Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 17 Issue 3(2025), Pages 52-60
https://doi.org/10.54671/BMAA-2025-3-4

ON THE EXISTENCE OF LIMIT CYCLES IN A FAMILY OF AUTONOMOUS THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

ROMAISSA MELLAL*, NABIL SELLAMI

Abstract. This study focuses on identifying limit cycles within a specific class of third-order autonomous differential equations described by

$$\dddot{x} + (\alpha x + \lambda)\ddot{x} + (\beta x + \mu)\dot{x} + \gamma x^2 + \lambda\mu x = \varepsilon^2\phi(x, \dot{x}, \ddot{x}, \varepsilon),$$

where $\phi = \phi(x, \dot{x}, \ddot{x}, \varepsilon)$ is a nonlinear perturbation, ε is a small parameter, and the parameters $\alpha, \lambda, \beta, \mu$, and γ are real with $\mu > 0$.
By employing first-order averaging theory, the problem of identifying limit cycles is reduced to an algebraic problem of determining non-degenerate zeros of a nonlinear function. This approach provides sufficient conditions ensuring the existence of limit cycles in the original differential equation. The efficacy of this work and the validity of the theoretical results are substantiated through illustrative applications.

1. Introduction

Limit cycles are isolated periodic solutions arising in nonlinear differential equations. This concept, first introduced by Henri Poincar in 1881 through his work "*Mmoire sur les courbes dfinies par une quation diffrentielle*" [24], represents a fundamental area of study in dynamical systems theory and plays a crucial role in understanding oscillatory phenomena across various scientific disciplines.

Third-order differential equations frequently appear in various scientific contexts such as mechanics and biological systems. Consequently, numerous studies have addressed periodic solutions of such equations using diverse analytical techniques, including [2, 3, 4]. Among these methods, averaging theory has proven particularly effective, as seen in [5, 8, 9, 16, 19, 20, 21, 25, 29].

Three-dimensional quadratic differential systems constitute an important class of nonlinear dynamical systems that have attracted considerable attention in mathematical research due to their rich dynamic behavior and broad range of applications. Numerous examples of such systems have been extensively studied in the literature.

2010 *Mathematics Subject Classification.* 34C07, 34C25, 34C29, 37G15.
Key words and phrases. Limit cycle; Averaging theory; Third-order differential equation; Perturbation.

*Corresponding Author: R. Mellal.
Submitted June 29, 2025. Accepted August 10, 2025. Published August 20, 2025.
Communicated by D. Anderson.

Classical examples include the Rikitake [26], Lorenz [22], Genesio [13], Chen [7], and Liu systems [15]. Recently, several new three-dimensional quadratic differential systems have been investigated, as in [1, 11, 12, 14].

The Hopf bifurcation of periodic orbits in the equation

$$\dddot{x} + (a_1x + a_0)\ddot{x} + (b_1x + b_0)\dot{x} + c_2x^2 + c_1x + c_0 = 0 \tag{1.1}$$

was studied in [10].

In [18], the authors examined the equation

$$\dddot{x} + (a_1x + a_0)\ddot{x} + (b_1x + b_0)\dot{x} + x^2 = 0. \tag{1.2}$$

In [17], periodic orbits were investigated for the equation

$$\dddot{x} + (a_1x + a_0)\ddot{x} + (b_1x + b_0)\dot{x} + c_2x^2 + a_0b_0x = \varepsilon^2 F(t, x, \dot{x}, \ddot{x}, \varepsilon), \tag{1.3}$$

where F is $\frac{2\pi}{\sqrt{b_0}}$-periodic in t, $b_0 > 0$, and ε is a small parameter.

In this work, we investigate the emergence of limit cycles in perturbed third-order autonomous differential systems described by

$$\dddot{x} + (\alpha x + \lambda)\ddot{x} + (\beta x + \mu)\dot{x} + \gamma x^2 + \lambda\mu x = \varepsilon^2\phi(x, \dot{x}, \ddot{x}, \varepsilon), \tag{1.4}$$

where $\phi = \phi(x, \dot{x}, \ddot{x}, \varepsilon)$ denotes a nonlinear autonomous perturbation function. The coefficients $\alpha, \lambda, \beta, \mu, \gamma \in \mathbb{R}$ with $\mu > 0$. Here, ε represents a small perturbation parameter.

Note that equation (1.4) is the autonomous version of equation (1.3) studied in [17], and our work completes the study initiated in that paper.

This work extends prior results on non-autonomous systems [17] by addressing the autonomous case, revealing that even simple autonomous perturbations can induce rich oscillatory behavior, providing new insights into the dynamics of these important mathematical systems.

Our research shows that the perturbation of the associated homogeneous third-order differential equation of (1.4) (i.e., equation (1.4) with $\varepsilon = 0$) or its equivalent three-dimensional quadratic differential system by an autonomous function can also produce limit cycles, analogous to the case of non-autonomous perturbations established in [17].

2. Main results

Let us define the function

$$\mathcal{F}(r_0) = \frac{\sqrt{\mu}}{2\pi} \int_0^{\frac{2\pi}{\sqrt{\mu}}} \cos(\sqrt{\mu}\theta)\, Q(A, B, C)\, d\theta, \tag{2.1}$$

where

$$Q(A, B, C) = A\big(-\gamma A - \beta B - \alpha C\big) + \phi_0(A, B, C),$$

with ϕ_0 being the zero-order term in ε from Taylor expansion of $\phi(\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon)$ at $\varepsilon = 0$, and

$$\begin{aligned} A &= \frac{-\sqrt{\mu}r_0\cos(\sqrt{\mu}\theta) + \lambda r_0\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}\,(\lambda^2 + \mu)}, \\ B &= \frac{\lambda r_0\cos(\sqrt{\mu}\theta) + \sqrt{\mu}r_0\sin(\sqrt{\mu}\theta)}{\lambda^2 + \mu}, \\ C &= \frac{\mu r_0\cos(\sqrt{\mu}\theta) - \lambda\sqrt{\mu}r_0\sin(\sqrt{\mu}\theta)}{\lambda^2 + \mu}. \end{aligned} \tag{2.2}$$

The theorem that follows constitutes the principal finding of this investigation.

Theorem 2.1. *Suppose $\lambda \neq 0$ and $\mu > 0$. If $\mathcal{F}(r_0)$ has a simple zero r_0^*, then equation* (1.4) *admits a periodic solution $\varepsilon x(t,\varepsilon)$ such that $x(t,\varepsilon)$ converges to*

$$x_0^*(t) = \frac{r_0^*\left(-\sqrt{\mu}\cos(\sqrt{\mu}t) + \lambda\sin(\sqrt{\mu}t)\right)}{\sqrt{\mu}\,(\lambda^2+\mu)}, \tag{2.3}$$

a solution of

$$\dddot{x} + \lambda\ddot{x} + \mu\dot{x} + \lambda\mu x = 0, \tag{2.4}$$

as $\varepsilon \to 0$.

This theorem is proved in subsection 4.1.

To apply Theorem 2.1, we present the following two corollaries.

Corollary 2.2. *When $\lambda \neq 0$, $\mu > 0$ and the function ϕ satisfies*

$$\phi\left(x,\dot{x},\ddot{x},\varepsilon\right) = x\dot{x}\ddot{x}\left(1-x^3\right)\left(\ddot{x}^2-1\right) + O\left(\varepsilon\right),$$

then equation (1.4) *admits a limit cycle $\varepsilon x_1(t,\varepsilon)$, such that $x_1(t,\varepsilon)$ converges to the solution*

$$x_{01}^*(t) = \frac{-\sqrt{2\mu}\cos(\sqrt{\mu}t) + \sqrt{2}\lambda\sin(\sqrt{\mu}t)}{\mu\sqrt{\lambda^2+\mu}} \tag{2.5}$$

of equation (2.4)*, as $\varepsilon \to 0$.*

Corollary 2.3. *For $\lambda = 3$, $\mu = 1$ and*

$$\phi\left(x,\dot{x},\ddot{x},\varepsilon\right) = \left(1-x^3\right)\left(\dot{x}^2-1\right)\left(\ddot{x}-1\right) + O\left(\varepsilon\right),$$

two limit cycles $\varepsilon x_2(t,\varepsilon)$ and $\varepsilon x_3(t,\varepsilon)$ of equation (1.4) *emerge, such that $x_2(t,\varepsilon)$ and $x_3(t,\varepsilon)$ converge respectively to the solutions*

$$x_{02}^*(t) = \frac{\sqrt{10+5\sqrt{2}}\,(-\cos t + 3\sin t)}{5} \tag{2.6}$$

and

$$x_{03}^*(t) = \frac{\sqrt{10-5\sqrt{2}}\,(-\cos t + 3\sin t)}{5} \tag{2.7}$$

of equation

$$\dddot{x} + 3\ddot{x} + \dot{x} + 3x = 0, \tag{2.8}$$

as $\varepsilon \to 0$.

Corollaries 2.2 and 2.3 are demonstrated in subsections 4.2 and 4.3 respectively.

3. Main Tool

Here, we provide the essential first-order averaging theory results that are crucial for our study. For more on averaging theory, see [28] and [30].

Consider a perturbed differential equation

$$\dot{\mathbf{x}}(t) = g_0\left(\mathbf{x},t\right) + \varepsilon g_1(\mathbf{x},t) + \varepsilon^2 g_2(\mathbf{x},t,\varepsilon), \tag{3.1}$$

with ε ranging from 0 to sufficiently small nonzero values.

The mappings $g_0, g_1 \colon \mathcal{D}\times\mathbb{R}\to\mathbb{R}^n$ and $g_2\colon \mathcal{D}\times\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ belong to the class $\mathcal{C}^2$ and exhibit $T-$periodicity in t; here, $\mathcal{D}\subset\mathbb{R}^n$ is an open set.

When $\varepsilon = 0$, the system

$$\dot{\mathbf{x}}(t) = g_0\left(\mathbf{x},t\right) \tag{3.2}$$

is assumed to have an m-dimensional submanifold of periodic solutions.

For a solution $\mathbf{x}(t, \mathbf{w})$ of (3.2) with initial conditions $\mathbf{x}(0, \mathbf{w}) = \mathbf{w}$, the linearized system takes the form

$$\dot{\mathbf{y}} = D_{\mathbf{x}} g_0 \left(\mathbf{x}(t, \mathbf{w}), t\right) \mathbf{y}. \tag{3.3}$$

Let $\mathcal{M}_{\mathbf{w}}(t)$ represent a fundamental matrix of (3.3), and define the projection $\mathcal{P} \colon \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ of $\mathbb{R}^n$ onto its first m coordinates.

The next theorem provides conditions for the existence of periodic solutions in the perturbed system (3.1) involving an averaging function $\mathcal{F}$ defined using the fundamental matrix.

Theorem 3.1. *For an open, bounded set $\mathcal{U} \subset \mathbb{R}^m$ and a $\mathcal{C}^2$ function $h \colon \overline{\mathcal{U}} \to \mathbb{R}^{n-m}$, we assume*

i) The set $\mathcal{W} = \left\{\mathbf{w}_\delta = (\delta, h(\delta)), \delta \in \overline{\mathcal{U}}\right\} \subset \mathcal{D}$ generates T-periodic solutions $\mathbf{x}(t, \mathbf{w}_\delta)$ of (3.2).

ii) For each $\mathbf{w}_\delta \in \mathcal{W}$, system (3.3) admits a fundamental matrix $\mathcal{M}_{\mathbf{w}_\delta}(t)$ where $\mathcal{M}^{-1}_{\mathbf{w}_\delta}(0) - \mathcal{M}^{-1}_{\mathbf{w}_\delta}(T)$ has an $m \times (n-m)$ zero matrix in its upper right block and a nonsingular $((n-m) \times (n-m))$ matrix Δ_δ in its lower right block.

Define the averaging function $\mathcal{F} \colon \overline{\mathcal{U}} \to \mathbb{R}^m$ by

$$\mathcal{F}(\delta) = \mathcal{P}\left(\frac{1}{T}\int_0^T \mathcal{M}^{-1}_{\mathbf{w}_\delta}(t)\, g_1(\mathbf{x}(t, \mathbf{w}_\delta), t)\, dt\right). \tag{3.4}$$

If there exists $\delta^* \in \mathcal{U}$ where $\mathcal{F}(\delta^*) = 0$ and $\det\left(\frac{d\mathcal{F}}{d\delta}(\delta^*)\right) \neq 0$, the system (3.1) has a $T-$periodic solution $\psi(t, \varepsilon)$ satisfying $\psi(0, \varepsilon) \to \mathbf{w}_{\delta^*}$ as $\varepsilon \to 0$.

This formulation originates from the classical work of Malkin [23] and Roseau [27]. In [6], a new proof of Theorem 3.1 is presented.

4. Proofs

4.1. Proof of Theorem 2.1.

Proof. By setting $(x, \dot{x}, \ddot{x}) = (x, y, z)$, we transform equation (1.4) into the following system of first-order equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\lambda\mu x - \mu y - \lambda z - \gamma x^2 - \beta x y - \alpha x z + \varepsilon^2 \phi(x, y, z, \varepsilon). \end{cases} \tag{4.1}$$

Applying the rescaling $(x, y, z) \mapsto (\varepsilon X, \varepsilon Y, \varepsilon Z)$, yields

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = Z \\ \dot{Z} = -\lambda\mu X - \mu Y - \lambda Z + \varepsilon\left(-\gamma X^2 - \beta XY - \alpha XZ + \phi_0(X, Y, Z)\right) + O(\varepsilon^2), \end{cases} \tag{4.2}$$

where $\phi_0 = \phi_0(X, Y, Z)$ is defined in section 2.

Now, we implement the linear transformation

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 1 \\ \lambda\sqrt{\mu} & \sqrt{\mu} & 0 \\ \mu & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

which converts the linear component of system (4.2) into its real Jordan canonical form. We obtain

$$\left\{\begin{array}{l}\dot{U}=-\sqrt{\mu}\,V+\varepsilon P(U,V,W)+O(\varepsilon^2)\\ \dot{V}=\sqrt{\mu}\,U\\ \dot{W}=-\lambda W+\varepsilon P\left(U,V,W\right)+O\left(\varepsilon^2\right),\end{array}\right. \tag{4.3}$$

where $P(U,V,W)=a\left(-\gamma a-\beta b-\alpha c\right)+\phi_0(a,b,c)$ such that

$$\begin{aligned} a&=\frac{-\sqrt{\mu}\,U+\lambda V+\sqrt{\mu}\,W}{\sqrt{\mu}\left(\lambda^2+\mu\right)}\\ b&=\frac{\lambda U+\sqrt{\mu}\,V-\lambda W}{\lambda^2+\mu}\\ c&=\frac{\mu U-\lambda\sqrt{\mu}\,V+\lambda^2 W}{\lambda^2+\mu}.\end{aligned}$$

Converting (U,V,W) to the cylindrical coordinates (r,θ,υ) via

$$U=r\cos(\sqrt{\mu}\,\theta),\ V=r\sin(\sqrt{\mu}\,\theta),\ W=\upsilon,$$

leads to

$$\left\{\begin{array}{l}\dot{r}=\varepsilon\cos(\sqrt{\mu}\,\theta)P\left(r\cos(\sqrt{\mu}\,\theta),r\sin(\sqrt{\mu}\,\theta),\upsilon\right)+O(\varepsilon^2)\\ \dot{\theta}=1-\varepsilon\frac{\sin(\sqrt{\mu}\,\theta)}{r\sqrt{\mu}}P\left(r\cos(\sqrt{\mu}\,\theta),r\sin(\sqrt{\mu}\,\theta),\upsilon\right)+O(\varepsilon^2)\\ \dot{\upsilon}=-\lambda\upsilon+\varepsilon\,P\left(r\cos(\sqrt{\mu}\,\theta),r\sin(\sqrt{\mu}\,\theta),\upsilon\right)+O(\varepsilon^2).\end{array}\right. \tag{4.4}$$

Dividing by $\dot{\theta}$, the last system is reduced to

$$\left\{\begin{array}{l}\frac{dr}{d\theta}=\varepsilon\cos\left(\sqrt{\mu}\theta\right)\ Q\left(\theta,r,\upsilon\right)+O\left(\varepsilon^2\right)\\ \frac{d\upsilon}{d\theta}=-\lambda\upsilon+\varepsilon\left(1-\frac{\lambda\upsilon\sin\left(\sqrt{\mu}\theta\right)}{r\sqrt{\mu}}\right)Q\left(\theta,r,\upsilon\right)+O\left(\varepsilon^2\right),\end{array}\right. \tag{4.5}$$

with $Q\left(\theta,r,\upsilon\right)=P\left(r\cos(\sqrt{\mu}\,\theta),r\sin(\sqrt{\mu}\,\theta),\upsilon\right).$
The system (4.5) takes the form of system (3.1) where

$$t=\sqrt{\mu}\,\theta,\ \mathbf{x}=\left(\begin{array}{c}r\\ \upsilon\end{array}\right),\ g_0=\left(\begin{array}{c}0\\ -\lambda\upsilon\end{array}\right)\quad\text{and}$$
$$g_1=\left(\begin{array}{c}\cos(\sqrt{\mu}\,\theta)\,Q\left(\theta,r,\upsilon\right)\\ \left(1-\frac{\lambda\upsilon\sin(\sqrt{\mu}\,\theta)}{r\sqrt{\mu}}\right)Q(\theta,r,\upsilon)\end{array}\right).$$

In the following, we have to apply Theorem 3.1 to system (4.5). If $\varepsilon=0$, the system (4.5) becomes

$$\left\{\begin{array}{l}\frac{dr}{d\theta}=0\\ \frac{d\upsilon}{d\theta}=-\lambda\upsilon,\end{array}\right. \tag{4.6}$$

which has the $\frac{2\pi\lambda}{\sqrt{\mu}}$-periodic solutions $(r(\theta),\upsilon(\theta))=(r_0,0)$, for $r_0>0$.

To verify the conditions of Theorem 3.1, note that $m=1$ and $n=2$.
For $r_1>0$ and $r_2>0$, define $\mathcal{U}=]r_1,r_2[\subset\mathbb{R}$, $\delta=r_0\in[r_1,r_2]$, and the function

$$\begin{aligned} h\colon[r_1,r_2]&\to\mathbb{R}\\ r_0&\mapsto h\left(r_0\right)=0.\end{aligned}$$

The set

$$\mathcal{W}=\left\{\mathbf{w}_\delta=(r_0,0),r_0\in[r_1,r_2]\right\},$$

and the projection

$$\begin{aligned}\mathcal{P}\colon \mathbb{R}\times\mathbb{R} &\to \mathbb{R}\\ (r,\upsilon) &\mapsto \mathcal{P}(r,\upsilon)=r.\end{aligned}$$

The fundamental matrix $\mathcal{M}_{\mathbf{w}_\delta}(\theta)$ of the linearized system (4.6) at $\mathbf{w}_\delta=(r_0,0)$ is

$$\mathcal{M}_{\mathbf{w}_\delta}(\theta)=\begin{pmatrix}1 & 0\\ 0 & e^{-\lambda\theta}\end{pmatrix}.$$

We have

$$\mathcal{M}^{-1}_{\mathbf{w}_\delta}(0)-\mathcal{M}^{-1}_{\mathbf{w}_\delta}\left(\frac{2\pi}{\sqrt{\mu}}\right)=\begin{pmatrix}0 & 0\\ 0 & 1-e^{\frac{2\pi\lambda}{\sqrt{\mu}}}\end{pmatrix},$$

which verifies condition (ii) of Theorem 3.1, since $\lambda\neq 0$ and $\mu>0$.

From (3.4), we get $\mathcal{F}(\delta)=\mathcal{F}(r_0)$, where $\mathcal{F}(r_0)$ is given by (2.1).

By Theorem 3.1, for each simple zero r_0^* of $\mathcal{F}(r_0)$, the system (4.5) admits a limit cycle $(r(\theta,\varepsilon),\upsilon(\theta,\varepsilon))$ such that

$$(r(0,\varepsilon),\upsilon(0,\varepsilon))\to(r_0^*,0),\ \text{when}\ \varepsilon\to 0.$$

Reverting the transformations, there exists a limit cycle $(r(t,\varepsilon),\theta(t,\varepsilon),\upsilon(t,\varepsilon))$ of system (4.4), with initial conditions converging to $(r_0^*,0,0)$ when $\varepsilon\to 0$.

Consequently, system (4.3) has a limit cycle $(U(t,\varepsilon),V(t,\varepsilon),W(t,\varepsilon))$, satisfying

$$(U(0,\varepsilon),V(0,\varepsilon),W(0,\varepsilon))\to(r_0^*,0,0),\ \text{when}\ \varepsilon\to 0.$$

Finally, the original differential equation (1.4) possesses a limit cycle $\varepsilon x(t,\varepsilon)$, such that $x(t,\varepsilon)$ converges to the periodic solution (2.3) of equation (2.4), as $\varepsilon\to 0$.

Thus, we have established Theorem 2.1. □

4.2. **Proof of Corollary 2.2.**

Proof. To prove this corollary, we apply Theorem 2.1.
Taking $\phi_0(x,\dot{x},\ddot{x})=x\dot{x}\ddot{x}\left(1-x^3\right)\left(\ddot{x}^2-1\right)$ then the function $\mathcal{F}(r_0)$ is

$$\mathcal{F}_1(r_0)=\frac{r_0^3\lambda\left(2\lambda^2+2\mu-r_0^2\mu\right)}{\left(\lambda^2+\mu\right)^3},$$

which has the real positive simple zero $r_{01}^*=\frac{\sqrt{2\mu(\lambda^2+\mu)}}{\mu}$ with

$$\frac{d\mathcal{F}_1}{dr_0}(r_{01}^*)=\frac{-\lambda}{2\mu(\lambda^2+\mu)}\neq 0.$$

By Theorem 2.1, it follows the existence of a limit cycle of equation (1.4) as shown in Figure 1, the formula (2.5) is obtained by substituting the above zero r_{01}^* in (2.3). □

4.3. **Proof of Corollary 2.3.**

Proof. If $\lambda=3,\mu=1$ and $\phi_0(x,\dot{x},\ddot{x})=\left(1-x^3\right)\left(\dot{x}^2-1\right)(\ddot{x}-1)$, then from (2.1), we have

$$\mathcal{F}_2(r_0)=-\frac{1}{16000}r_0^5+\frac{1}{200}r_0^3-\frac{1}{20}r_0,$$

which has two positive simple zeros $r_{02}^*=2\sqrt{10+5\sqrt{2}}$ and $r_{03}^*=2\sqrt{10-5\sqrt{2}}$ with

$$\frac{d\mathcal{F}_2}{dr_0}(r_{02}^*)=\frac{-1-\sqrt{2}}{5}\neq 0\quad\text{and}\quad\frac{d\mathcal{F}_2}{dr_0}(r_{03}^*)=\frac{-1+\sqrt{2}}{5}\neq 0.$$

The proof follows directly from Theorem 2.1, we have two limit cycles of equation (1.4) represented in Figure 2, the formulas (2.6) and (2.7) are obtained by substituting the above zeros r^*_{02} and r^*_{03} respectively in (2.3). $\square$

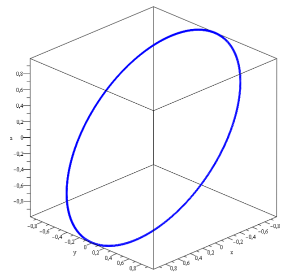

FIGURE 1. Limit cycle of the system of Corollary 2.2 with $\alpha = \lambda = \beta = \mu = \gamma = 1$ and $\varepsilon = 0.01$.

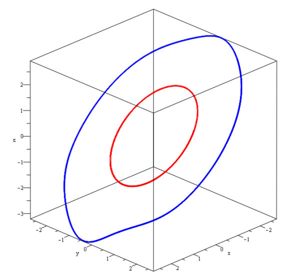

FIGURE 2. Limit cycles of the system of Corollary 2.3 with $\varepsilon = 0.01$.

5. Conclusion

In this paper, we have extended the analysis of periodic solutions in third-order differential systems by focusing on autonomous perturbations of a class of third-order autonomous differential equations. Using first-order averaging theory, we have established explicit criteria for the existence of limit cycles via the identification of simple zeros of a scalar averaging function. Our results complement and generalize the earlier work of Llibre and Makhlouf [17], who studied the non-autonomous case with time-dependent perturbations.

Whereas their framework deals with periodic orbits arising from non-autonomous quadratic systems and requires analyzing zeros of a two-dimensional averaged function, our approach demonstrates that autonomous nonlinear perturbations alone suffice to generate rich oscillatory behavior, often with simpler scalar conditions.

Furthermore, the constructive examples and corollaries we present showcase the practical applicability and effectiveness of our method in identifying multiple limit cycles.

Overall, our contribution advances the theory of limit cycles for third-order autonomous systems, providing new insights and a complementary perspective that bridges and enhances existing results in the literature.

Statements and Declarations. The authors have no competing interests to declare that are relevant to the content of this article.

Acknowledgments. The authors express their gratitude to the anonymous referee for their valuable feedback, which greatly contributed to enhancing this article.

References

[1] A. A. Abddulkareem, A. I. Amen and N. H. Hussein, *Limit Cycles of the Three-dimensional Quadratic Differential System via Hopf Bifurcation*. Baghdad Sci.J, **21 (9)** (2024), Article number 2970.

[2] A. A. Adeyanju and C. Tunc, *Uniform-ultimate boundedness, stability and periodicity of solutions to certain differential equation of third order*, Montes Taurus J. Pure Appl. Math. **6 (2)** (2024) 1–14.

[3] S. Akram, A. Nawaz, H. Kalsoom , M. Idrees and Y. M. Chu, *Existence of Multiple Periodic Solutions for Cubic Nonautonomous Differential Equation*. Mathematical Problems in Engineering Volume 2020, Article ID 7618097, 14 pages.

[4] M. Aldiaiji, B. Qaraad, L. F. Iambor, S. S. Rabie,E. M. Elabbasy, *Oscillation of Third-Order Differential Equations with Advanced Arguments*. Mathematics 2024, 12, 93. https://doi.org/10.3390/math12010093

[5] A. I. Amen, *Dynamics and Zero Hopf Periodic Solutions of Chaos Laser System*. Int. J. Appl. Comput. Math, **11** (2025), Article number 75. https://doi.org/10.1007/s40819-025-01888-9

[6] A. Buica, J. P. Francoise and J. Llibre, *Periodic Solutions of Nonlinear Periodic Differential Systems with a Small Parameter*. Comm. Pure Appl. Anal **6 (1)** (2007), 103–111.

[7] G. Chen and T. Ueta, *Yet another chaotic attractor*. Int. J. Bifur. Chaos **9** (1999), 1465–1466.

[8] A. Cima, J. Llibre and M. A Teixeira, *Limit Cycles of some Polynomial Differential System in Dimension 2, 3 and 4 Via Averaging theory,* Applicable Analysis **87** (2008), 149–164.

[9] Z. Diab, J. L. G. Guirao, J. A. Vera, *A Note on the Periodic Solutions for a Class of Third Order Differential Equations*. Symmetry 2021, 13, 31. https://dx.doi.org/10.3390/sym13010031

[10] F. S. Dias and L. F. Mello, *Nonlinear Analysis of a Quadratic System Obtained from a Scalar Third Order Differential Equation*. Elect. J. Diff. Equ **161** (2010), 1–25.

[11] B. Ferec, M. ulj and M. Mencinger, *On the Integrability of Persistent Quadratic Three-Dimensional Systems*. Mathematics **12 (9)** (2024), Article number 1338, https://dx.doi.org/10.3390/math12091338.

[12] I. A. Garca, *Small Amplitude Periodic Orbits in Three-Dimensional Quadratic Vector Fields with a Zero-Hopf Singularity*. J Dyn Diff Equat **36** (2024), 1325–1346.

[13] R. Genesio and A. Tesi, *Harmonic balance methods for analysis of chaotic dynamics in nonlinear systems*. Automatica **28** (1992), 531–548.

[14] A. M. Husien and A. I. Amen, *Hopf and Zero-Hopf Bifurcation Analysis for a Chaotic System*. International Journal of Bifurcation and Chaos **34 (8)** (2024), Article ID 2450104.

[15] C. Liu, T. Liu, L. Liu and K. Liu, *A new chaotic attractor*. Chaos Solitons Fractals **22** (2004), 1031–1038.

[16] J. Llibre, B. D. Lopes and J. R. de Moraes, *Periodic Solutions of Continuous Third-Order Differential Equations with Piecewise Polynomial Nonlinearities*. International Journal of Bifurcation and Chaos, Vol. 30, No. 11 (2020) 2050158 (7 pages).

[17] J. Llibre, A. Makhlouf, *Periodic Orbits of a Non-Autonomous Quadratic Differential System Obtained from Third-Order Differential Equations*. Dynamical Systems **27 (2)** (2012), 161–168.

[18] J. Llibre, A. Makhlouf, *Zero-Hopf Periodic Orbit of a Quadratic System of Differential Equations Obtained from a Third Order Differential Equation.* Differ Equ Dyn Syst **27** (2019), 75–82.

[19] J. Llibre, M. Messias and A. de C. Reinol, *ZeroHopf Bifurcations in Three-Dimensional Chaotic Systems with One Stable Equilibrium.* International Journal of Bifurcation and Chaos, Vol. 30, No. 13 (2020) 2050189 (11 pages)

[20] J. Llibre, L. Roberto, *On the Periodic Orbits of the Third-Order Differential Equation* $x''' - \mu x'' + x' - \mu x = \varepsilon F(x, x', x'')$. Applied Mathematics Letters **26** (2013), 425–430.

[21] J. Llibre, J. Yu, and X. Zhang, *Limit Cycles for a Class of Third Order Differential Equations.* Rocky Mountain J. Math. **40 (2)** (2010), 581–594.

[22] E. N. Lorenz, *Limit Cycles for a Class of Third Order Differential Equations.* J. Atmos. Sci. **20** (1963), 130–141.

[23] I. G. Malkin, *Some Problems of the Theory of Nonlinear Oscillations,* (Russian) Gostexizdat, Moscow, 1956.

[24] H. Poincaré, *Mémoire sur les Courbes Définies par une Équation Différentielle.* J. Math. Pures. Appl **3 (7)** (1881), 375–422.

[25] N. Rezaiki and A. Boulfoul, *Periodic Solutions of Nonsmooth Third-Order Differential Equations* J. Innov. Appl. Math. Comput. Sci. **4 (1)** (2024), 1–10.

[26] T. Rikitake, *Oscillations of a system of disk dynamos.* Proc. R. Cambridge Philos. Soc. **54** (1958), 89–105.

[27] M. Roseau, *Vibrations Non-Linéaires et Thé orie de la Stabilité,* (French) Springer Tracts in Natural Philosophy **8**, Springer-Verlag, Berlin, New York, 1966.

[28] J. A. Sanders, and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems.* Applied Mathematical Sciences **59**, Springer, New York, 1985.

[29] A. E. Tabet, A. Makhlouf and S. Kassa, *3-Dimensional zeroHopf bifurcation via averaging theory of fourth order.* So Paulo J. Math. Sci. **19** (2025), Article number 13. https://doi.org/10.1007/s40863-025-00489-0

[30] F. Verhulst, *Nonlinear Differentail Equations and Dynamical Systems.* Universitext, 2^{nd} edition, Springer, Berlin, 1996.

Romaissa Mellal
Laboratory of Analysis and Control of Differential Equations "ACED", Department of Mathematics, University of May 8th 1945, P.O. Box 401, Guelma 24000, Algeria
E-mail address: `mellal.romaissa@univ-guelma.dz, romaissa.mellal@yahoo.fr`

Nabil Sellami
Laboratory of Analysis and Control of Differential Equations "ACED", Department of Mathematics, University of May 8th 1945, P.O. Box 401, Guelma 24000, Algeria
E-mail address: `sellami.nabil@univ-guelma.dz, sellaminab@yahoo.fr`