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ON THE SOLVABILITY OF ONE DIMENSIONAL INVERSE PROBLEM FOR FRACTIONAL WAVE EQUATION WITH MEMORY TERM AND INTEGRAL OVERDETERMINATION CONDITIONS

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Abstract. The aim of this work is to determine the source term and to solve the problem of interest. First, we study the direct problem in a second order fractional wave equation subject to periodic, Dirichlet and nonlocal initial conditions. Finally, To solve the problem of interest, we transform the considered problem to an equivalent problem. Using the Fourier method, the equivalent problem is reduced to a system of integrals equations. By a contraction mapping, The existence and uniqueness of the solution of the system of integral equations is proved.Then, the existence and uniqueness of the solution of the inverse problem is obtained.

1. Introduction and problem statement

The study of fractional wave equations has gained significant attention in recent decades due to their ability to model complex physical phenomena that exhibit nonlocal and memory effects, characteristics that are often observed in a wide range of real-world systems. These equations arise in various fields such as physics, engineering, and applied mathematics, where they offer more accurate representations of anomalous diffusion, viscoelastic materials, and electromagnetic wave propagation, among others. The general form of fractional differential equations, which incorporate fractional derivatives, has its roots in the works of Riemann, Liouville, and others in the 19th century, though it was not until the 1970s that fractional calculus became a more formalized and widely studied area, spurred by the efforts of scholars like Samko, Kilbas, and Marichev [18]. In the context of wave phenomena, fractional equations extend classical models like the wave equation to account for memory and hereditary effects, making them ideal for describing systems where traditional integer-order models fail to capture observed dynamics. In many practical scenarios, however, the goal is not only to understand the direct behavior of such systems, but also to identify underlying unknown parameters, such as source terms,

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from observed data. This challenge, known as the inverse problem, is a central topic in applied mathematics and physics due to its importance in areas like remote sensing, medical imaging, and geophysics [17]. Solving inverse problems often involves the development of novel mathematical techniques, as these problems are typically ill-posed and require special approaches to guarantee well-defined solutions. The study of inverse problems for fractional wave equations has attracted increasing interest [11], particularly for their relevance in applications where non-local interactions play a key role. In this regard we mention the work of E. Azizbayov and Y. Mehraliyev [2], in which they considered the problem:

$$\begin{cases} c(t)u_t(x,t) = u_{xx}(x,t) + a(t)u(x,t) + b(t)g(x) + f(x,t), & \\ \text{for } (x,t) \in \Omega = [0,1] \times [0,T], & \\ u(x,0) + \delta u(x,T) + \int_0^T p(t)u(x,t) = \varphi(x), & (0 \le x \le 1), \\ u(0,t) = u(1,t), & (0 \le t \le T), \\ \int_0^1 u(x,t)dx = 0, & (0 \le t \le T), \\ u(x_i,t) = h_i(t), & (i = 1,2;\ 0 \le t \le T), \end{cases}$$

and proved the existence of classical solution by the Fourier method. In this paper the motivation is to study and find a classical solution to the inverse problem of fractional wave equation with memory term.In this work we study the linear wave equation

$$D_{0,t}^{\alpha} u(x,t) - u_{xx}(x,t) = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} u(x,s)ds + b(t)u(x,t) + a(t)g(x) + f(x,t), \quad (x,t) \in \Omega, \tag{1.1}$$

with nonlocal initial conditions

$$u(x,0) + \int_0^T k_1(t)u(x,t) = \varphi(x), \quad \forall x \in [0,1], \tag{1.2}$$

$$u_t(x,0) + \int_0^T k_2(t)u(x,t) = \psi(x), \quad \forall x \in [0,1], \tag{1.3}$$

the Dirichlet condition

$$u(0,t) = 0, \qquad \forall t \in [0,T], \tag{1.4}$$

and the nonlocal boundary condition

$$u_x(0,\ t) = u_x(1,t), \qquad \forall t \in [0,T], \tag{1.5}$$

where $\Omega = [0,1] \times [0,T]$, with $T < +\infty$, $f(x,t)$, $\varphi(x)$, $\psi(x)$, $g(x)$, $k_1(t)$ and $k_2(t)$ are given functions, β is a positive constant with $1 < \beta < 2$. $D_{0,t}^{\alpha} u$ stand for the Caputo fractional derivatives of order α, with $1 < \alpha < 2$. For (1.1-1.5) the problem direct is the determination of $u(x,t)$ in Ω such that $u \in C^{2,2}(\Omega)$ and $D_{0,t}^{\alpha} u \in C(\Omega)$. when the functions $f(x,t)$, $\varphi(x)$, $\psi(x)$, $g(x)$, $a(t)$, $b(t)$ are given and continuous.While the inverse problem consists of determining $a(t)$, $b(t)$ and $u(x,t)$ from the nonlocal initial conditions (1.2-1.3) and the nonlocal boundary conditions (1.4-1.5). This problem is not uniquely solvable. To have the inverse problem is uniquely solvable, we impose the overdetermination conditions

$$u(1,t) + \int_0^1 u(x,t)dx = h(t), \qquad h(t) \neq 0 \text{ for all } t \in [0,T], \tag{1.6}$$

$$\int_0^1 xu(x,t)dx = m(t), \qquad m(t) \neq 0 \text{ for all } t \in [0,T], \tag{1.7}$$

where $D_{0,t}^{\alpha}u$ denote the derivative at the Caputo sense defined by

$$D_{0,t}^{\alpha}u = I^{2-\alpha}u_{tt}(x,t) = \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}u_{ss}(x,s)ds.$$

and $1 < \alpha < 2$.
We assume that the following matching conditions for the functions $\varphi(x)$, $\psi(x)$ are satisfied

$$\begin{cases} \varphi(0) = \psi(0) = 0, \\ \varphi'(0) = \varphi'(1), \\ \psi'(0) = \psi'(1), \\ \varphi(1) + \int_0^1 \varphi(x)dx = h(0) + \int_0^T k_1(t)h(t), \\ \psi(1) + \int_0^1 \psi(x)dx = h'(0) + \int_0^T k_2(t)h(t). \\ \int_0^1 \varphi(x)dx = m(0) + \int_0^T k_1(t)m(t) \end{cases}, \tag{1.8}$$

and

$$M(t) = m(t)\left(g(1) + \int_0^1 g(x)dx\right) - h(t)\int_0^1 xg(x)dx \neq 0 \tag{1.9}$$

The aim is to determine the source term and to find a solution to the associated inverse problem. We first analyze the direct problem, deriving the solution to the fractional wave equation under various boundary conditions and examining the theoretical implications of each. Building on classical techniques, we transform the original problem into an equivalent formulation that simplifies the analysis and leads to a system of integral equations. The use of Fourier methods in this context is not new, as it has been widely employed in the study of wave equations and their inverse counterparts [5]. By applying the contraction mapping principle, we establish the existence and uniqueness of the solution to the integral equation system. Finally, we extend this result to the inverse problem itself, proving the existence and uniqueness of the solution for the source term. The approach presented in this work contributes to the growing body of research on fractional wave equations and inverse problems, providing both theoretical insights and practical tools for solving complex problems in various scientific and engineering disciplines. In this context, it's important to highlight the monograph [10, 6, 2, 9, 14] et all, in [10], the statement of problem and the proof techniques used in the study are different from representation in this paper. The organization of this paper is as follows. In Section 1, the considered problem is stated. Sections 2 deals with the solvability of the direct problem. While section 3 gives the solvability of the considered inverse problem.
For the study of this problem, we first recall the following results.

Lemma 1.1. *The solution of equation*

$$v(t) + \lambda\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}vds = q(t),$$

for $\lambda \in \mathbb{R}$ *and* $\beta > 0$ *satisfies the integral equation*

$$v(t) = q(t) - \lambda\int_0^t (t-s)^{\beta-1}h(s)E_\beta\left(-\lambda(t-s)^\beta, \beta\right)ds.$$

where $q \in L^1(\mathbb{R})$ *and* E_β *is the Mittag-Leffler Function defined by*

$$E_\beta(z,\mu) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\beta n + \mu)}.$$

Lemma 1.2. *see* [15].
1) There exist $C > 0$, *such that*

$$|E_\alpha(z,\mu)| \leq \frac{C}{1+|z|}, \qquad \mathcal{R}e(z) \leq 0$$

2)see [3], [13]

$$\frac{1}{\Gamma(\beta)} \int_0^t \tau^{\mu-1} E_\alpha(\lambda\tau^\alpha,\mu)\,(t-\tau)^{\beta-1}\,d\tau = t^{\beta+\mu-1} E_\alpha(\lambda t^\alpha, \mu+\beta).$$

3)

$$\frac{1}{\Gamma(\beta)} + zE_\alpha(z,\alpha+\beta) = E_\alpha(z,\beta), \qquad \alpha > 0,\ \beta > 0$$

Existence and uniqueness of the solution of the direct problem

Definition 1.1. *The function* $u(x,t)$, *is said to be a classical solution of the problem (1.1-1.5) if* $u(x,t) \in C^{2,2}(\Omega)$, $D_{0,t}^\alpha u \in C(\Omega)$ *and verifying the conditions (1.1-1.5).*

To solve the homogeneous problem (1.1-1.5) by Fourier's method, we arrive at the following spectral problem

$$\begin{cases} X'' + \lambda^2 X = 0, & \forall x \in [0,1] \\ X(0) = 0, & \\ X'(0) = X'(1). & \end{cases} \tag{1.10}$$

This problem is not self adjoint. We can prove that the set

$$S = \{X_0 = x,\ X_{2k} = x\cos\lambda_k x,\ X_{2k-1}(x) = \sin\lambda_k x\},$$

form Riesz basis in $L^2(0,1)$ biorthogonal to the set

$$\widetilde{S} = \{Y_0 = 2,\ \ Y_{2k}(x) = 4\cos\lambda_k x,\ Y_{2k-1} = 4(1-x)\sin\lambda_k x\}, \quad \lambda_k = 2\pi k, \quad k = 1,2,3,...$$

For more details, the reader can consult [8, 7, 12]. Therefore for the solvability of the problem (1.1-1.5), we shall seek the function $u(x,t)$ in the form

$$u(x,t) = u_0(t)\,x + \sum_{k\geq 1} u_{2k}(t) X_{2k} + u_{2k-1}(t) X_{2k-1}, \tag{1.11}$$

where

$$u_i(t) = \int_0^1 u(x,t) Y_i(x) dx, \qquad i = 0,1,2,...$$

these functions $u_i(t)$ are twice differentiables on $[0,T]$. Substituting (1.11) into (1.1), we get

$$D_{0,t}^\alpha u_0(t) = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} u_0(s)\,ds + b(t)u_0(t) + a(t)g_0 + f_0(t), \tag{1.12}$$

$$D_{0,t}^\alpha u_{2k}(t) + \lambda_k^2 u_{2k}(t) = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} u_{2k}(s)ds + b(t)u_{2k}(t) + a(t)g_{2k} + f_{2k}(t), \tag{1.13}$$

$$D_{0,t}^\alpha u_{2k-1}(t) + \lambda_k^2 u_{2k-1}(t) =$$

$$\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} u_{2k-1}(s)ds + b(t)u_{2k-1}(t) + 2\lambda_k u_{2k}(t) + a(t)g_{2k-1} + f_{2k-1}(t). \quad (1.14)$$

and the following conditions are satisfied

$$u_i(0) + \int_0^T k_1(t)u_i(t)dt = \varphi_i, \qquad i \in \{0, 2k, 2k-1\} \quad (1.15)$$

$$u_i'(0) + \int_0^T k_2(t)u_i(t)dt = \psi_i, \qquad i \in \{0, 2k, 2k-1\} \quad (1.16)$$

where

$$\begin{cases} f_i(t) = \int_0^1 f(x,t)Y_i(x)dx, \\ g_i = \int_0^1 g(x)Y_i(x)dx, \\ \varphi_i = \int_0^1 \varphi(x)Y_i(x)\,dx, \\ \psi_i = \int_0^1 \psi(x)Y_i(x)\,dx. \end{cases}$$

Applying operator $I^{\alpha-1}$ to both sides in (1.12-1.14), and using the Dirichlet formula, the solution of the first equation is given by

$$\begin{aligned} u_0(t) &= \varphi_0 + t\psi_0 - \int_0^T k_1(t)u_0(t)dt - \int_0^T tk_2(t)u_0(t)dt \\ &+ \int_0^t \frac{(t-s)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} u_0(s)\,ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} b(s)\,u_0(s)\,ds \\ &+ g_0 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)\,ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_0(s)\,ds, \end{aligned} \quad (1.17)$$

the solutions of the second and third equations (1.13), (1.14) are given by

$$\begin{aligned} u_{2k}(t) &= \varphi_{2k} + t\psi_{2k} - \int_0^T k_1(t)u_{2k}(t)dt - \int_0^T tk_2(t)u_{2k}(t)dt \\ &-\lambda_k^2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_{2k}(s)ds + \int_0^t \frac{(t-s)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} u_{2k}(s)\,ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} b(s)u_{2k}(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(g_{2k}a(s) + f_{2k}(s)\right) ds, \end{aligned}$$

and

$$\begin{aligned} u_{2k-1}(t) &= \varphi_{2k-1} + t\psi_{2k-1} - \int_0^T k_1(t)u_{2k-1}(t)dt - \int_0^T tk_2(t)u_{2k-1}(t)dt \\ &-\lambda_k^2 \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_{2k-1}(s)ds + \int_0^t \tfrac{(t-s)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} u_{2k-1}(s)\,ds + \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} b(s)u_{2k-1}(s)ds \\ &+2\lambda_k \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_{2k}(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(g_{2k-1}a(s) + f_{2k-1}(s)\right) ds. \end{aligned}$$

According to lemma 1.1 and 1.2, the previous equalities can be expressed as follows

$$\begin{aligned} u_{2k}(t) &= -\lambda_k^2 t^\alpha E_\alpha\left(-\lambda_k^2 t^\alpha, \alpha+1\right) \left(\varphi_{2k} - \int_0^T k_1(t)u_{2k}(t)dt\right) \\ &-\lambda_k^2 t^{\alpha+1} E_\alpha\left(-\lambda_k^2 t^\alpha, \alpha+2\right) \left(\psi_{2k} - \int_0^T k_2(t)u_{2k}(t)dt\right) \\ &-\lambda_k^2 \int_0^t (t-s)^{2\alpha-\beta+1} E_\alpha\left(-\lambda_k^2 (t-s)^\alpha, 2\alpha-\beta+2\right) u_{2k}(s)\,ds \end{aligned}$$

$$-\lambda_k^2 \int_0^t (t-s)^{2\alpha-1} E_\alpha\left(-\lambda_k^2 (t-s)^\alpha, 2\alpha\right) b(s) u_{2k}(s)\, ds$$

$$-\lambda_k^2 \int_0^t (t-s)^{2\alpha-1} E_\alpha\left(-\lambda_k^2 (t-s)^\alpha, 2\alpha\right) \left(g_{2k} a(s) + f_{2k}(s)\right) ds. \tag{1.18}$$

and

$$u_{2k-1}(t) = -\lambda_k^2 t^\alpha E_\alpha\left(-\lambda_k^2 t^\alpha, \alpha+1\right) \left(\varphi_{2k-1} - \int_0^T k_1(t) u_{2k-1}(t) dt\right)$$

$$-\lambda_k^2 t^{\alpha+1} E_\alpha\left(-\lambda_k^2 t^\alpha, \alpha+2\right) \left(\psi_{2k-1} - \int_0^T k_2(t) u_{2k-1}(t) dt\right)$$

$$-\lambda_k^2 \int_0^t (t-s)^{2\alpha-\beta+1} E_\alpha\left(-\lambda_k^2 (t-s)^\alpha, 2\alpha-\beta+2\right) u_{2k-1}(s)\, ds$$

$$-2\lambda_k^3 \int_0^t (t-s)^{2\alpha-1} E_\alpha\left(-\lambda_k^2 (t-s)^\alpha, 2\alpha\right) u_{2k}(s) ds$$

$$-\lambda_k^2 \int_0^t (t-s)^{2\alpha-1} E_\alpha\left(-\lambda_k^2 (t-s)^\alpha, 2\alpha\right) b(s) u_{2k-1} ds$$

$$-\lambda_k^2 \int_0^t (t-s)^{2\alpha-1} E_\alpha\left(-\lambda_k^2 (t-s)^\alpha, 2\alpha\right) \left(g_{2k-1} a(s) + f_{2k-1}(s)\right) ds. \tag{1.19}$$

For the solvability of this problem, we denote by $B^3_{2,T}$ ([16, 19]) the set of all functions $u(x,t)$ of the form

$$u(x,t) = u_0(t)x + \sum_{k\geq 1} u_{2k}(t) X_{2k} + u_{2k-1}(t) X_{2k-1},$$

defined on Ω such that $u_0(t)$, $u_{2k}(t)$, $u_{2k-1}(t)$ are continuous on $[0,T]$ and

$$\|u_0(t)\|_{C[0,T]} + \sqrt{\sum_{k\geq 1} \lambda_k^6 \|u_{2k}(t)\|^2_{C[0,T]}} + \sqrt{\sum_{k\geq 1} \lambda_k^6 \|u_{2k-1}(t)\|^2_{C[0,T]}} < +\infty.$$

The norm of this space is defined by

$$\|u(x,t)\|_{B^3_{2,T}} = \|u_0(t)\|_{C[0,T]} + \sqrt{\sum_{k\geq 1} \lambda_k^6 \|u_{2k}(t)\|^2_{C[0,T]}} + \sqrt{\sum_{k\geq 1} \lambda_k^6 \|u_{2k-1}(t)\|^2_{C[0,T]}},$$

it is obvious that these spaces are Banach spaces.

Theorem 1.3. *Let the following conditions be satisfied*
H1) $f \in C\left(\overline{\Omega}\right)$, $f_x,\ f_{xx} \in C\left(\overline{\Omega}\right)$, $f_{xxx} \in L^2(\Omega)$, *such that*

$$f(0,t) = f_{xx}(0,t) = 0,\ \ f_x(0,t) = f_x(1,t).$$

H2) $g,\ g',\ g'' \in C([0,1])$, $g''', g'''' \in L^2(0,1)$, *and the following conditions*

$$g(0) = g''(0) = 0,\ \ g'(0) = g'(1)$$

hold
H3) $\varphi \in C^2([0,1])$, $\varphi'''(x) \in L^2[0,1]$, $\psi \in C^2([0,1])$, *and the compatibility conditions (1.8) are satisfied and*

$$\varphi''(0) = \psi''(0) = 0$$

Then the problem (1.1-1.5) has a unique solution in the ball $S(0,R)=\left\{z\in E^3_{2,T},\ \|z\|_{B^3_{2,T}}\leq R,\ R>0\right\}$ *of the space* $B^3_{2,T}$.

Proof. To solve the system (1.17-1.19) in the space $B^3_{2,T}$, we consider the operator equation

$$Lu=w(x,t)=w_0(t)x+\sum_{k\geq0}w_{2k}(t)X_{2k}+\sum_{k\geq1}w_{2k-1}(t)X_{2k-1},\tag{1.20}$$

the functions $w_0(t)$, $w_{2k}(t)$ and $w_{2k-1}(t)$ are equal to the right-hand sides of 1.17, 1.18 and 1.19, respectively.

According to lemma 1.2 we can deduce the following estimates

$$\begin{aligned}\|w_0\|_{C[0,T]}\leq&\left(T\|k_1(t)+Tk_2(t)\|_{C[0,T]}+\frac{T^{\alpha-\beta+2}}{\Gamma(\alpha-\beta+3)}\right)\|u_0(t)\|_{C[0,T]}\\&+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|b(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}+|\varphi_0|+T|\psi_0|\\&+\frac{T^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)}\|f_0(t)\|_{L^2([0,T])}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}g_0\|a(t)\|_{C[0,T]},\end{aligned}\tag{1.21}$$

$$\begin{aligned}\|w_{2k}\|_{C[0,T]}\leq&\,CT\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\left(\|k_1(t)\|_{C[0,T]}+T\|k_2(t)\|_{C[0,T]}\right)\right)\|u_{2k}(t)\|_{C[0,T]}\\&+\frac{T^{\alpha-1}}{\alpha}\|b(t)\|_{C[0,T]}\|u_{2k}(t)\|_{C[0,T]}+C|\varphi_{2k}|+CT|\psi_{2k}|\\&+C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\|f_{2k}(t)\|_{L^2([0,T])}+\frac{CT^{\alpha}}{\alpha}g_{2k}\|a(t)\|_{C[0,T]},\end{aligned}$$

and

$$\begin{aligned}\|w_{2k-1}\|_{C[0,T]}\leq&\,CT\left(\tfrac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\left(\|k_1(t)\|_{C[0,T]}+T\|k_2(t)\|_{C[0,T]}\right)\right)\|u_{2k-1}(t)\|_{C[0,T]}\\&+C\frac{T^{\alpha}}{\alpha}\|b(t)\|_{C[0,T]}\|u_{2k-1}(t)\|_{C[0,T]}+\frac{2CT^{\alpha}}{\lambda_k}\|u_{2k}(t)\|_{C[0,T]}+C|\varphi_{2k-1}|+CT|\psi_{2k-1}|\\&+C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\|f_{2k-1}(t)\|_{L^2([0,T])}+\frac{CT^{\alpha}}{\alpha}g_{2k-1}\|a(t)\|_{C[0,T]}.\end{aligned}$$

From the previous inequalities we deduce

$$\begin{aligned}&\sqrt{\sum_{k\geq1}\lambda_k^6\|w_{2k}\|^2_{C[0,T]}}\leq\\&\sqrt{5}CT\left(\tfrac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\left(\|k_1(t)\|_{C[0,T]}+T\|k_1(t)\|_{C[0,T]}\right)+\tfrac{T^{\alpha-1}}{\alpha}\|b(t)\|_{C[0,T]}\right)\sqrt{\textstyle\sum_{k\geq1}\lambda_k^6\|u_{2k}\|^2_{C[0,T]}}\\&+\sqrt{5}\frac{CT^{\alpha}}{\alpha}\|a\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^6|g_{2k}|^2}+\sqrt{5}C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\sqrt{\sum_{k\geq1}\int_0^T\lambda_k^6|f_{2k}|^2dt}\\&+\sqrt{5}C\sqrt{\sum_{k\geq1}\lambda_k^6|\varphi_{2k}|^2}+\sqrt{5}CT\sqrt{\sum_{k\geq1}\lambda_k^6|\psi_{2k}|^2}\end{aligned}\tag{1.22}$$

and

$$\sqrt{\sum_{k\geq1}\lambda_k^6\|w_{2k-1}\|^2_{C[0,T]}}\leq$$

$$\sqrt{6}CT\left(\tfrac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\|k_1(t)\|_{C[0,T]}+T\|k_2(t)\|_{C[0,T]}+\tfrac{T^{\alpha-1}}{\alpha}\|b(t)\|_{C[0,T]}\right)\sqrt{\textstyle\sum_{k\geq1}\lambda_k^6\|u_{2k-1}\|^2_{C[0,T]}}$$
$$+2\sqrt{6}CT^{\alpha}\sqrt{\sum_{k\geq1}\lambda_k^6\|u_{2k}\|^2_{C[0,T]}}+\sqrt{6}C\sqrt{\sum_{k\geq1}\lambda_k^6|\varphi_{2k-1}|^2}+\sqrt{6}CT\sqrt{\sum_{k\geq1}\lambda_k^6|\psi_{2k-1}|^2}$$
$$+\sqrt{6}\frac{CT^{\alpha}}{\alpha}\|a\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^6|g_{2k-1}|^2}+\sqrt{6}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\sqrt{\sum_{k\geq1}\int_0^T\lambda_k^6|f|^2_{2k-1}dt}, \quad (1.23)$$

Then, summing the last inequalities with (1.21), we obtain

$$\|Lu\|_{B^3_{2,T}}\leq A(T)\|u\|_{B^3_{2,T}}+B(T).$$

where

$$A(T)=$$
$$T\max\left\{2\sqrt{6}CT^{\alpha-1},\ \frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+3)}+\|k_1(t)+tk_2(t)\|_{C[0,T]}+\frac{T^{\alpha-1}}{\Gamma(\alpha+1)}\|b(t)\|_{C[0,T]},\right.$$
$$\left.2\sqrt{6}C\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\|k_1(t)\|_{C[0,T]}+T\|k_1(t)\|_{C[0,T]}+\tfrac{T^{\alpha-1}}{\alpha}\|b(t)\|_{C[0,T]}\right)\right\}$$

and

$$B(T)=\|\varphi\|_{L^2[0,1]}+4\sqrt{5}C\|\varphi'''(x)\|_{L^2[0,1]}+4\sqrt{6}C\|-3\varphi''(x)+(1-x)\varphi'''(x)\|_{L^2[0,1]}$$
$$+T\|\psi\|_{L^2[0,1]}+4\sqrt{6}CT\|-3\psi''(x)+(1-x)\psi'''(x)\|_{L^2[0,1]}+4\sqrt{5}CT\|\psi'''(x)\|_{L^2(0,1)}$$
$$+4\sqrt{6}\tfrac{CT^{\alpha}}{\alpha}\|a\|_{C[0,T]}\|-3g''(x)+(1-x)g'''(x)\|_{L^2[0,1]}+4\sqrt{5}\tfrac{CT^{\alpha}}{\alpha}\|a\|_{C[0,T]}\|g'''(x)\|_{L^2[0,1]}$$
$$+4\sqrt{6}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\|-3f_{xx}(x)+(1-x)f_{xxx}(x)\|_{L^2[0,1]}\right\|_{L^2(0,T)}$$
$$+4\sqrt{5}C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\|f_{xxx}(x)\|_{L^2[0,1]}\right\|_{L^2(0,T)}+\frac{T^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)}\|f(x,t)\|_{L^2(\Omega)}.$$

We choose $R=\rho B(T)$, where $\rho>1$ and if

$$A(T)\leq\frac{\rho-1}{\rho}, \quad (1.24)$$

then the operator L mapping the elements of the ball $S(0,R)$ into itself.
From the definition of L, we have for $u_1,\ u_2\in S(0,R)$

$$\|Lu_1-Lu_2\|_{B^3_{2,T}}\leq A(T)\|u_1-u_2\|_{B^3_{2,T}}, \quad (1.25)$$

then according to (1.24), the operator L is a contraction, so it has a unique fixed point in the ball $S(0,R)$ of the space $B^3_{2,T}$ wich is a solution of operator equation (1.20), then the function u as an element of the space $B^3_{2,T}$ is continuous and has continuous derivatives u_x, u_{xx} in Ω. It is easy to verify that the conditions (1.1-1.5) are satisfied in the classical sense. □

2. Solvability of the inverse problem

Definition 2.1. *The triplet of functions* $\{u(x,t),\ a(t),\ b(t)\}$ *is said to be a classical solution of the problem (1.1-1.7) if* $u(x,t)\in C^{2,2}(\Omega)$, $D^{\alpha}_{0,t}u\in C(\Omega)$, $a(t),\ b(t)\in C[0,T]$ *and satisfies the conditions (1.1-1.7).*

2.1. **Equivalent problem.**

Lemma 2.1. *Assume that $f(x,t) \in C(\Omega)$, $\varphi(x)$, $\psi(x)$, $g(x) \in C[0,1]$, $h(t)$, $m(t) \in C^2[0,T]$ and the compatibility conditions (1.8) hold. Then the problem (1.1-1.7) is equivalent to (1.1-1.5) and the conditions*

$$D_{0,t}^{\alpha} h(t) - u_{xx}(1,t) = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} h(s)ds + b(t)h(t)$$
$$+ a(t)\left(g(1) + \int_0^1 g(x)dx\right) + f(1,t) + \int_0^1 f(x,t)dx. \tag{2.1}$$

$$D_{0,t}^{\alpha} m(t) - u_x(1,t) + u(1,t) = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} m(s)ds + b(t)m(t)$$
$$+ a(t)\int_0^1 xg(x)dx + \int_0^1 xf(x,t)dx. \tag{2.2}$$

Proof. $\Longrightarrow$) Let $\{u(x,t),\ a(t),\ b(t)\}$ is a classical solution of problem (1.1-1.7), we suppose that $h(t)$, $m(t) \in C^2[0,T]$. From (1.1) we have

$$D_{0,t}^{\alpha} u(1,t) - u_{xx}(1,t) =$$
$$\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} u(1,s)ds + b(t)u(1,t) + a(t)g(1) + f(1,t).$$

integrating (1.1) over $[0,1]$, we have

$$D_{0,t}^{\alpha} \int_0^1 u(x,t)dx = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} \int_0^1 u(x,s)dxds + b(t)\int_0^1 u(x,t)dx$$
$$+ a(t)\int_0^1 g(x)dx + \int_0^1 f(x,t)dx.$$

By summing the last equalities, we get

$$D_{0,t}^{\alpha}\left(u(1,t) + \int_0^1 u(x,t)dx\right) - u_{xx}(1,t) =$$
$$\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}\left(u(1,s) + \int_0^1 u(x,s)dx\right)ds + b(t)\left(u(1,t) + \int_0^1 u(x,t)dx\right)$$
$$+ a(t)\left(g(1) + \int_0^1 g(x)dx\right) + f(1,t) + \int_0^1 f(x,t)dx, \tag{2.3}$$

taking into account (1.6) we conclude that (2.1) is satisfied.

Multiplying the both sides of Eq.(1.1) by x and integrating from 0 to 1 with respect to x, taking into account (1.7), we get (2.2)

$\Longleftarrow$) Suppose that $\{u(x,t),\ a(t),\ b(t)\}$ is a classical solution of problem (1.1-1.5) and the conditions (2.1) and (2.2) are satisfied. From (2.1) and (2.3) we have

$$D_{0,t}^{\alpha}\left(u(1,t) + \int_0^1 u(x,t)dx - h(t)\right) = b(t)\left(u(1,t) + \int_0^1 u(x,t)dx - h(t)\right)$$
$$+ \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}\left(u(1,s) + \int_0^1 u(x,s)dx - h(s)\right)ds, \tag{2.4}$$

Let $k(t) = u(1,t) + \int_0^1 u(x,t)dx - h\left(t\right)$, using the compatibility conditions (1.8), we have $k(0) = k'(0) = 0$. Then (2.4) becomes

$$k(t) = \int_0^t \left(\frac{(t-s)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} b(s) \right) k(s)ds, \tag{2.5}$$

According to Gronwell lemma we have $k(t) = 0$. Therefore

$$u(1,t) + \int_0^1 u(x,t)dx = h\left(t\right).$$

By multiplying both sides of Eq.(1.1) by x and integrating from 0 to 1 with respect to x and taking into account (2.2) we get

$$D_{0,t}^{\alpha} \left(\int_0^1 xu(x,t)dx - m(t) \right) = b(t) \left(\int_0^1 xu(x,t)dx - m(t) \right).$$

Similarlly, we can deduce that

$$\int_0^1 xu(x,t)dx = m\left(t\right).$$

The proof is complete. □

2.2. **Existence and uniqueness of the classical solution.** The first component of the classical solution is given by

$$u(x,t) = u_0(t)x + \sum_{k\geq 1} u_{2k}(t)X_{2k} + u_{2k-1}(t)X_{2k-1}, \tag{2.6}$$

where $u_0(t)$, $u_{2k}(t)$, $u_{2k-1}(t)$ are defined by (1.17), (1.18) and (1.19) respectively. From (2.1), (2.2), (1.18) and (1.9), the second and the third components of the classical solution are given by

$$\begin{aligned} a(t) = \frac{m(t)}{M(t)} &\left(D_{0,t}^{\alpha} \left(h(t)\right)(t) - \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} h(s)ds - f(1,t) - \int_0^1 f(x,t)dx \right) \\ -\frac{h(t)}{M(t)} &\left(D_{0,t}^{\alpha} \left(m(t)\right)(t) - \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} m(s)ds - \int_0^1 xf(x,t)dx \right) \\ &+ \frac{m(t)}{M\left(t\right)} \sum_{k\geq 1} \lambda_k^2 u_{2k}(t)dt + \frac{h(t)}{M\left(t\right)} \sum_{k\geq 1} \lambda_k u_{2k-1}(t)dt \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} b(t) = \frac{-\int_0^1 xg(x)dx}{M(t)} &\left(D_{0,t}^{\alpha} \left(h(t)\right)(t) - \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} h(s)ds - f(1,t) - \int_0^1 f(x,t)dx \right) \\ + \frac{g(1) + \int_0^1 g(x)dx}{M(t)} &\left(D_{0,t}^{\alpha} \left(m(t)\right)(t) - \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} m(s)ds - \int_0^1 xf(x,t)dx \right) \\ &- \frac{\int_0^1 xg(x)dx}{M\left(t\right)} \sum_{k\geq 1} \lambda_k^2 u_{2k}(t) - \frac{g(1) + \int_0^1 g(x)dx}{M(t)} \sum_{k\geq 1} \lambda_k u_{2k-1}(t), \end{aligned} \tag{2.8}$$

where $M\left(t\right) = \left(g(1) + \int_0^1 g(x)dx\right) m(t) - \left(\int_0^1 xg(x)dx\right) h(t) \neq 0$.

Therefore, to solve the problem (1.1-1.5), (2.1,2.2), the system (1.12-1.14) and (2.7,2.8) must be solved.

The uniqueness of the solution of the considered problem based on the following lemma

Lemma 2.2. *if the triplet* $\{u(x,t),a(t),b(t)\}$ *is a classical solution of the problem (1.1-1.5), (2.1 ,2.2), then*

$$\begin{cases} u_0(t) = \int_0^1 u(x,t)Y_0(x)dx, \\ u_{2k}(t) = \int_0^1 u(x,t)Y_{2k}(x)dx, \\ u_{2k-1}(t) = \int_0^1 u(x,t)Y_{2k-1}(x)dx, \end{cases} \tag{2.9}$$

are solution of the (1.12-1.14) on $[0,T]$.

Proof. If $\{u(x,t),\ a(t),\ b(t)\}$ is a classical solution of the problem (1.1-1.5), (2.1,2.2), then taking the scalair product in $L^2[0,1]$ of equation (1.1) with $Y_i(x)$, we obtain

$$\int_0^1 D_{0,t}^\alpha uY_{2k}(x)dx - \int_0^1 u_{xx}Y_{2k}(x)dx = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} \int_0^1 u(x,t)Y_{2k}(x)dx$$

$$+ \int_0^1 b(t)u(x,tY_{2k}(x)dx + \int_0^1 a(t)g(x)Y_{2k}(x)dx + \int_0^1 f(x,t)Y_{2k}(x)dx, \quad k \geq 0,$$

and

$$\int_0^1 D_{0,t}^\alpha uY_{2k-1}(x)dx - \int_0^1 u_{xx}Y_{2k-1}(x)dx = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} \int_0^1 u(x,t)Y_{2k-1}(x)dx$$

$$+ \int_0^1 b(t)u(x,tY_{2k-1}(x)dx + \int_0^1 a(t)g(x)Y_{2k-1}(x)dx + \int_0^1 f(x,t)Y_{2k-1}(x)dx.$$

Integrating over $[0,1]$, using the conditions (1.4) and (1.5) we obtain

$$\begin{cases} \int_0^1 u_{xx}Y_0(x)dx = 0, \\ \int_0^1 u_{xx}Y_{2k}(x)dx = -\lambda_k^2 u_{2k}, \\ \int_0^1 u_{xx}Y_{2k-1}(x)dx = -2\lambda_k u_{2k} - \lambda_k^2 u_{2k-1}. \end{cases}$$

Using the fact that $\int_0^1 D_{0,t}^\alpha uY_j(x)dx = D_{0,t}^\alpha \int_0^1 u(x,t)\, Y_j(x)dx = D_{0,t}^\alpha u_j$, we conclude that equations (1.12-1.14) are satisfied. Simillarly, according to conditions (1.2), (1.3) we get that conditions (1.15) and (1.16) are satisfied. Then the functions (2.9) are solutions of the system (1.12-1.14) on the interval $[0,T]$.
The proof of the lemma 2.2 is complete. □

Remark. *According to lemma 2.2, to prove the uniqueness of the solution to the problem (1.1-1.5, 2.1, 2.2), it suffices to prove the uniqueness of the solution to the system (1.12-1.14).*

It is easy to show that if

$$\begin{cases} u_{2k}(t) = \int_0^1 u(x,t)Y_{2k}(x)dx, & k \geq 0, \\ u_{2k-1}(t) = \int_0^1 u(x,t)Y_{2k-1}(x)dx, & k \geq 1, \end{cases}$$

are solutions of the system (1.12-1.14), then the functions
$u(x,t) = u_0(t)x + \sum\limits_{k\geq 1} u_{2k}(t)X_{2k} + u_{2k-1}(t)X_{2k-1}$, $a(t)$ and $b(t)$ defined by (2.7) and (2.8) respectively are solutions of the system (1.1-1.5, 2.1, 2.2).
To solve the system (1.12-1.14) and (2.7, 2.8) we need these spaces. Denote by $E_{2,T}^3$

the space $B^3_{2,T}\times C\,[0,T]\times C\,[0,T]$ of the functions $v(x,t)=\{u(x,t),a(t),b(t)\}$ whose standard norm

$$\|v(x,t)\|_{E^3_{2,T}}=\|u(x,t)\|_{B^3_{2,T}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\,,$$

it is obvious that these spaces are Banach spaces.
In the space $E^3_{2,T}$, we consider the operator $L(u,a,b)=\{L_1(u,a,b),L_2(u,a,b),L_3(u,a,b)\}\,,$ where

$$\begin{cases} L_1\left(u,a,b\right)=w(x,t)=w_0(t)x+\sum\limits_{k\geq 0}w_{2k}(t)X_{2k}+\sum\limits_{k\geq 1}w_{2k-1}(t)X_{2k-1}, \\ L_2\left(u,a,b\right)=c(t), \\ L_3\left(u,a,b\right)=d(t), \end{cases}$$

the functions $w_i(t)$, $c(t)$ and $d(t)$, are defined by the left sides of (1.17-1.19) and (2.7,2.9) respectrively.

Theorem 2.3. *If the conditions of theorem 1.3 are satisfied. Then the problem (1.1-1.5) and (2.1, 2.2) has a unique solution in the ball*
$\Phi\left(0,R\right)=\left\{z\in E^3_{2,T},\ \|z\|_{E^3_{2,T}}\leq R,\quad R>0\right\}$, *of the space* $E^3_{2,T}$.

Proof. From (1.21-1.23), we deduce the following estimates

$$\|L_1u\|_{B^3_{2,T}}\leq A_0\left(T\right)\|u\|_{B^3_{2,T}}+B_0(T)\|b(t)\|_{C[0,T]}\|u\|_{B^3_{2,T}}+C_0(T)\|a(t)\|_{C[0,T]}+D_0(T). \tag{2.10}$$

where

$$A_0\left(T\right)=\max\left\{T\|k_1(t)+tk_2(t)\|_{C[0,T]}+\frac{T^{\alpha-\beta+2}}{\Gamma\left(\alpha-\beta+3\right)},\right.$$
$$\sqrt{5}CT\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+2T^{\alpha}+\left(\|k_1(t)\|_{C[0,T]}+T\|k_2(t)\|_{C[0,T]}\right)+2\sqrt{6}CT^{\alpha}\right),$$
$$\left.\sqrt{6}C\,T\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+2T^{\alpha}+\left(\|k_1(t)\|_{C[0,T]}+T\|k_2(t)\|_{C[0,T]}\right)\right)\right\},$$
$$B_0(t)=\max\left\{\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)},\ \sqrt{6}C\frac{T^{\alpha}}{\alpha}\right\}$$
$$+4\sqrt{5}C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\|f_{xxx}(x)\|_{L^2(\Omega)}+\frac{T^{2\alpha-1}}{\left(2\alpha-1\right)\Gamma\left(\alpha\right)}\|f(x,t)\|_{L^2(\Omega)}\,.$$
$$C_0(T)=\max\left\{4\sqrt{6}\frac{CT^{\alpha}}{\alpha}\|3g''(x)-(1-x)g'''(x)\|_{L^2[0,1]}+4\sqrt{5}\frac{CT^{\alpha}}{\alpha}\|g'''(x)\|_{L^2[0,1]}\,,\right.$$
$$\left.\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)}\|g\|\right\}$$
$$D_0(t)=\max\left\{\|\varphi(x)\|_{L^2(0,1)}+T\|\psi(x)\|_{L^2(0,1)}\,,\right.$$
$$\left.4\sqrt{5}C\|\varphi'''(x)\|_{L^2[0,1]}+4\sqrt{6}C\|3\varphi''(x)-(1-x)\varphi'''(x)\|_{L^2[0,1]}\right\}$$
$$+4\sqrt{6}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\|3f_{xx}(x)-(1-x)f_{xxx}(x)\|_{L^2(\Omega)}$$
$$+4\sqrt{5}C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\|f_{xxx}(x)\|_{L^2(\Omega)}+\frac{T^{2\alpha-1}}{\left(2\alpha-1\right)\Gamma\left(\alpha\right)}\|f(x,t)\|_{L^2(\Omega)}\,.$$

According to (2.7) and (2.9) we have

$$\|a(t)\|_{C[0,T]} \leq$$
$$\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-2}} \sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k}^2(t)\right\|_{C[0,T]}}$$
$$+\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-4}} \sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k-1}^2(t)\right\|_{C[0,T]}}$$
$$+\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]} \left\|D_{0,t}^{\alpha}h(t)-\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}h(s)ds - f(1,t) - \int_0^1 f(x,t)dx\right\|_{C[0,T]}$$
$$+\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]} \left\|D_{0,t}^{\alpha}m(t)-\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}m(s)ds - \int_0^1 xf(x,t)dx\right\|_{C[0,T]}$$

and

$$\|b(t)\|_{C[0,T]} \leq$$
$$\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-2}} \sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k}^2(t)\right\|_{C[0,T]}}$$
$$+\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-4}} \sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k-1}^2(t)\right\|_{C[0,T]}},$$
$$+\left\|\frac{\int_0^1 xg(x)dx}{M(t)}\right\|_{C[0,T]} \left\|D_{0,t}^{\alpha}h(t)-\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}h(s)ds - f(1,t) - \int_0^1 f(x,t)dx\right\|_{C[0,T]}$$
$$+\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]} \left\|D_{0,t}^{\alpha}m\left(t\right)-\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}m(s)ds - \int_0^1 xf(x,t)dx\right\|_{C[0,T]},$$

from (1.22) and (1.23)

$$\|a(t)\|_{C[0,T]} \leq \sqrt{5}CT\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\times$$
$$\left(\tfrac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\|k_1(t)\|_{C[0,T]}+T\|k_1(t)\|_{C[0,T]}\right)\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k}\right\|_{C[0,T]}^2}$$
$$+\sqrt{6}CT\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\times$$
$$\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\|k_1(t)\|_{C[0,T]}+T\|k_2(t)\|_{C[0,T]}\right)\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k-1}\right\|_{C[0,T]}^2}$$
$$+2\sqrt{6}\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-4}}CT^{\alpha}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k}\right\|_{C[0,T]}^2}$$
$$+\sqrt{5}\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]} \sqrt{\sum_{k\geq 1}\lambda_k^{-2}}CT\tfrac{T^{\alpha-1}}{\alpha}\|b(t)\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k}\right\|_{C[0,T]}^2}$$

$$+\sqrt{6}CT\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\frac{T^{\alpha-1}}{\alpha}\left\|b(t)\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{6}\left\|u_{2k-1}\right\|_{C[0,T]}^2}$$

$$+\sqrt{5}\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\frac{CT^{\alpha}}{\alpha}\left\|a\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{6}\left|g_{2k}\right|^2}$$

$$+\sqrt{6}\frac{CT^{\alpha}}{\alpha}\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\left\|a\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{6}\left|g_{2k-1}\right|^2}$$

$$+\sqrt{5}C\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\left(\sqrt{\sum_{k\geq1}\lambda_k^{6}\left|\varphi_{2k}\right|^2}+T\sqrt{\sum_{k\geq1}\lambda_k^{6}\left|\psi_{2k}\right|^2}\right)$$

$$+\sqrt{6}C\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\left(\sqrt{\sum_{k\geq1}\lambda_k^{6}\left|\varphi_{2k-1}\right|^2}+T\sqrt{\sum_{k\geq1}\lambda_k^{6}\left|\psi_{2k-1}\right|^2}\right)$$

$$+\sqrt{5}C\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\sqrt{\sum_{k\geq1}\int_0^T\lambda_k^{6}\left|f_{2k}\right|^2dt}$$

$$+\sqrt{6}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\sqrt{\sum_{k\geq1}\int_0^T\left|\lambda_k^{3}f_{2k-1}\right|^2dt}$$

$$+\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^{\alpha}\left(h(t)\right)(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}h(s)ds-f(1,t)-\int_0^1f(x,t)dx\right\|_{C[0,T]}$$

$$+\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^{\alpha}\left(m(t)\right)(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}m(s)ds-\int_0^1xf(x,t)dx\right\|_{C[0,T]},$$

the previous inequality can be expressed as

$$\left\|a(t)\right\|_{C[0,T]}\leq A_1(T)\left(\sqrt{\sum_{k\geq1}\lambda_k^{6}\left\|u_{2k}\right\|_{C[0,T]}^2}+\sqrt{\sum_{k\geq1}\lambda_k^{6}\left\|u_{2k-1}\right\|_{C[0,T]}^2}\right)$$

$$+B_1(T)\left\|b(t)\right\|_{C[0,T]}\left(\sqrt{\sum_{k\geq1}\lambda_k^{6}\left\|u_{2k}\right\|_{C[0,T]}^2}+\sqrt{\sum_{k\geq1}\lambda_k^{6}\left\|u_{2k-1}\right\|_{C[0,T]}^2}\right)$$

$$+C_1(T)\left\|a(t)\right\|_{C[0,T]}+D_1(T),$$

where

$$A_1(T)=$$

$$\max\left\{\sqrt{5}CT\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\left(\tfrac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\left\|k_1(t)\right\|_{C[0,T]}+T\left\|k_1(t)\right\|_{C[0,T]}\right),\right.$$

$$\sqrt{6}CT\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\left(\left\|k_1(t)\right\|_{C[0,T]}+T\left\|k_2(t)\right\|_{C[0,T]}\right)\right),$$

$$\left.2\sqrt{6}\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}CT^{\alpha}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\right\},$$

$$B_1(T)=\max\left\{\sqrt{5}\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}C\frac{T^\alpha}{\alpha}\sqrt{\sum_{k\geq1}\lambda_k^{-2}},\sqrt{6}C\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\frac{T^\alpha}{\alpha}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\right\}$$

$$C_1(T)=\max\left\{\sqrt{5}\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\frac{CT^\alpha}{\alpha}\left\|g'''(x)\right\|_{L^2([0,1])},\right.$$

$$\left.\sqrt{6}\frac{CT^\alpha}{\alpha}\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\left\|3g''(x)-(1-x)g'''(x)\right\|_{L^2([0,1])}\right\},$$

and

$$D_1(T)=\sqrt{5}C\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\left\|\varphi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{5}CT\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\left\|\psi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{5}C\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|f_{xxx}(x)\right\|_{L^2(\Omega)}$$

$$+\sqrt{6}C\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\left\|-3\varphi''(x)+(1-x)\varphi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{6}CT\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\left\|3\psi''(x)-(1-x)\psi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{6}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\left\|3f_{xx}(x)-(1-x)f_{xxx}(x)\right\|_{L^2(\Omega)}$$

$$+\left\|\frac{m(t)}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^\alpha h(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}h(s)ds-f(1,t)-\int_0^1 f(x,t)dx\right\|_{C[0,T]}$$

$$+\left\|\frac{h(t)}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^\alpha m(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}m(s)ds-\int_0^1 xf(x,t)dx\right\|_{C[0,T]}.$$

Simillarly, we have

$$\|b(t)\|_{C[0,T]}\leq\sqrt{5}CT\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-2}}\times$$

$$\left(\tfrac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\|k_1(t)\|_{C[0,T]}+T\|k_1(t)\|_{C[0,T]}\right)\sqrt{\sum_{k\geq1}\lambda_k^6\|u_{2k}\|_{C[0,T]}^2}$$

$$+\sqrt{6}CT\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq1}\lambda_k^{-4}}\times$$

$$\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\|k_1(t)\|_{C[0,T]}+T\|k_2(t)\|_{C[0,T]}\right)\sqrt{\sum_{k\geq1}\lambda_k^6\|u_{2k-1}\|_{C[0,T]}^2}$$

$$+2\sqrt{6}CT^{\alpha}\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k}\right\|_{C[0,T]}^2}$$

$$+\sqrt{5}CT\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}\frac{T^{\alpha-1}}{\alpha}\left\|b(t)\right\|_{C[0,T]}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k}\right\|_{C[0,T]}^2}$$

$$+\sqrt{6}CT\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}\frac{T^{\alpha-1}}{\alpha}\left\|b(t)\right\|_{C[0,T]}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left\|u_{2k-1}\right\|_{C[0,T]}^2}$$

$$+\sqrt{5}\frac{CT^{\alpha}}{\alpha}\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}\left\|a\right\|_{C[0,T]}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|g_{2k}\right|^2}$$

$$+\sqrt{6}\frac{CT^{\alpha}}{\alpha}\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}\left\|a\right\|_{C[0,T]}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|g_{2k-1}\right|^2}$$

$$+\sqrt{5}C\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|\varphi_{2k}\right|^2}$$

$$+\sqrt{5}CT\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|\psi_{2k}\right|^2}$$

$$+\sqrt{6}C\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|\varphi_{2k-1}\right|^2}$$

$$+\sqrt{6}CT\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|\psi_{2k-1}\right|^2}$$

$$+\sqrt{5}C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|f_{2k}\right|^2}$$

$$+\sqrt{6}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\sqrt{\sum_{k\geq 1}\lambda_k^{6}\left|f_{2k-1}\right|^2}$$

$$+\left\|\frac{\int_0^1 xg(x)dx}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^{\alpha}h(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}h(s)ds-f(1,t)-\int_0^1 f(x,t)dx\right\|_{C[0,T]}$$

$$+\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^{\alpha}m(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}m(s)ds-\int_0^1 xf(x,t)dx\right\|_{C[0,T]}.$$

We denote by

$$A_2(T)=$$

$$\max\left\{\sqrt{5}CT\left\|\frac{\int_0^1 xg(x)dx}{M\left(t\right)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\left(\tfrac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\left\|k_1(t)\right\|_{C[0,T]}+T\left\|k_1(t)\right\|_{C[0,T]}\right),\right.$$

$$\sqrt{6}CT\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\left(\frac{T^{\alpha-\beta+1}}{\alpha-\beta+2}+\left\|k_1(t)\right\|_{C[0,T]}+T\left\|k_2(t)\right\|_{C[0,T]}\right),$$

$$2\sqrt{6}CT^{\alpha}\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\Bigg\},$$

$$B_2(T)=\max\Bigg\{\sqrt{5}CT\left\|\frac{\int_0^1 xg(x)dx}{M\,(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}\frac{T^{\alpha-1}}{\alpha}},$$

$$\sqrt{6}CT\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}\frac{T^{\alpha-1}}{\alpha}}\Bigg\},$$

$$C_2(T)=\max\Bigg\{\sqrt{5}\frac{CT^{\alpha}}{\alpha}\left\|\frac{\int_0^1 xg(x)dx}{M\,(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\left\|g'''(x)\right\|_{L^2([0,1])},$$

$$\sqrt{6}\frac{CT^{\alpha}}{\alpha}\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\left\|-3g''(x)+(1-x)g'''(x)\right\|_{L^2([0,1])}\Bigg\},$$

and

$$D_2(T)=\sqrt{5}C\left\|\frac{\int_0^1 xg(x)dx}{M\,(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\left\|\varphi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{5}CT\left\|\frac{\int_0^1 xg(x)dx}{M\,(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\left\|\psi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{6}C\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\left\|3\varphi''(x)-(1-x)\varphi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{6}CT\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\left\|3\psi''(x)-(1-x)\psi'''(x)\right\|_{L^2([0,1])}$$

$$+\sqrt{5}C\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\frac{\int_0^1 xg(x)dx}{M\,(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-2}}\left\|f_{xxx}(x)\right\|_{L^2(\Omega)}$$

$$+\sqrt{6}\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\sqrt{\sum_{k\geq 1}\lambda_k^{-4}}\left\|3f_{xx}(x)-(1-x)f_{xxx}(x)\right\|_{L^2(\Omega)}$$

$$+\left\|\frac{\int_0^1 xg(x)dx}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^{\alpha}h(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}h(s)ds-f(1,t)-\int_0^1 f(x,t)dx\right\|_{C[0,T]}$$

$$+\left\|\frac{g(1)+\int_0^1 g(x)dx}{M(t)}\right\|_{C[0,T]}\left\|D_{0,t}^{\alpha},m(t)-\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}m(s)ds-\int_0^1 xf(x,t)dx\right\|_{C[0,T]}$$

then

$$\|b(t)\|_{C[0,T]}\leq A_2\,(T)\,\|u\|_{B^3_{2,T}}+B_2(T)\,\|b(t)\|_{C[0,T]}\,\|u\|_{B^3_{2,T}}+C_2(T)\,\|a(t)\|_{C[0,T]}+D_2(T).$$

Combining the previous inequalities with (2.10) we have

$$\|Lu\|_{E^3_{2,T}}\leq A_3(T)\,\|u\|_{B^3_{2,T}}+B_3(T)\,\|b(t)\|_{C[0,T]}\,\|u\|_{B^3_{2,T}}+C_3(T)\,\|a(t)\|_{C[0,T]}+D_3(T),\tag{2.11}$$

where

$$\begin{cases} A_3(T) = A_0(T) + A_1(T) + A_2(T), \\ B_3(T) = B_0(T) + B_1(T) + B_2(T), \\ C_3(T) = C_0(T) + C_1(T) + C_2(T), \\ D_3(T) = D_0(T) + D_1(T) + D_2(T). \end{cases}$$

If $R = \rho D_3(T)$, $\rho > 1$, then the operator L is a contraction mapping.
First, we show that the operator L mapping the elements of the ball Φ_ρ into itself. It is clear that the operator L satisfies (2.11), we suppose that

$$A_3(T) + C_3(T) \leq \frac{\rho - 1}{2\rho} \text{ and } B_3(T)D_3(T) \leq \frac{\rho - 1}{2\rho^2}, \tag{2.12}$$

then

$$\|L(u,a)\|_{E_T^3} \leq \rho D_3(T). \tag{2.13}$$

Consequently, the operator L acts in the ball. It remains to prove that the operator L is a contraction. Indeed, similarly, with aid of (2.11) we have for any (u_1, b_1, a_1), $(u_2, b_{2,} a_2) \in \Phi_\rho$,

$$\|L(u_1, b_1, a_1) - L(u_2, b_{2,} a_2)\|_{E_{2,T}^3} \leq$$

$$\frac{\rho - 1}{2\rho}\left[\|u_1 - u_2\|_{B_{2,T}^3} + \|b_1(t) - b_2(t)\|_{C[0,T]} + \|a_1 - a_2\|_{C[0,T]}\right], \tag{2.14}$$

is satisfied. Then it follows from (2.12) together with (2.13) and (2.14) that the operator L maps the ball Φ_ρ into itself and is contractive. Consequently, the operator L has a unique fixed point $\{u, b, a\}$ in the ball Φ_ρ, that is the unique solution of the system (1.12-1.14).
Since $u \in B_{2,T}^3$, then u is continuous and has continuous derivatives u_x, u_{xx} in Ω ([4, page 49]). According to (1.12-1.14), the functions $D_{0,t}^\alpha u(t)$, $b(t)$, $a(t)$ are continuous on $[0, T]$. It is straightforward to verify that the equation (1.1) and the conditions (1.2-1.5), (2.1) and (1.7) are satisfied in the ordinary sense. The proof of the theorem is complete. □

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