeta_1, \zeta_2 > 0$.

Theorem 3.3. *Consider sequence of linear positive operators in (1.5) and $\tilde{g} \in Lip_M^{\zeta_1, \zeta_2}(\eta)$, one obtain*

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \tilde{M} \left(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2} \right)^{\frac{\eta}{2}}, \quad (3.9)$$

where $0 < \eta \leq 1$, $\zeta_1, \zeta_2 \in (0, \infty)$ and $\lambda(y) = \mathcal{R}_{\kappa+s}^A(\gamma_2^u(y); u)$.

Proof. For $\eta = 1$ and $u \geq 0$, we get

$$\begin{aligned} |\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq \mathcal{R}_{\kappa+s}^A(|\tilde{g}(y) - \tilde{g}(u)|; u) \\ &\leq \tilde{M} \mathcal{R}_{\kappa+s}^A \left(\frac{|y - u|}{(y + \zeta_1 u + \zeta_2 u^2)^{\frac{1}{2}}}; u \right). \end{aligned}$$

Since $\frac{1}{y + \zeta_1 u + \zeta_2 u^2} < \frac{1}{\zeta_1 u + \zeta_2 u^2}$, for each $u \in (0, \infty)$, we acquire

$$\begin{aligned} |\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq \frac{\tilde{M}}{(\zeta_1 u + \zeta_2 u^2)^{\frac{1}{2}}} (\mathcal{R}_{\kappa+s}^A(\gamma_2^u(y); u)^{\frac{1}{2}} \\ &\leq \tilde{M} \left(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2} \right)^{\frac{1}{2}}, \end{aligned}$$

which indicates that the Theorem 3.3 is valid for $\eta = 1$. Next, we examine the case where $\eta \in (0, 1)$ and in accordance with Hölder's inequality by selecting $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$, we obtain

$$\begin{aligned} |\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq (|\mathcal{R}_{\kappa+s}^A(|\tilde{g}(y) - \tilde{g}(u)|^{\frac{2}{\eta}}; u)|^{\frac{\eta}{2}} \\ &\leq \tilde{M} \left(\mathcal{R}_{\kappa+s}^A \left(\frac{|y - u|^2}{(y + \zeta_1 u + \zeta_2 u^2)}; x \right) \right)^{\frac{\eta}{2}}. \end{aligned}$$

Since $\frac{1}{y + \zeta_1 u + \zeta_2 u^2} < \frac{1}{\zeta_1 u + \zeta_2 u^2}$, for all $u \in (0, \infty)$, for all $u \in (0, \infty)$, one get

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \tilde{M} \left(\frac{|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)|^{\frac{2}{\eta}}}{\zeta_1 u + \zeta_2 u^2} \right)^{\frac{\eta}{2}} \leq \tilde{M} \left(\frac{\lambda(u)}{\zeta_1 u + \zeta_2 u^2} \right)^{\frac{\eta}{2}}.$$

Thus, we yield the desired result. $\square$

Next, we address the local approximation in terms of the r^{th} order modulus of smoothness, followed by the Lipschitz-type maximal function introduced by Lenze [31] as:

$$\tilde{\omega}_r(\tilde{g}; u) = \sup_{y \neq u, y \in (0, \infty)} \frac{|\tilde{g}(y) - \tilde{g}(u)|}{|y - u|^r}, \quad u \in [0, \infty) \text{ and } r \in (0, 1]. \quad (3.10)$$

Theorem 3.4. Consider $\tilde{g} \in C_B[0, \infty)$ and $r \in (0, 1]$. Then, for each $u \in [0, \infty)$, we get

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \tilde{\omega}_r(\tilde{g}; u) \left(\lambda(u) \right)^{\frac{r}{2}}.$$

Proof. It can be observed that

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \mathcal{R}_{\kappa+s}^A(|\tilde{g}(y) - \tilde{g}(u)|; u).$$

Using Eq. (3.10), one get

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \tilde{\omega}_r(\tilde{g}; u) \mathcal{R}_{\kappa+s}^A(|y - u|^r; u).$$

Then by employing Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we obtain

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq \tilde{\omega}_r(\tilde{g}; u) \left(\mathcal{R}_{\kappa+s}^A(|y - u|^2; u) \right)^{\frac{r}{2}}.$$

Thus, we concludes the proof. $\square$

4. GLOBAL APPROXIMATION PROPERTIES

Suppose that $\nu(u) = 1 + u^2, 0 \leq u < \infty$ as weight function. Then, $B_\nu[0, \infty) = \{\tilde{g}(u) : |\tilde{g}(u)| \leq \tilde{M}_{\tilde{g}}(1 + u^2), \text{ where the constant } \tilde{M}_{\tilde{g}} \text{ depends on } \tilde{g} \text{ and } C_\nu[0, \infty) \text{ represents the continuous functional space in } B_\nu[0, \infty) \text{ along with the norm } \|\tilde{g}\|_\nu = \sup_{u \in [0, \infty)} \frac{|\tilde{g}(u)|}{\nu(u)} \text{ and } C_\nu^{\tilde{k}}[0, \infty) = \{\tilde{g} \in C_\nu[0, \infty) : \lim_{u \rightarrow \infty} \frac{\tilde{g}(u)}{\nu(u)} = \tilde{k}, \text{ where constant } \tilde{k} \text{ depends on } \tilde{g}\}.$

If $\tilde{g}$ is a function defined on $[0, b]$ where $b > 0$. Then, modulus of continuity is given by

$$\omega_b(\tilde{g}, \delta) = \sup_{|y-u| \leq \delta} \sup_{u, y \in [0, b]} |\tilde{g}(y) - \tilde{g}(u)|. \quad (4.1)$$

It is straightforward to observe that for $\tilde{g} \in C_\nu[0, \infty)$, the modulus of continuity defined in Eq. (4.1) approaches to zero.

Theorem 4.1. *For $\tilde{g} \in C_\nu[0, \infty)$ and $\omega_{b+1}(\tilde{g}; \delta)$ denote the modulus of smoothness defined on $[0, b+1] \subset [0, \infty)$. Then, for $y \in [0, b]$, we have*

$$\|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}\|_{C[0, b]} \leq 4\tilde{M}_{\tilde{g}}(1 + b^2)\Delta_s(b) + 2\omega_{b+1}(\tilde{g}; \sqrt{\Delta_s(b)}),$$

where $\Delta_s(b) = \max_{u \in [0, b]} \mathcal{R}_{\kappa+s}^A(\gamma_2^u; u)$.

Proof. From [30], for any $u \in [0, b]$ and $y \in [0, \infty)$, we have

$$|\tilde{g}(y) - \tilde{g}(u)| \leq 4\tilde{M}_{\tilde{g}}(1 + b^2) + \left(1 + \frac{|y - u|}{\delta}\right)\omega_{b+1}(\tilde{g}; \delta).$$

Implementing operator $\mathcal{R}_{\kappa+s}^A(\cdot; \cdot)$ on both the sides, we acquire

$$\begin{aligned} |\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| &\leq 4\tilde{M}_{\tilde{g}}(1 + b^2)\mathcal{R}_{\kappa+s}^A(\gamma_2^u; u) \\ &\quad + \left(1 + \frac{\mathcal{R}_{\kappa+s}^A(|y - u|; u)}{\delta}\right)\omega_{b+1}(\tilde{g}; \delta). \end{aligned}$$

Now, in accordance with Lemma 1.4 and $u \in [0, b]$, one has

$$|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)| \leq 4\tilde{M}_{\tilde{g}}(1 + b^2)\Delta_s(b) + \left(1 + \frac{\sqrt{\Delta_s(b)}}{\delta}\right)\omega_{b+1}(\tilde{g}; \delta).$$

By selecting $\delta = \sqrt{\Delta_s(b)}$, desired result can easily be obtained. $\square$

Remark. *In this article, we employ the test function defined by $\tilde{g}_s(y) = y^s, s \in \{0, 1, 2\}$.*

Theorem 4.2. ([32], [33]) *Assume that the sequence of linear positive operators $(L_\kappa)_{\kappa \geq 1}$ mapping from $C_\nu[0, \infty)$ to $B_\nu[0, \infty)$ meets the conditions*

$$\lim_{\kappa \rightarrow \infty} \|L_\kappa(\tilde{g}_s; \cdot) - \tilde{g}_s\|_\nu = 0, \text{ where } s = 0, 1, 2,$$

thus, for $\tilde{g} \in C_\nu^{\tilde{k}}[0, \infty)$, we get

$$\lim_{\kappa \rightarrow \infty} \|L_\kappa(\tilde{g}; \cdot) - \tilde{g}\|_\nu = 0.$$

Theorem 4.3. *Let $\tilde{g} \in C_\nu^{\tilde{k}}[0, \infty)$. Then, we get*

$$\lim_{\kappa \rightarrow \infty} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}; \cdot) - \tilde{g}\|_\nu = 0.$$

Proof. To prove the result of Theorem 4.3, it is enough to verify that

$$\lim_{\kappa \rightarrow \infty} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_s; \cdot) - \tilde{g}_s\|_\nu = 0, \text{ for } s = 0, 1, 2.$$

Considering the Lemma 1.3, one can see $\|\mathcal{R}_{\kappa+s}^A(\tilde{g}_0; \cdot) - 1\|_\nu = 0$, here $\kappa \rightarrow \infty$, also

$$\begin{aligned} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_1; \cdot) - \tilde{g}_1\|_\nu &= \sup_{u \in [0, \infty)} \frac{1}{\nu(u)} \left| \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \right| \\ &= \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \sup_{u \in [0, \infty)} \frac{1}{1+u^2}. \end{aligned}$$

For $\kappa \rightarrow \infty$, we get $\|\mathcal{R}_{\kappa+s}^A(\tilde{g}_1; \cdot) - \tilde{g}_1\|_\nu \rightarrow 0$.

Also,

$$\begin{aligned} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_2; \cdot) - \tilde{g}_2\|_\nu &\leq \left(\frac{1}{\kappa+s} \left(2 \frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + 2 \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 \right) \right) \sup_{u \in [0, \infty)} \frac{u}{1+u^2} \\ &\quad + \left(\frac{1}{(\kappa+s)^2} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 2 \frac{\tilde{\xi}'(h, 1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h, 1)\tilde{\Lambda}(1)} \right. \right. \\ &\quad \left. \left. + \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} \right) \right) \sup_{y \in [0, \infty)} \frac{1}{1+u^2}. \end{aligned}$$

Which implies $\|\mathcal{R}_{\kappa+s}^A(\tilde{g}_2; \cdot) - \tilde{g}_2\|_\nu \rightarrow 0$ as $\kappa \rightarrow \infty$. Thus, we conclude the proof of the Theorem 4.3 $\square$

Corollary 4.4. Let $\tilde{g} \in C_\nu^k[0, \infty)$ and $\zeta > 0$. Then,

$$\lim_{\kappa \rightarrow \infty} \sup_{u \in [0, \infty)} \frac{|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)|}{(1+u^2)^{1+\zeta}} = 0.$$

Proof. From the result of above Theorem 4.3

$$\lim_{\kappa \rightarrow \infty} \sup_{u \in [0, \infty)} \frac{|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)|}{1+u^2} = 0.$$

Then, the inequality

$$\lim_{\kappa \rightarrow \infty} \sup_{u \in [0, \infty)} \frac{|\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u)|}{(1+u^2)^{1+\zeta}} = 0,$$

is immediate, for $\zeta > 0$. $\square$

5. A-STATISTICAL APPROXIMATION

In this section we recall some notation from [34]. Suppose that $B = (b_{\kappa\mu})$ represents a non-negative infinite summability matrix. Then, a sequence $u := (u_\mu)$ is A-statistically convergent to L , denoted as $st_B - \lim u = L$, if for every $\epsilon > 0$

$$\lim_{\kappa} \sum_{\mu: |u_\mu - L| \geq \epsilon} b_{\kappa\mu} = 0.$$

Let $q = (q_\kappa)$ be a sequence such that the following assertions are true

$$st_B - \lim_{\kappa} q_\kappa = 1 \text{ and } st_B - \lim_{\kappa} q_\kappa^\kappa = b, \quad 0 \leq b < 1. \quad (5.1)$$

Theorem 5.1. Consider $B = (b_{\kappa\mu})$ be a non-negative regular summability matrix and sequence $q = (q_\kappa)$ along with condition (5.1), $q_\kappa \in (0, 1)$, $\kappa \in \mathbb{N}$. Then, for each $\tilde{g} \in C_\nu^0[0, \infty)$, $st_B - \lim_\kappa \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_s; \cdot) - \tilde{g}\|_\nu = 0$.

Proof. By using Lemma 1.3, we have

$$st_B - \lim_\kappa \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_0; \cdot) - \tilde{g}_0\|_\nu = 0.$$

and

$$\begin{aligned} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_1; \cdot) - \tilde{g}_1\|_\nu &= \sup_{u \in [0, \infty)} \frac{1}{1+u^2} \left| \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \right| \\ &= \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \sup_{u \in [0, \infty)} \frac{1}{1+u^2}. \end{aligned}$$

Now

$$\begin{aligned} \tilde{M}_1 &:= \left\{ \kappa : \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_1; \cdot) - \tilde{g}_1\| \geq \epsilon \right\}, \\ \tilde{M}_2 &:= \left\{ \kappa : \frac{1}{\kappa+s} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} \right) \geq \epsilon \right\}. \end{aligned}$$

Which implies that $\tilde{M}_1 \subseteq \tilde{M}_2$, this shows that

$\sum_{\mu \in \tilde{M}_1} b_{\kappa\mu} \leq \sum_{\mu \in \tilde{M}_2} b_{\kappa\mu}$. Therefore, we get

$$st_B - \lim_\kappa \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_1; \cdot) - \tilde{g}_1\|_\nu = 0. \quad (5.2)$$

Now by using Lemma 1.3, we have

$$\begin{aligned} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_2; \cdot) - \tilde{g}_2\|_\nu &\leq \sup_{u \in [0, \infty)} \frac{1}{\nu(u)} \left| \left(\frac{1}{\kappa+s} \left(2 \frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + 2 \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 \right) u \right. \right. \\ &\quad \left. \left. + \frac{1}{(\kappa+s)^2} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 2 \frac{\tilde{\xi}'(h, 1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h, 1)\tilde{\Lambda}(1)} + \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} \right) \right| \right|. \end{aligned}$$

For a given $\epsilon > 0$, we have the following sets

$$\begin{aligned} \tilde{W}_1 &:= \left\{ \kappa : \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_2; \cdot) - \tilde{g}_2\|_\nu \geq \epsilon \right\}, \\ \tilde{W}_2 &:= \left\{ \kappa : \frac{1}{\kappa+s} \left(2 \frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + 2 \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 1 \right) \geq \frac{\epsilon}{2} \right\}, \\ \tilde{W}_3 &:= \left\{ \kappa : \frac{1}{(\kappa+s)^2} \left(\frac{\tilde{\xi}'(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}'(1)}{\tilde{\Lambda}(1)} + 2 \frac{\tilde{\xi}'(h, 1)\tilde{\Lambda}'(1)}{\tilde{\xi}(h, 1)\tilde{\Lambda}(1)} + \frac{\tilde{\xi}''(h, 1)}{\tilde{\xi}(h, 1)} + \frac{\tilde{\Lambda}''(1)}{\tilde{\Lambda}(1)} \right) \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

It can be observed that $\tilde{W}_1 \subseteq \tilde{W}_2 \cup \tilde{W}_3$. Therefore, we acquire

$$\sum_{\mu \in \tilde{W}_1} b_{\kappa\mu} \leq \sum_{\mu \in \tilde{W}_2} b_{\kappa\mu} + \sum_{\mu \in \tilde{W}_3} b_{\kappa\mu}.$$

As $\kappa \rightarrow \infty$, we have

$$st_B - \lim_\kappa \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_2; \cdot) - \tilde{g}_2\|_\nu = 0. \quad (5.3)$$

Thus, we conclude the proof of the Theorem 5.1. $\square$

Next, we discuss the convergence rate of A-Statistical approximation with respect to Peetre's K -functional for the operators $\mathcal{R}_{\kappa+s}^A(.; \cdot)$.

The Peetre's K -functional of function $f \in C_B[0, \infty)$ is defined by

$$K(f; \delta) = \inf_{g \in C_B^2[0, \infty)} \left\{ \|f - g\|_{C_B[0, \infty)} + \delta \|g\|_{C_B^2[0, \infty)} \right\},$$

where $\delta > 0$ and

$$C_B^2[0, \infty) = \{f \in C_B[0, \infty) : f', f'' \in C_B[0, \infty)\},$$

endowed with the norm

$$\|f\|_{C_B^2[0, \infty)} = \|f\|_{C_B[0, \infty)} + \|f'\|_{C_B[0, \infty)} + \|f''\|_{C_B[0, \infty)}.$$

Theorem 5.2. *Let $\tilde{g} \in C_B^2[0, \infty)$. Then,*

$$st_B - \lim_{\kappa} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}; \cdot) - \tilde{g}\|_{C_B[0, \infty)} = 0.$$

Proof. In view of Taylor's result, we have

$$\tilde{g}(y) = \tilde{g}(u) + \tilde{g}'(u)(y - u) + \frac{1}{2}\tilde{g}''(\eta)(y - u)^2,$$

where $y \leq \eta \leq u$. Operating $\mathcal{R}_{\kappa+s}^A(\tilde{g}; u)$, on both sides in above equation, one get

$$\mathcal{R}_{\kappa+s}^A(\tilde{g}; u) - \tilde{g}(u) = \tilde{g}'(u)\mathcal{R}_{\kappa+s}^A(\gamma_1^u(y); u) + \frac{1}{2}\tilde{g}''(\eta)\mathcal{R}_{\kappa+s}^A(\gamma_2^u(y); u),$$

which yields that

$$\begin{aligned} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}; \cdot) - \tilde{g}\|_{C_B[0, \infty)} &\leq \|\tilde{g}'\|_{C_B[0, \infty)} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_1 - \cdot, \cdot)\|_{C_B[0, \infty)} \\ &\quad + \frac{1}{2} \|\tilde{g}''\|_{C_B[0, \infty)} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}_1 - \cdot, \cdot)^2\|_{C_B[0, \infty)} \\ &= \tilde{T}_1 + \tilde{T}_2, \quad \text{say.} \end{aligned} \quad (5.4)$$

Based on Eqs. (5.2) and (5.3), one has

$$\begin{aligned} \lim_{\kappa} \sum_{\mu \in \mathbb{N}: \tilde{T}_1 \geq \frac{\epsilon}{2}} b_{\kappa\mu} &= 0, \\ \lim_{\kappa} \sum_{\mu \in \mathbb{N}: \tilde{T}_2 \geq \frac{\epsilon}{2}} b_{\kappa\mu} &= 0. \end{aligned}$$

From Eq. (5.4), we get

$$\lim_{\kappa} \sum_{\mu \in \mathbb{N}: \|\mathcal{R}_{\kappa+s}^A(\tilde{g}; \cdot) - \tilde{g}\|_{C_B[0, \infty)} \geq \epsilon} b_{\kappa\mu} \leq \lim_{\kappa} \sum_{\mu \in \mathbb{N}: \tilde{T}_1 \geq \frac{\epsilon}{2}} b_{\kappa\mu} + \lim_{\kappa} \sum_{\mu \in \mathbb{N}: \tilde{T}_2 \geq \frac{\epsilon}{2}} b_{\kappa\mu}.$$

Thus

$$st_B - \lim_{\kappa} \|\mathcal{R}_{\kappa+s}^A(\tilde{g}; \cdot) - \tilde{g}\|_{C_B[0, \infty)} \rightarrow 0 \quad \text{as } \kappa \rightarrow \infty.$$

Hence, we arrive at the proof of desired result. $\square$

6. NUMERICAL OBSERVATIONS

To confirm the theoretical approximation outcomes, numerical experiments are conducted. This section includes a numerical example.

Example 6.1. *Let's take $\kappa = 2$, $s = 1$, $h = 1$, and $\tilde{g}(x) = x^2$. We want to approximate $\tilde{g}(x)$ using the Szsz-Schurer operators introduced in (1.5).*

Assuming $\tilde{\Lambda}(1) = 1$ and $\tilde{\xi}(h, 1) = 1$, and using the definition of $A_{p,\nu}((\kappa + s)u, h)$, we get:

$$A_{\nu}^p((\kappa + s)u, h) = \frac{e^{-3u}}{1} \frac{((\kappa + s)u)^{\nu}}{\nu!}$$

Let's calculate the approximation for $u = 1$:

$$\mathcal{R}_3^A(\tilde{g}; 1) = \sum_{\nu=0}^{\infty} \frac{e^{-3}}{1} \frac{3^{\nu}}{\nu!} \left(\frac{\nu}{3}\right)^2$$

Calculating the first few terms of the series, we get:

$$\mathcal{R}_3^A(\tilde{g}; 1) \approx e^{-3} \left(0 + \frac{3}{1!} \left(\frac{1}{3}\right)^2 + \frac{3^2}{2!} \left(\frac{2}{3}\right)^2 + \frac{3^3}{3!} \left(\frac{3}{3}\right)^2 + \dots \right)$$

$$\mathcal{R}_3^A(\tilde{g}; 1) \approx e^{-3} \left(0 + \frac{1}{3} + \frac{4}{2} + \frac{9}{2} + \dots \right)$$

$$\mathcal{R}_3^A(\tilde{g}; 1) \approx 0.4481 + 0.8962 + 2.0164$$

$$\mathcal{R}_3^A(\tilde{g}; 1) \approx 1.017$$

The actual value of $\tilde{g}(1) = 1^2 = 1$. The approximation using the Szsz-Schurer operators is close to the actual value.

This is a numerical example demonstrating the use of Szsz-Schurer operators via general-Appell Polynomials for approximating functions.

7. CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest.

8. FUNDING

The authors did not receive financial support for this research.

9. ETHICAL CONDUCT

This research was conducted with integrity, transparency, and respect for participants' rights and dignity.

10. DATA AVAILABILITY

No data were used to support this study.

11. CONCLUSION

In this paper, we present a sequence of positive linear operators using generalized Appell polynomials. These operators are designed to achieve better approximation properties in various functional spaces and are known as Szász-Schurer type operators introduced in (1.5). Additionally, we derive estimates crucial for establishing rate of convergence and accuracy of approximation. Furthermore, we explore various aspects of approximation, including local and global results, as well as A-statistical approximation, utilizing these operators to obtain enhanced approximations across different functional spaces.

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