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A MILLER-ROSS-TYPE POISSON DISTRIBUTION CONNECTED WITH CERTAIN CLASS OF STARLIKE FUNCTIONS

WALEED AL-RAWASHDEH

Abstract. In this paper, we make use of Miller-Ross function and Poisson distribution to introduce a novel class of starlike bi-univalent functions that is subordinated to Gegenbauer polynomials, which we denote as $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$. In addition, we find bounds for the growth and distortion of functions belonging to our class and some of its various subclasses. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to our class and some of its various subclasses. Furthermore, we derive the initial coefficients bounds and the Fekete-Szegö inequality associated with the logarithmic function for functions belonging to our class.

1. Introduction

Special functions are well-established streams of research in geometric function theory, applied mathematics, representation theory, engineering, physics and astronomy, quantum chemistry, computer science and many other mathematical sciences. One of the reasons for the frequent occurrence of special functions is that; solutions of extremal problems can often be expressed in terms of special functions. Another reason is that some important conformal mappings are given by special functions. For instance, the conformal mapping of an annulus onto the complement of two closed segments on the real axis and the conformal mapping of a square onto a rectangle are given by elliptic functions.

In this paper, we consider the Miller-Ross function which is introduced by Miller and Ross, [32]. For any complex numbers α, β and z, this special function is defined by

$$M_{\alpha,\beta}(z) = z^\alpha \sum_{n=0}^{\infty} \frac{(\beta z)^n}{\Gamma(n+\alpha+1)}, \quad \text{where} \ \ \Re(\alpha) > 0,\ \Re(\beta) > 0.$$

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The probability distributions of a random variable constitutes a core concept within the domains of statistics and probability theory. Their probability mass functions have been crucial in probability theory and many other mathematical sciences. There has been tremendous studies make use of distribution series in mathematical sciences. Using the probability distributions, many researchers have investigated some important features in the field of geometric function theory such as coefficient estimates and the Fekete and Szegö functional problem, see for example ([42], page2).

In addition, a discrete random variable X is said to adhere to a Poisson distribution with the parameter $m > 0$ if its probability mass function can be represented in the following manner

$$P(X = k) = \frac{e^{-m} m^k}{k!}, \quad \text{where} \quad k = 0, 1, 2, 3, \cdots$$

In this paper, we make use of convolution of the Macluarin series representation of Borel distribution and Miller-Ross function to introduce a novel class of starlike bi-univalent functions associated with Gegenbauer polynomials.

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where} \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. As known univalent functions are injective (one-to-one) functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, according to Koebe one-quarter Theorem [13], the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius 1/4. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \quad r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{A}$ which are given by equation (1.1). For more information about univalent and bi-univalent functions we refer the readers to the articles [27], [30], [37], [39] the monograph [13], [19] and the references therein.

The research in the geometric function theory has been very active in recent years, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class.

Coefficient related investigations of functions belong to the class Σ began around the 1970. It is worth mentioning that, in the year 1967, Lewin [27] studied the class of bi-univalent functions and derived the bound for $|a_2|$. Later on, in the year 1969, Netanyahu [37] showed that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions belong to the class Σ. In addition, in the year 1979, Brannan and Clunie [8] proved that $|a_2| \leq \sqrt{2}$ for functions in the class Σ. Since then, many researchers investigated the coefficient bounds for various subclasses of the bi-univalent function class Σ. However, not much is known about the bounds of the general coefficients $|a_2|$ for $n \geq 4$. In fact, the coefficient estimate problem for the general coefficient $|a_n|$ is still an open problem.

In the year 1933, Fekete and Szegö [17] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{A}$ with any complex λ is called the Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [2], [4], [9], [11], [17], [20], [22], [30], [46] and the references therein.

2. Preliminaries

In this section we present some information that are crucial for the main results of this paper. In the year 1903, Mittag-Leffler [33] defined a special function, named after him, as follows

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)}, \quad \text{where} \quad z \in \mathbb{C}, \text{ and } \Re\{\alpha\} > 0.$$

This last series converges in the whole complex plane for all values of $\Re(\alpha) > 0$ and diverges for $z \neq 0$ when $\Re\{\alpha\} < 0$. Additionally, when $\Re\{\alpha\} = 0$, the radius of convergence is given by $e^{\pi/2|Im(z)|}$. In 1905, Wiman ([52], [53]) introduced and studied the Mittag-Leffler function of two parameters as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+n\alpha)}, \quad \text{where} \quad z \in \mathbb{C}, \ \Re\{\alpha\} > 0, \Re\{\beta\} > 0.$$

For additional insights into Mittag-Leffler functions and their various applications, we recommend that readers consult the works of [6], [21], [23], [26], [28], [38], [54] and the references provided therein.

Recently, in their monograph, [32], Miller and Ross introduced a special function, which is later called the Miller-Ross function. This function is defined as

$$M_{\alpha,\beta}(z) = z^\alpha e^{\beta z} X^*(\alpha, \beta z),$$

where $z, \alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and X^* is the incomplete gamma function. Moreover, $M_{\alpha,\mu}(z)$ is the solution of the following ordinary differential equation

$$y' - \beta y = \frac{z^{\alpha-1}}{\Gamma(\alpha)}.$$

Furthermore, the Miller-Ross function can be written as

$$M_{\alpha,\beta}(z) = z^{\alpha} E_{1,1+\alpha}(\beta z),$$

where $E_{1,1+\alpha}(\beta z)$ is the Mittag-Leffler function of two parameters.

It is clear that $M_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Therefore, it is natural to consider the following normalization of the Miller-Ross function:

$$\begin{aligned}\mathbb{M}_{\alpha,\beta}(z) &= z^{1-\alpha}\Gamma(1+\alpha) M_{\alpha,\mu}(z)\\ &= z + \sum_{n=2}^{\infty} \frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n+\alpha)} z^n.\end{aligned}$$

It is worth mentioning that Eker and Ece, [15], proved that for $\beta > 0$, if $\alpha > (2+\sqrt{2})\beta - 1$ then the normalized Miller-Ross function is starlike in the unit disk $\mathbb{D}$. They also proved that if $\alpha > 2\beta - 1$, then the normalized Miller-Ross function is univalent and starlike in the disk $\mathbb{D}_{1/2} = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$. In addition, they proved if $\alpha > (2+\sqrt{2})\beta - 1$ then the normalized Miller-Ross function is univalent and convex in the disk $\mathbb{D}_{1/2}$.

In this paper, we restrict our attention to the case of real numbers $\alpha > -1$ and $\beta > 0$. it is well-known that the probability mass function of the Miller-Ross-type Poisson distribution is given by

$$\mathcal{P}_{\alpha,\beta}(m;k) := \frac{(m\beta)^k m^{\alpha}}{\Gamma(\alpha+k+1) M_{\alpha,\beta}(m)}, \quad \text{where } \alpha > -1,\ \beta > 0. \tag{2.1}$$

Recently, [42], Şeker et al. used the probability mass function (2.1) to introduce the Miller-Ross-type distribution series as follows

$$\mathcal{M}^{m}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{(m\beta)^{n-1} m^{\alpha}}{\Gamma(\alpha+n) M_{\alpha,\beta}(m)}. \tag{2.2}$$

Note that, if we put $\alpha = 0$ and $\beta = 1$ in the power series (2.2), we obtain the power series representation of the Poisson distribution which is introduced by Porwal [40].

The convolution of specific analytic functions has been of very important in geometric function theory. It can be used to express a wide variety of differential and integral operators. The convolution, also referred to as Hadamard product, of two analytic functions $f(z)$ as described in the equation (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is expressed as follows:

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Moreover, the convolution operation provides a deeper mathematical exploration and enhances our understanding of the geometric and symmetric properties of functions within the space $\mathcal{H}$. Its significance in operator theory and geometric function theory is well-established and thoroughly discussed in the available literature. For those seeking further insights into convolution within geometric function theory, we recommend consulting the monograph [13], as well as the articles [4], [34] and the associated references therein.

Now, using the convolution, we introduce the following linear operator

$$\Psi^m_{\alpha,\beta} : \mathcal{A} \to \mathcal{A}$$

which is define as:

$$\begin{aligned}\Psi^m_{\alpha,\beta} f(z) &= \mathcal{M}^m_{\alpha,\beta}(z) * f(z)\\ &= z + \sum_{n=2}^{\infty} \Psi_n a_n z^n,\end{aligned} \tag{2.3}$$

where

$$\Psi_n = \frac{(m\beta)^{n-1} m^\alpha}{\Gamma(\alpha+n) M_{\alpha,\beta}(m)}.$$

For more information about the Miller-Ross-type Poisson distribution and its applications in the geometric function theory we refer the interested readers to the articles [7], [15], [16], [34], [42], [43], [47] and the related references included therein.

In the year 1994, Szynal [48] introduced and studied a subclass $\mathfrak{F}(\delta)$ of the class $\mathcal{A}$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} K(z,x)\, d\sigma(x), \tag{2.4}$$

where $K(z,x) = \dfrac{z}{(z^2-2xz+1)^\delta}$, $\delta \geq 0$, $-1 \leq t \leq 1$, and σ is the probability measure on $[-1,1]$. Moreover, the function $K(z,x)$ has the following Taylor-Maclaurin series expansion

$$K(z,x) = z + \Delta(1,\delta,x)z^2 + \Delta(2,\delta,x)z^3 + \Delta(3,\delta,x)z^4 + \cdots,$$

where $\Delta(n,\delta,x)$ denotes the Gegenbauer polynomials of order δ and degree n in x. Furthermore, For any real numbers $\delta, x \in \mathbb{R}$,with $\alpha \geq 0$ and $-1 \leq x \leq 1$, and $z \in \mathbb{D}$ the generating function of Gegenbauer polynomials is given by

$$G_\delta(z,x) = (z^2 - 2xz + 1)^{-\delta}.$$

Moreover, for any fixed x the function $G_\delta(z,x)$ is analytic on the unit disk $\mathbb{D}$ and its Taylor-Maclaurin series is given by

$$G_\delta(z,x) = \sum_{n=0}^{\infty} \Delta(n,\delta,x) z^n. \tag{2.5}$$

In addition, Gegenbauer polynomials can be defined in terms of the following recurrence relation:

$$\Delta(n,\delta,x) = \frac{2x(n+\delta-1)\Delta(n-1,\delta,x) - (n+2\delta-2)\Delta(n-1,\delta,x)}{n}, \tag{2.6}$$

with initial values,

$$\Delta(0,\delta,x)=1,\quad \Delta(1,\delta,x)=2\delta x,\quad \text{and}\quad \Delta(2,\delta,x)=2\delta(\delta+1)x^2-\delta. \tag{2.7}$$

It is well-known that the Gegenbauer polynomials and their special cases, are orthogonal polynomials, such as Legendre polynomials $L_n(x)$ and the Chebyshev polynomials of the second kind $T_n(x)$ where the values of δ are $\delta = 1/2$ and $\delta = 1$ respectively, more precisely

$$L_n(x,z)=G_{\frac{1}{2}}(z,x),\quad \text{and}\quad T_n(x,z)=G_1(z,x).$$

For more information about the Gegenbauer polynomials and their special cases, we refer the readers to the articles [3], [5], [20], [24], [30], [46], the monograph [13], [19], [45], and the references therein.

The characterization of starlike functions is essential in complex analysis, particularly in the geometric function theory. A function f is said to be starlike if for any $\mu \in f(\mathbb{D})$ and $k \in [0,1]$, then $k\mu$ belongs to $f(\mathbb{D})$. Furthermore, the class of starlike functions of order α represented by $\mathcal{S}^*(\alpha)$, where the parameter α satisfies the condition $0 \le \alpha < 1$. A function $f \in \mathcal{S}$ is considered to be a member of the class $\mathcal{S}^*(\alpha)$ if the following condition holds for all $\zeta \in \mathbb{D}$:

$$\mathcal{R}\rceil \left\{ \frac{\zeta f'(\zeta)}{f(\zeta)} \right\} > \alpha.$$

When considering the case where $\alpha = 0$, it follows that $\mathcal{S}^*(0)$ coincides with the classical class of starlike functions, denoted as $\mathcal{S}^*$. The class $\mathcal{S}^*(\phi)$ was introduced by Ma and Minda [29] and is defined as

$$\mathcal{S}^*(\phi)=\left\{ f\in\mathcal{S} : \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \phi(\zeta) \right\},$$

where ϕ is a univalent function characterized by having a positive real part within the open unit disk $\mathbb{D}$. Moreover, the image $\phi(\mathbb{D})$ is symmetric with respect to the real axis, and it satisfies the conditions $\phi'(0) > 0$ and is starlike with respect to the value $\phi(0) = 1$.

Notably, substituting ϕ with specific functions yields various recognized subclasses of starlike functions. For instance, Sokól and Stankiewicz [44] defined the class $\mathcal{S}^*(\sqrt{1+z})$. Furthermore, Mendiratta et al. [31] introduced the class $\mathcal{S}^*_e$, which arises from the choice of $\phi(z) = e^z$. Another example is the class $\mathcal{S}^*_{SG}$, which is derived from the function $\phi(z)\dfrac{2}{(1+e^{-z})}$, as presented by Goel and Kumar [18]. When $\phi(z)$ is taken to be $1 + \sin z$, the resulting class is $\mathcal{S}^*_{Sin}$, introduced by Cho et al. [12]. Furthermore, Kumar and Bango [25] defined the class $\mathcal{S}^*_{log}$, which is obtained by selecting $\phi(z) = 1 - \log(1 + z)$. For more information about starlike functions we refer the interested readers to consult the articles For more information about classes of starlike analytic functions, we refer the readers, for example, to the articles [5], [6], [31], [34], [41], [44], [51] and the references provided therein.

Moreover, let f and g be analytic functions in $\mathbb{D}$. We say the function f is subordinate to the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwartz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$

then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D}$. For more information about the Subordination Principle we refer the readers to the monographs [14], [32] and [36].

The main purpose of this article is to make use of Miller-Ross function and Poisson distribution to introduce a new class of starlike bi-univalent functions that is subordinated to Gegenbauer polynomials, which we denote as $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$, which we define as follows.

Definition 2.1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$ if it satisfies the following subordinations:*

$$\frac{z\left(\Psi^m_{\alpha,\beta}f(z)\right)'}{\Psi^m_{\alpha,\beta}f(z)} \prec G_\delta(x,z), \tag{2.8}$$

and

$$\frac{w\left(\Psi^m_{\alpha,\beta}g(w)\right)'}{\Psi^m_{\alpha,\beta}g(w)} \prec G_\delta(x,w), \tag{2.9}$$

where $\alpha > -1$, $\beta > 0$, $\delta > 0$, $\frac{1}{2} < x \leq 1$ and the function $g(w) = f^{-1}(w)$ is given by the equation (1.2).

In our analysis, we introduce a parameter δ that plays a crucial role in categorizing our class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$. The choice of δ can significantly influence the properties and behaviors of this class, leading us to identify distinct subclasses based on its values. For example, taking values of δ as $\delta = 1/2$ and $\delta = 1$, give us the orthogonal polynomials Legendre polynomials $L_n(x,z)$ and the Chebyshev polynomials of the second kind $T_n(x,z)$, respectively. Therefore, we get the following subclass of starlike bi-univalent functions defined using Miller-Ross-type Poisson distribution that are subordinate to the Legendre polynomials and the Chebyshev polynomials of the second kind.

Example 2.2. *Let f be a bi-univalent function. Then f is said to be in the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; L_n(x,z)$ if it satisfies the following subordinations:*

$$\frac{z\left(\Psi^m_{\alpha,\beta}f(z)\right)'}{\Psi^m_{\alpha,\beta}f(z)} \prec L_n(x,z), \tag{2.10}$$

and

$$\frac{w\left(\Psi^m_{\alpha,\beta}g(w)\right)'}{\Psi^m_{\alpha,\beta}g(w)} \prec L_n(x,w), \tag{2.11}$$

where $\alpha > -1$, $\beta > 0$, $\frac{1}{2} < x \leq 1$ and the function $g(w) = f^{-1}(w)$ is given by the equation (1.2).

Example 2.3. *Let f be a bi-univalent function. Then f is said to be in the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; T_n(x,z)$ if it satisfies the following subordinations:*

$$\frac{z\left(\Psi^m_{\alpha,\beta}f(z)\right)'}{\Psi^m_{\alpha,\beta}f(z)} \prec T_n(x,z), \tag{2.12}$$

and

$$\frac{w\left(\Psi^m_{\alpha,\beta} g(w)\right)'}{\Psi^m_{\alpha,\beta} g(w)} \prec T_n(x,w), \tag{2.13}$$

where $\alpha > -1$, $\beta > 0$, $\frac{1}{2} < x \leq 1$ *and the function* $g(w) = f^{-1}(w)$ *is given by the equation (1.2).*

The subsequent lemma, extensively elaborated upon in existing literature (refer to, for example, [22]), represents well-established principles that hold significant importance for the research we are currently presenting.

Lemma 2.4. *If* L *belongs to the Caratheodory class* $\mathcal{P}$, *then for* $z \in \mathbb{D}$ *the function* Ω *can be written as*

$$L(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

In addition, $|p_n| \leq 2$ *for* $n \geq 1$. *Moreover, for any* $\lambda \in \mathbb{C}$, *we have*

$$|p_2 - p_1^2| \leq 2\max\{1, |1 - 2\lambda|\}.$$

In particular, if λ *is a real number, then*

$$|p_2 - p_1^2| \leq \begin{cases} -4\lambda + 2, & \text{if} \;\; \lambda \leq 0, \\ 2, & \text{if} \;\; 0 \leq \lambda \leq 1, \\ 4\lambda - 2, & \text{if} \;\; \lambda \geq 1. \end{cases}$$

The primary goal of this study is to determine the bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$, which we introduce using a Miller-Ross function and Poisson distribution that associated with Gegenbauer polynomials. In addition, we examine the corresponding Fekete-Szegö problem for functions belong to the presenting class. Moreover, we determine the coefficient bounds and the Fekete-Szegö problem that are associated with the logarithmic function. We also provide relevant connections of our main results with those considered in earlier investigations.

3. Initial Coefficient Bounds and Fekete-Szegö problem

In this section, we provide estimates for the initial Taylor-Maclaurin coefficients for the functions belong to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$ which are given by equation (1.1) as well as some of its various subclasses. Moreover, we present the Fekete-Szegö inequalities for the functions belong to our class and some of its various special cases.

Theorem 3.1. *Let the function* f *be a bi-univalent function given by equation (1.1). If* f *belongs to the class* $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$, *then*

$$|a_2| \leq \frac{\delta(2x)^{3/2}}{\sqrt{|8x^2\psi_3 - (2x^2(1+4\delta) - 2x - 1)\psi_2^2|}}, \tag{3.1}$$

and

$$|a_3| \leq \frac{4\delta^2 x^2}{\psi_2^2} + \frac{\delta x}{\psi_3}. \tag{3.2}$$

Proof. Let f belong to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$. Then, using Definition 2.1, there are two Schwarz functions $u(z)$ and $v(w)$ on the unit disk $\mathbb{D}$ such that

$$\frac{z\left(\Psi^m_{\alpha,\beta}f(z)\right)'}{\Psi^m_{\alpha,\beta}f(z)} = G_\delta(x, u(z)), \tag{3.3}$$

and

$$\frac{w\left(\Psi^m_{\alpha,\beta}g(w)\right)'}{\Psi^m_{\alpha,\beta}g(w)} = G_\delta(x, v(w)). \tag{3.4}$$

Now, we define the following analytic functions

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{3.5}$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \tag{3.6}$$

It is clear that, these functions $h(z)$ and $k(w)$ are analytic in the open unit disk $\mathbb{D}$ and belong to the Caratheodory class. In addition, $p(0) = 1 = q(0)$, they have positive real parts, $|p_j| \le 2$ and $|q_j| \le 2$ for all $j \in \mathbb{N}$.

Therefore, we can rewrite the equations (3.5) and (3.6) in the following manner

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{p_1}{2}z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \frac{1}{2}\left(\frac{p_1^3}{4} - p_1 p_2 + p_3\right)z^3 + \cdots \tag{3.7}$$

and

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{q_1}{2}w + \frac{1}{2}\left(q_2 - \frac{q_1^2}{2}\right)w^2 + \frac{1}{2}\left(\frac{q_1^3}{4} - q_1 q_2 + q_3\right)w^3 + \cdots \tag{3.8}$$

Now, by consulting the equations (2.5) and (3.7), the right-hand side of equation (3.3) can be written as follows

$$G_\delta(x, u(z)) = 1 + \frac{p_1}{2}\Delta(1,\delta,x)z + \left[\frac{\Delta(1,\delta,x)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\Delta(2,\delta,x)}{4}p_1^2\right]z^2 + \cdots \tag{3.9}$$

Similarly, by consulting the equations (2.5) and (3.8), the right-hand side of equation (3.4) can be written as follows

$$\begin{aligned} G_\delta(x, v(w)) = 1 &+ \frac{q_1}{2}\Delta(1,\delta,x)w \\ &+ \left[\frac{\Delta(1,\delta,x)}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{\Delta(2,\delta,x)}{4}q_1^2\right]w^2 + \cdots \end{aligned}$$

On the other hand, using the Miller-Ross Poisson distribution (2.3), we can write the left-hand side of equation (3.3) as follows

$$\frac{z\left(\Psi^m_{\alpha,\beta}f(z)\right)'}{\Psi^m_{\alpha,\beta}f(z)} = 1 + \psi_2 a_2 z + (2\psi_3 a_3 - \psi_2^2 a_2)z^2 + \cdots \tag{3.10}$$

Similarly, using the Miller-Ross Poisson distribution (2.3), we can write the left-hand side of equation (3.4) as follows

$$\frac{w\left(\Psi^m_{\alpha,\beta}g(w)\right)'}{\Psi^m_{\alpha,\beta}g(w)} = 1 - \psi_2 a_2 w + [-2\psi_3 a_3 + (4\psi_3 - \psi_2^2)a_2]z^2 + \cdots \tag{3.11}$$

Now, invoking (3.3) upon comparing the coefficients in both sides of (3.9) and (3.10) we get the following two equations

$$\psi_2 a_2 = \frac{p_1}{2}\Delta(1,\delta,x), \tag{3.12}$$

$$2\psi_3 a_3 - \psi_2^2 a_2^2 = \frac{\Delta(1,\delta,x)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\Delta(2,\delta,x)}{4}p_1^2. \tag{3.13}$$

Similarly, invoking (3.4) upon comparing the coefficients in both sides of (**??**) and (3.11) we get the following two equations

$$-\psi_2 a_2 = \frac{q_1}{2}\Delta(1,\delta,x), \tag{3.14}$$

$$-2\psi_3 a_3 + (4\psi_3 - \psi_2^2)a_2^2 = \frac{\Delta(1,\delta,x)}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{\Delta(2,\delta,x)}{4}q_1^2. \tag{3.15}$$

Now, the addition of squares of equations (3.12) and (3.14), we get the following

$$8\psi_2^2 a_2^2 = (p_1^2 + q_1^2)(\Delta(1,\delta,x))^2. \tag{3.16}$$

On the other hand, adding equation (3.12) to equation (3.14) gives $p_1 = -q_1$ and $p_1^2 + q_1^2 = 2p_1^2$. Therefore, the last equation can be written as follows

$$4\psi_2^2 a_2^2 = p_1^2(\Delta(1,\delta,x))^2. \tag{3.17}$$

Moreover, addition of the equations (3.13) and (3.15) gives the following equation

$$(8\psi_3 - 4\psi_2^2)a_2^2 = \Delta(1,\delta,x)\left((p_2 + q_2) - \frac{(p_1^2 + q_1^2)}{2}\right) + \Delta(2,\delta,x)\frac{(p_1^2 + q_1^2)}{2}.$$

Hence, using the fact $p_1^2 + q_1^2 = 2p_1^2$ then substituting the value of p_1^2 from equation (3.17), the last equation can be written as follows

$$\begin{aligned}&\left\{[\Delta(1,\delta,x)]^2(8\psi_3 - 4\psi_2^2) + 4\psi_2^2[\Delta(1,\delta,x) - \Delta(2,\delta,x)]\right\}a_2^2\\ &= (p_2 + q_2)[\Delta(1,\delta,x)]^3.\end{aligned} \tag{3.18}$$

Therefore, using the initial values of the Gegenbauer polynomials (2.7), then using the facts $|p_2| \le 2$ and $|q_2| \le 2$, the last equation gives

$$|a_2|^2 \le \frac{8|\delta|^3|x|^3}{|\delta x^2(8\psi_3 - 4\psi_2^2) + \psi_2^2(2x - 2(1+\delta)x^2 + 1)|}.$$

Simplifying the last inequality gives the desired estimation of $|a_2|$ as presented in inequality (3.1).

For the rest of this proof, we are looking for the estimation of $|a_3|$. Subtracting equation (3.15) from equation (3.13) then using the fact $p_1^2 - q_1^2 = 0$, we easily obtain the following equation

$$a_3 = a_2^2 + \Delta(1,\delta,x)\left(\frac{p_2 - q_2}{8\psi_3}\right) \tag{3.19}$$

Therefore, consulting equation (3.17), then using the facts $|q_2| \leq 2$ and $|p_i| \leq 2$, for $i = 1, 2$, we get the following inequality

$$|a_3| \leq \frac{(\Delta(1,\delta,x))^2}{\psi_2^2} + \frac{\Delta(1,\delta,x)}{2\psi_3}.$$

Finally, using the initial values of the Gegenbauer polynomials (2.7), we get the desired estimation of $|a_3|$. This completes the proof of Theorem 3.1. □

The following theorem considers the classical Fekete-Szegö problem for functions belong to our class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$.

Theorem 3.2. *Let f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$, then for some $\lambda \in \mathbb{R}$ the following inequality holds*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{4\delta x}{\psi_3}, & \text{if} \ \ |1-\lambda| \leq \frac{\psi_2^2}{2\delta x \psi_3} \\ \frac{8\delta^2 x^2 |1-\lambda|}{\psi_2^2}, & \text{if} \ \ |1-\lambda| \geq \frac{\psi_2^2}{2\delta x \psi_3}. \end{cases} \tag{3.20}$$

Proof. For a real value λ, using the equations (3.19) and (3.16), we easily obtain the following equations

$$a_3 - \lambda a_2^2 = (1-\lambda)a_2^2 + \frac{p_2 - q_2}{8\psi_3}\Delta(1,\delta,x) \tag{3.21}$$

$$= \frac{p_2 - q_2}{8\psi_3}\Delta(1,\delta,x) + (1-\lambda)\frac{p_1^2 + q_1^2}{8\psi_2^2}(\Delta(1,\delta,x))^2 \tag{3.22}$$

$$= \frac{\Delta(1,\delta,x)}{8}\left\{\frac{p_2 - q_2}{\psi_3} + (p_1^2 + q_1^2)\Omega\right\}, \tag{3.23}$$

where $\Omega = \dfrac{(1-\lambda)\Delta(1,\delta,x)}{\psi_2^2}$.

Therefore, we get the following inequalities

$$|a_3 - \lambda a_2^2| \leq \frac{|\Delta(1,\delta,x)|}{8}\left\{\frac{|p_2 - q_2|}{\psi_3} + |p_1^2 + q_1^2||\Omega|\right\} \tag{3.24}$$

$$\leq \frac{|\Delta(1,\delta,x)|}{8}|p_1^2 + q_1^2|\left\{\frac{1}{\psi_3} + |\Omega|\right\} \tag{3.25}$$

$$\leq 2|\Delta(1,\delta,x)| \max\left\{\frac{1}{\psi_3} + |\Omega|\right\}, \tag{3.26}$$

where the last inequality we used the fact that $|p_1| \leq 2$ and $|q_1| \leq 2$. Moreover, the last inequality can be written as follows

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{2|\Delta(1,\delta,x)|}{\psi_3}, & \text{if} \ \ |\Omega| \leq \frac{1}{\psi_3} \\ 2|\Delta(1,\delta,x)||\Omega|, & \text{if} \ \ |\Omega| \geq \frac{1}{\psi_3}. \end{cases} \tag{3.27}$$

Finally, using the initial values (2.7) and simplifying the last inequality (3.27), we get the desired result (3.30). This completes the theorem's proof. □

The following two corollaries give the initial coefficients bounds and the Fekete-Szegö inequality of functions belonging to the subclass $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; L_n(x,z))$ which consists of starlike bi-univalent functions that are subordinate to the Legendre polynomials. For more information about the Legendre polynomials we refer the Legendre polynomials we refer the interested readers to the articles [1], [5], [10], [35] and the related references included therein.

Corollary 3.3. *Let the function f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; L_n(x,z))$, then*

$$|a_2| \leq \frac{\sqrt{2}x^{3/2}}{\sqrt{|8x^2\psi_3 - (6x^2 - 2x - 1)\psi_2^2|}}, \tag{3.28}$$

and

$$|a_3| \leq \frac{2x^2\psi_3 + x\psi_2^2}{2\psi_2^2\psi_3}. \tag{3.29}$$

Corollary 3.4. *Let f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; L_n(x,z))$, then for some $\lambda \in \mathbb{R}$ the following inequality holds*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{2x}{\psi_3}, & \text{if } |1-\lambda| \leq \frac{\psi_2^2}{x\psi_3} \\ \frac{2x^2|1-\lambda|}{\psi_2^2}, & \text{if } |1-\lambda| \geq \frac{\psi_2^2}{x\psi_3}. \end{cases} \tag{3.30}$$

The following corollary present the initial coefficients bounds and the Fekete-Szegö inequality of functions belonging to the subclass $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; T_n(x,z))$ which consists of starlike bi-univalent functions that are subordinate to the Chebyshev polynomials. For more information about the Chebyshev polynomials we refer the Legendre polynomials we refer the interested readers to the articles [9], [30], [49], [50] and the related references included therein.

Corollary 3.5. *Let the function f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; T_n(x,z))$, then*

$$|a_2| \leq \frac{(2x)^{3/2}}{\sqrt{|8x^2\psi_3 - (10x^2 - 2x - 1)\psi_2^2|}}, \tag{3.31}$$

and

$$|a_3| \leq \frac{4x^2\psi_3 + x\psi_2^2}{\psi_2^2\psi_3}. \tag{3.32}$$

Corollary 3.6. *Let f be a bi-univalent function given by equation (1.1). If f belongs to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; T_n(x,z))$, then for some $\lambda \in \mathbb{R}$ the following inequality holds*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{4x}{\psi_3}, & \text{if } |1-\lambda| \leq \frac{\psi_2^2}{2x\psi_3} \\ \frac{8x^2|1-\lambda|}{\psi_2^2}, & \text{if } |1-\lambda| \geq \frac{\psi_2^2}{2x\psi_3}. \end{cases} \tag{3.33}$$

4. Initial Logarithmic Coefficient Bounds and Fekete-Szegö problem

In this section, we are looking to determine the coefficient bounds and the Fekete-Szegö inequalities associated with the logarithmic function. It is well-known that if f is analytic function in the open unit disk $\mathbb{D}$ such that $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, then the logarithmic coefficients of f are given by

$$\log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} 2d_n z^n, \quad \text{for} \quad z \in \mathbb{D}. \tag{4.1}$$

In addition, for a function f belongs to our class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$, the left-hand side of the subordination (2.8) we get f must be analytic in the open unit disk $\mathbb{D}$ and $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. Hence, for all functions f belonging to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$, the relation (4.1) is well defined. The following theorem gives the initial logarithmic coefficient bounds for functions belonging to our specific class.

Theorem 4.1. *Let f be a function belongs to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$ with the logarithmic coefficients given by the equation (4.1), then the following inequalities hold*

$$|d_1| \leq \frac{\delta x}{\psi_2}, \tag{4.2}$$

and

$$|d_2| \leq \frac{\delta x}{4\psi_3} \max\left|\frac{2(1-\delta)x^2 + (x-1)}{x} + \frac{4\delta x \psi_3}{\psi_2^2}\right|. \tag{4.3}$$

Proof. Let f be in the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta}; G_\delta(x,z))$ that has the form (2.8). Then equating the coefficients of the equation (4.1), we obtain the following equations

$$2d_1 = a_2, \tag{4.4}$$

and

$$4d_2 = 2a_3 - a_2^2. \tag{4.5}$$

Now, consulting the equations (3.12) and (3.13), we obtain the following equations

$$a_2 = \frac{\Delta(1,\delta,x)p_1}{2\psi_2}, \tag{4.6}$$

and

$$a_3 = \frac{[(\Delta(1,\delta,x))^2 - \Delta(1,\delta,x) + \Delta(2,\delta,x)]p_1^2 + \Delta(1,\delta,x)p_2}{8\psi_3}. \tag{4.7}$$

Therefore, using the equations (4.4) and (4.6), we obtain the following equation

$$d_1 = \frac{\Delta(1,\delta,x)p_1}{4\psi_2}. \tag{4.8}$$

Thus, using the initial values (2.7), the last equation gives the bound of $|d_1|$ as required in the inequality (4.2).

One the other hand, consulting the equations (4.5), (4.6) and (4.7) we obtain the following equation

$$d_2 = \frac{\Delta(1,\delta,x)}{16\psi_3}\left\{p_2 - Kp_1^2\right\}, \tag{4.9}$$

where

$$K=\frac{\Delta(1,\delta,x)\psi_3}{\psi_2^2}-\frac{\Delta(2,\delta,x)}{\Delta(1,\delta,x)}-\Delta(1,\delta,x)+1$$

Hence, using Lemma 2.4, we get the following inequality

$$|d_2|\le\frac{|\Delta(1,\delta,x)|}{8\psi_3}\max\{1,|1-2K|\}. \tag{4.10}$$

Finally, using the initial values (2.7), we get the required bound of $|d_2|$ that presented by the inequality (4.3). This completes the proof. □

The following theorem presents the initial logarithmic Fekete-Szegö inequality for functions belong to our specific class.

Theorem 4.2. *Let f be a function belongs to the class $\mathcal{S}^*(\Psi^m_{\alpha,\beta};G_\delta(x,z))$ with the logarithmic coefficients given by the equation (4.1). Then for some $\lambda\in\mathbb{C}$ the following inequality holds*

$$|d_2-\lambda d_1^2|\le\begin{cases}\frac{\delta x}{4\psi_3}, & \text{if} \quad 1+\lambda\in[\lambda_1,\lambda_2]\\ \frac{|\mathcal{B}|}{4\psi_3\psi_2^2}, & \text{if} \quad 1+\lambda\notin[\lambda_1,\lambda_2],\end{cases} \tag{4.11}$$

where

$$\mathcal{B}=[1+(1-4\delta)x-2(1+\delta)x^2]\delta\psi_2^2+4\delta^2\psi_3(1+\lambda)x^2$$
$$\lambda_1=\frac{[(6\delta+2)x^2-2x-1]\psi_2^2}{4\delta\psi_3x^2},$$
$$\lambda_2=\frac{[(6\delta+2)x^2-1]\psi_2^2}{4\delta\psi_3x^2}$$

Proof. For any complex number λ, consulting the equations (4.4) and (4.5), we obtain the following equation

$$d_2-\lambda d_1^2=\frac{1}{2}\left(a_3-\frac{(1+\lambda)a_2^2}{2}\right).$$

Therefore, using the equations (4.6) and (4.7), the last equation can be written as follows

$$d_2-\lambda d_1^2=\frac{\Delta(1,\delta,x)}{16\psi_3}\left\{p_2-\left(\frac{(1+\lambda)\psi_3\Delta(1,\delta,x)}{\psi_2^2}-\frac{\Delta(2,\delta,x)}{\Delta(1,\delta,x)}-\Delta(1,\delta,x)+1\right)p_1^2\right\}.$$

Hence, applying Lemma 2.4 on the last equation, we easily obtain

$$|d_2-\lambda d_1^2|\le\frac{|\Delta(1,\delta,x)|}{8\psi_3}\max\left\{1,|\Omega|\right\}, \tag{4.12}$$

where

$$\Omega=\frac{2(1+\lambda)\psi_3\Delta(1,\delta,x)}{\psi_2^2}+\frac{\Delta(1,\delta,x)-2\left(\Delta(1,\delta,x)\right)^2-2\Delta(2,\delta,x)}{\Delta(1,\delta,x)}.$$

Now, if $|\Omega|\le1$, then we get

$$\lambda_1\le(1+\lambda)\le\lambda_2,$$

where

$$\lambda_1=\frac{[(\Delta(1,\delta,x))^2-\Delta(1,\delta,x)+\Delta(2,\delta,x)]\psi_2^2}{\left(\Delta(1,\delta,x)\right)^2\psi_3}, \tag{4.13}$$

$$\lambda_2 = \frac{[(\Delta(1,\delta,x))^2 + \Delta(2,\delta,x)]\psi_2^2}{(\Delta(1,\delta,x))^2 \psi_3}. \tag{4.14}$$

Thus, using the equations (4.13), (4.14) and Lemma 2.4, inequality (4.12) can be written as follows

$$|d_2 - \lambda d_1^2| \le \begin{cases} \frac{|\Delta(1,\delta,x)|}{8\psi_3}, & \text{if} \quad 1+\lambda \in [\lambda_1, \lambda_2] \\ \frac{|\Delta(1,\delta,x)||\Omega|}{4\psi_3}, & \text{if} \quad 1+\lambda \notin [\lambda_1, \lambda_2]. \end{cases} \tag{4.15}$$

Finally, by using the initial values of Gegenbauer polynomials that are presented in equation (2.7), simplifying the right-hand side of the inequality (4.15), we arrive at the required result as presented in inequality (4.2). This signifies the completion of the proof. □

5. Conclusion

This research paper has investigated a new class of starlike bi-univalent functions using the Miller-Ross-type Poisson distribution, that is subordinated to Gegenbauer polynomials, which we denoted as $\mathcal{S}^*(\Psi_{\alpha,\beta}^m; G_\delta(x,z))$. The author successfully found the growth and distortion bounds for functions belonging to our class and some of its various subclasses. Moreover, we derived the classical Fekete-Szegö inequality of functions belonging to our class and some of its various subclasses. Furthermore, we found estimated for the initial coefficients and the classical Fekete-Szegö functional problem associated with the logarithmic function for functions belonging to our class. The findings of this research are expected to inspire the researchers to connect our class with other orthogonal polynomials. Also, the insights provided in this paper are anticipated to motivate researchers to broaden these concepts to encompass harmonic functions and symmetric q-calculus.

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7. Conflicts of interest

The author declares that there is no conflicts of interest.

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Waleed Al-Rawashdeh
Department of Mathematics
Zarqa University
2000 Zarqa, 13110 Jordan
E-mail address: `walrawashdeh@zu.edu.jo`