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A NONLINEAR HENSTOCK-TYPE INTEGRAL FOR RIESZ SPACE-VALUED FUNCTIONS

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Abstract. In this paper, we introduce a nonlinear extension of the Henstock integral for functions taking values in a Riesz space. By combining the conditions provided in Lee (1989) for the real case with Fremlins (D)-double sequence technique, we establish several fundamental properties of the integral, including a Cauchy-type criterion and the integrability of step functions and (D)-continuous functions. Furthermore, we demonstrate that the Saks-Henstock lemma holds for this integral and prove the absolute (D)-continuity of its primitive.

1. Introduction

The concept of integration is central to mathematical analysis, serving as a powerful tool for accumulation and measurement. Integrals can be defined through various frameworks, including descriptive, constructive, and measure-theoretic approaches. In the constructive framework, integration is defined using Riemann-like sums and tagged partitions, leading to classical integrals such as the Riemann, McShane, and Henstock integrals, along with their respective Stieltjes extensions [2, 8, 10]. These integrals are typically formulated over intervals and form the foundation of both real analysis and its broader extensions.

One such extension occurs in the study of the space of all Henstock-integrable functions over a given interval, where a Riesz representation theorem holds for every continuous linear functional. To incorporate nonlinear functionals within this framework, the Henstock integral was slightly modified to define a *nonlinear* Henstock-type integral [6, 9]. This modification not only generalizes the Henstock and Henstock-Stieltjes integrals but also preserves key properties that ensure the robustness and applicability of the integration theory. These developments naturally lead to the question of how such nonlinear integrals might behave when defined over more abstract algebraic structures such as Riesz spaces.

To address this, we turn our attention to integration in the context of Riesz spaces. A key difficulty in extending integration to Riesz spaces lies in the inapplicability of the classical ϵ-technique, which is crucial in real analysis. For instance,

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the property in $\mathbb{R}$ that $t = \inf A$ implies the existence of an element $a \in A$ such that $t + \epsilon > a$ (for any $\epsilon > 0$) does not hold in general Riesz spaces. To overcome this, Fremlin introduced the double sequence technique, where he defined the so called (D)-sequence [7]. A bounded double sequence (a_{ij}) in a Riesz space X is called a (D)-sequence if for each $i \in \mathbb{N}$, the sequence $(a_{ij})_j$ is decreasing and converges to 0. These (D)-sequences serve as substitutes for ϵ in real-valued analysis. Building on this, several authors such as Boccuto et al. [3, 4, 5, 12, 13], developed integral theories in Riesz and ordered spacesthough only for the linear case. In contrast, this paper aims to develop a *nonlinear* Henstock-type integral with values in Riesz spaces, utilizing the double sequence technique as a foundation.

2. Results

We refer to [1, 11, 14] for standard definitions and terminology in Riesz spaces. Throughout this paper, we always assume that the Riesz space X is Dedekind complete, meaning that every subset of X with an upper bound has a least upper bound (supremum), and X is weakly σ-distributive, that is,

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i\varphi(i)} \right) = 0, \quad \text{for all } (D)\text{-sequence } (a_{ij}) \text{ in } X.$$

Let $[a, b]$ be a closed and bounded interval on $\mathbb{R}$ and $\mathcal{I}[a, b]$ be the collection of all intervals $I \subseteq [a, b]$. For any positive function $\delta : [a, b] \to (0, \infty)$ (known as a gauge), we denote by $\mathcal{P}_\delta[a, b]$ the collection of all δ-fine partitions on $[a, b]$, i.e., all partitions $P = \{([u_i, v_i]; \xi_i) : i = 1, 2, \ldots, n\}$ on $[a, b]$ such that $\xi_i \in [u_i, v_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.

Now, let $\phi : X \times \mathcal{I}[a, b] \to X$. Throughout the rest of paper, we always assume that $\phi(x, [u, v]) = \phi(x, (u, v]) = \phi(x, [u, v)) = \phi(x, (u, v))$. For a function $f : [a, b] \to X$ and a (partial) partition $P = \{([u_i, v_i]; \xi_i) : i = 1, 2, \ldots, n\}$ on $[a, b]$, we define

$$S_\phi(P, f) = \sum_{i=1}^{n} \phi(f(\xi_i), [u_i, v_i]).$$

Using similar double sequence technique idea as in [5], we define a nonlinear Henstock-type integral with values in Riesz spaces as follows.

Definition 2.1. Let $\phi : X \times \mathcal{I}[a, b] \to X$. A function $f : [a, b] \to X$ is said to be ϕ-integrable on $[a, b]$ if there is $A \in X$ and a (D)-sequence (a_{ij}) in X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there is a gauge δ on $[a, b]$ such that for every $P \in \mathcal{P}_\delta[a, b]$, we have

$$|S_\phi(P, f) - A| \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Furthermore, the ϕ-integral value A is denoted by $(\phi) \displaystyle\int_a^b f$.

Note that if ϕ is of the form $\phi(x, [u, v]) = x(v - u)$, then the above definition is the definition of the Henstock integral for Riesz-valued functions. Moreover, if ϕ is of the form $\phi(x, [u, v]) = x(g(v) - g(u))$ for some function $g : [a, b] \to \mathbb{R}$, then the above definition becomes the definition of the Henstock-Stieltjes integral for Riesz-valued functions.

Lemma 2.2. *Let $\phi : X \times \mathcal{I}[a,b] \to X$. If the function $f : [a,b] \to X$ is ϕ-integrable on $[a,b]$, then the element A in Definition 2.1 is unique.*

Proof. Suppose that A and B are the ϕ-integral values of f on $[a,b]$. Further, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gauge δ on $[a,b]$ such that for any $P \in \mathcal{P}_\delta[a,b]$, we have

$$|S_\phi(P,f) - A| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \quad \text{and} \quad |S_\phi(P,f) - B| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Hence,

$$|A - B| \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

As X is weakly σ-distributive, we obtain $|A-B| \leq 0$. It follows $A = B$. □

In the real-valued case (see [9]), the function ϕ needs some conditions $(N1 - N5)$ in order to obtain basic properties of the ϕ-integral. In this paper, we generalize those conditions to the Riesz space setting as follows.

(N1) For every $I \in \mathcal{I}[a,b]$, $\phi(0,I) = 0$.

(N2) For every $I \in \mathcal{I}[a,b]$ and $z \in X^+$, there is a (D)-sequence (a_{ij}) in X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, $s \in X$ with $|s| \leq z$, and (D)-sequence (b_{ij}) in X, there is $\psi \in \mathbb{N}^{\mathbb{N}}$ such that

$$|\phi(s,I) - \phi(t,I)| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for every $t \in X$ with $|s-t| \leq \bigvee_{i=1}^{\infty} b_{i\psi(i)}$ and $|t| \leq z$.

(N3) For every $s \in X$,

$$\phi(s, I_1 \cup I_2) = \phi(s, I_1) + \phi(s, I_2),$$

for every disjoint intervals $I_1, I_2 \in \mathcal{I}[a,b]$.

(N4) For every $z \in X^+$, there is a (D)-sequence (a_{ij}) in X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and (D)-sequence (b_{ij}) in X, there is $\psi \in \mathbb{N}^{\mathbb{N}}$ such that

$$\left| \sum_{i=1}^{n} \phi(s_i, I_i) - \sum_{i=1}^{n} \phi(t_i, I_i) \right| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)},$$

for every $s_i, t_i \in X$ with $|s_i - t_i| \leq \bigvee_{i=1}^{\infty} b_{i\psi(i)}$, $|s_i| \leq z$, and $|t_i| \leq z$ for each $i = 1, 2, \ldots, n$, and pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in \mathcal{I}[a,b]$.

(N5) For every $z \in X^+$, there is a (D)-sequence (a_{ij}) in X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $\mu > 0$ such that

$$\left| \sum_{i=1}^{n} \phi(s_i, I_i) \right| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)},$$

for every $s_i \in X$ with $|s_i| \leq z$ for each $i = 1, 2, \ldots, n$, and pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in \mathcal{I}[a,b]$ with a total length less than μ.

In this paper, we will not use (N1) and (N2), as they are just consequences of (N3) and (N4), respectively. Note also that if ϕ satisfies (N3) and f is a constant function, then f is ϕ-integrable on $[a,b]$ and

$$(\phi)\int_a^b f = \phi(f(a),[a,b]).$$

Now, we begin our results by examining the properties of the integral when a function is integrable with respect to both ϕ and ψ.

Theorem 2.3. *Let $\phi,\psi : X\times\mathcal{I}[a,b]\to X$. If $f:[a,b]\to X$ is both ϕ-integrable and ψ-integrable on $[a,b]$, then f is also $(\alpha\phi+\psi)$-integrable on $[a,b]$ for each $\alpha\in\mathbb{R}$ and*

$$(\alpha\phi+\psi)\int_a^b f = \alpha(\phi)\int_a^b f + (\psi)\int_a^b f.$$

Proof. Suppose that A and B are the ϕ-integral value and ψ-integral value of f on $[a,b]$, respectively. Further, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi\in\mathbb{N}^{\mathbb{N}}$, there exists a gauge δ on $[a,b]$ such that for any $P\in\mathcal{P}_\delta[a,b]$, we have

$$|S_\phi(P,f)-A|\le\bigvee_{i=1}^{\infty}a_{i\varphi(i)}\quad\text{and}\quad|S_\psi(P,f)-B|\le\bigvee_{i=1}^{\infty}a_{i\varphi(i)}.$$

Fix $\varphi\in\mathbb{N}^{\mathbb{N}}$ and let δ be a gauge on $[a,b]$ satisfying the above condition. Note that for any $P\in\mathcal{P}_\delta[a,b]$, $S_{\alpha\phi+\psi}(P,f)=\alpha S_\phi(P,f)+S_\psi(P,f)$. Hence,

$$|S_{\alpha\phi+\psi}(P,f)-(\alpha A+B)|\le\bigvee_{i=1}^{\infty}(|\alpha|+1)a_{i\varphi(i)}.$$

It follows that f is also $(\alpha\phi+\psi)$-integrable on $[a,b]$ and

$$(\alpha\phi+\psi)\int_a^b f = \alpha(\phi)\int_a^b f + (\psi)\int_a^b f. \qquad\square$$

Theorem 2.4. *Let $\phi,\psi : X\times\mathcal{I}[a,b]\to X$ satisfy $\phi(\cdot,I)\le\psi(\cdot,I)$ for every $I\in\mathcal{I}[a,b]$. If $f:[a,b]\to X$ is both ϕ-integrable and ψ-integrable on $[a,b]$, then*

$$(\phi)\int_a^b f\le(\psi)\int_a^b f.$$

Proof. By Theorem 2.3, since f is both ϕ-integrable and ψ-integrable on $[a,b]$, f is also $(\phi-\psi)$-integrable on $[a,b]$ and

$$(\phi-\psi)\int_a^b f = (\phi)\int_a^b f - (\psi)\int_a^b f.$$

Suppose that A is the $(\phi-\psi)$-integral value of f on $[a,b]$. Further, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi\in\mathbb{N}^{\mathbb{N}}$, there exists a gauge δ on $[a,b]$ such that for any $P\in\mathcal{P}_\delta[a,b]$, we have

$$|S_{\phi-\psi}(P,f)-A|\le\bigvee_{i=1}^{\infty}a_{i\varphi(i)}.$$

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$ and let δ be a gauge on $[a,b]$ satisfying the above condition. Since $(\phi-\psi)(\cdot,I) \leq 0$ for any $I \in \mathcal{I}[a,b]$, we have that,

$$A \leq A - S_{\phi-\psi}(P,f) \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for any $P \in \mathcal{P}_\delta[a,b]$. As X is weakly σ-distributive, we conclude that $A \leq 0$ and hence,

$$(\phi)\int_a^b f \leq (\psi)\int_a^b f. \qquad \square$$

As the name suggests, the nonlinear integral may not be linear as can be seen in the following example. However, if we assume that the function $\phi(\cdot,I)$ is linear for every $I \in \mathcal{I}[a,b]$, the integral becomes linear.

Example 2.5. Given a function $\phi : \mathbb{R}\times[0,1] \to \mathbb{R}$ with $\phi(s,[u,v]) = (e^s-1)(v-u)$ for every $[u,v] \in \mathcal{I}[a,b]$ and $s \in \mathbb{R}$. Define a function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} n, & \text{if } x \in \left(\frac{1}{4^n}, \frac{1}{4^{n-1}}\right] \text{ for some } n \in \mathbb{N} \\ 0, & \text{if } x = 0. \end{cases}$$

It can be shown that f is ϕ-integrable on $[0,1]$, but $2f$ is not.

Theorem 2.6. *Let $\phi : X \times \mathcal{I}[a,b] \to X$ satisfy that for every $I \in \mathcal{I}[a,b]$, $\phi(\cdot,I)$ is linear, i.e. $\phi(\alpha x+y,I) = \alpha\phi(x,I)+\phi(y,I)$ for every $x,y \in X$ and $\alpha \in \mathbb{R}$. If $f,g : [a,b] \to X$ are ϕ-integrable on $[a,b]$, then $\alpha f+g$ is also ϕ-integrable on $[a,b]$ for all $\alpha \in \mathbb{R}$ and*

$$(\phi)\int_a^b (\alpha f+g) = \alpha(\phi)\int_a^b f + (\phi)\int_a^b g.$$

Proof. Suppose that A and B are the ϕ-integral values of f and g on $[a,b]$, respectively. Further, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gauge δ on $[a,b]$ so that for any $P \in \mathcal{P}_\delta[a,b]$, we have

$$|S_\phi(P,f)-A| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \quad \text{and} \quad |S_\phi(P,g)-B| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$ and let δ be a gauge on $[a,b]$ satisfying the above condition. For any $P \in \mathcal{P}_\delta[a,b]$, $S_\phi(P,\alpha f+g) = \alpha S_\phi(P,f)+S_\phi(P,g)$ since $\phi(\cdot,I)$ is linear for any $I \in \mathcal{I}[a,b]$. Hence,

$$|S_\phi(P,\alpha f+g)-(\alpha A+B)| \leq \bigvee_{i=1}^{\infty}(|\alpha|+1)a_{i\varphi(i)}.$$

It follows that $\alpha f+g$ is also ϕ-integrable on $[a,b]$ and

$$(\phi)\int_a^b (\alpha f+g) = \alpha(\phi)\int_a^b f + (\phi)\int_a^b g. \qquad \square$$

Next, we show that the Cauchy criterion for nonlinear integrals remains valid, and notably, this can be established without relying on any of the conditions (N1)(N5).

Theorem 2.7 (Cauchy Criterion)**.** *Let $\phi : X \times \mathcal{I}[a,b] \to X$. A function $f : [a,b] \to X$ is ϕ-integrable on $[a,b]$ if and only if there is a (D)-sequence (a_{ij}) in X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there is a gauge δ on $[a,b]$ such that for every $P_1, P_2 \in \mathcal{P}_\delta[a,b]$, we have*

$$|S_\phi(P_1, f) - S_\phi(P_2, f)| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Proof. ($\Rightarrow$) Suppose that A is the ϕ-integral value of f on $[a,b]$. Further, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gauge δ on $[a,b]$ such that for any $P_1, P_2 \in \mathcal{P}_\delta[a,b]$, we have

$$|S_\phi(P_1, f) - A| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \quad \text{and} \quad |S_\phi(P_2, f) - A| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$ and let δ be a gauge on $[a,b]$ satisfying the above condition. Then for any $P_1, P_2 \in \mathcal{P}_\delta[a,b]$,

$$|S_\phi(P_1, f) - S_\phi(P_2, f)| \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

Therefore, the forward implication holds.
($\Leftarrow$) Now, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gauge δ on $[a,b]$ satisfying the following condition, denoted as (C): for any $P_1, P_2 \in \mathcal{P}_\delta[a,b]$, one has

$$|S_\phi(P_1, f) - S_\phi(P_2, f)| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$. Define M_φ as the set of all gauges δ on $[a,b]$ satisfying (C), and let $N_\varphi = \{S_\phi(P, f) : P \in \mathcal{P}_\delta[a,b], \delta \in M_\varphi\}$. Note that for any $\delta_1, \delta_2 \in M_\varphi$, we have $|S_\phi(P_1, f) - S_\phi(P_2, f)| \leq 2\bigvee_{i=1}^{\infty} a_{i\varphi(i)}$ for all $P_1 \in \mathcal{P}_{\delta_1}[a,b]$ and $P_2 \in \mathcal{P}_{\delta_1}[a,b]$. Consequently, N_φ is bounded. Since X is Dedekind complete, $\sup N_\varphi$ and $\inf N_\varphi$ exist. Moreover,

$$\sup N_\varphi - \inf N_\varphi \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

As X is weakly σ-distributive, we obtain

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \sup N_\varphi \leq \bigvee_{\varphi \in \mathbb{N}^{\mathbb{N}}} \inf N_\varphi + \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} (\sup N_\varphi - \inf N_\varphi) \leq \bigvee_{\varphi \in \mathbb{N}^{\mathbb{N}}} \inf N_\varphi.$$

On the other hand, note that for any $\psi \in \mathbb{N}^{\mathbb{N}}$, $\gamma \in M_\psi \cap M_\varphi$, and $P \in \mathcal{P}_\gamma[a,b]$, $\inf N_\psi \leq S_\phi(P, f) \leq \sup N_\varphi$. Hence,

$$\bigvee_{\varphi \in \mathbb{N}^{\mathbb{N}}} \inf N_\varphi \leq \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \sup N_\varphi.$$

Consequently,

$$\bigvee_{\varphi \in \mathbb{N}^{\mathbb{N}}} \sup N_\varphi = \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \inf N_\varphi.$$

Denote this common value by A. Note that for any $\delta \in M_\varphi$ and $P \in \mathcal{P}_\delta[a,b]$,

$$S_\phi(P,f) - A \leq \sup N_\varphi - \bigvee_{\varphi \in \mathbb{N}^\mathbb{N}} \inf N_\varphi \leq \sup N_\varphi - \inf N_\varphi \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)} \quad \text{and}$$

$$A - S_\phi(P,f) \leq \bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \sup N_\varphi - \inf N_\varphi \leq \sup N_\varphi - \inf N_\varphi \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

Hence,

$$|S_\phi(P,f) - A| \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

It follows that f is ϕ-integrable on $[a,b]$. Therefore, the backward implication also holds. ☐

Remark 2.8. Let $\phi : X \times \mathcal{I}[a,b] \to X$ and $f : [a,b] \to X$ be a ϕ-integrable function on $[a,b]$. For any $\varphi \in \mathbb{N}^\mathbb{N}$, define M_φ be the set of all gauge δ on $[a,b]$ satisfying (C) as in the proof of Theorem 2.7, and let $N_\varphi = \{S_\phi(P,f) : P \in \mathcal{P}_\delta[a,b], \delta \in M_\varphi\}$. The following result holds:

$$\bigvee_{\varphi \in \mathbb{N}^\mathbb{N}} \sup N_\varphi = \bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \inf N_\varphi = (\phi) \int_a^b f.$$

Theorem 2.9. *Let $\phi : X \times \mathcal{I}[a,b] \to X$. If $f : [a,b] \to X$ is ϕ-integrable on $[a,b]$, then f is also ϕ-integrable on $[c,d]$ for every $[c,d] \subseteq [a,b]$.*

Proof. Let $[c,d] \subseteq [a,b]$. By the Cauchy Criterion, since f is integrable on $[a,b]$, there is a (D)-sequence (a_{ij}) in X such that for any $\varphi \in \mathbb{N}^\mathbb{N}$, there exists a gauge δ on $[a,b]$ such that for any $P_1, P_2 \in \mathcal{P}_\delta[a,b]$, we have

$$|S_\phi(P_1,f) - S_\phi(P_2,f)| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Fix $\varphi \in \mathbb{N}^\mathbb{N}$ and let δ be a gauge on $[a,b]$ satisfying the above condition. Define gauges α, β, and γ as the restrictions of δ on $[a,c]$, $[c,d]$, and $[d,b]$, respectively. Fix $Q_1 \in \mathcal{P}_\alpha[a,c]$ and $Q_2 \in \mathcal{P}_\gamma[d,b]$. Note that for any $P_1, P_2 \in \mathcal{P}_\beta[c,d]$, $Q_1 \cup P_1 \cup Q_2$, $Q_1 \cup P_2 \cup Q_2 \in \mathcal{P}_\delta[a,b]$. Hence,

$$|S_\phi(P_1,f) - S_\phi(P_2,f)| = |S_\phi(Q_1 \cup P_1 \cup Q_2, f) - S_\phi(Q_1 \cup P_2 \cup Q_2, f)| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

By the Cauchy Criterion, f is ϕ-integrable on $[c,d]$. ☐

Note that by Theorem 2.9, if f is ϕ-integrable on $[a,b]$ and $c \in (a,b)$, then f is ϕ-integrable on $[a,c]$ dan $[c,b]$. The converse is also true when we assume that ϕ satisfies (N3).

Theorem 2.10. *Let $\phi : X \times \mathcal{I}[a,b] \to X$ satisfy (N3), and let $c \in [a,b]$. If $f : [a,b] \to X$ is ϕ-integrable on $[a,c]$ and $[c,b]$, then f is also ϕ-integrable on $[a,b]$ and*

$$(\phi) \int_a^b f = (\phi) \int_a^c f + (\phi) \int_c^b f.$$

Proof. Suppose that A and B is the ϕ-integral values of f on $[a,c]$ and $[c,b]$, respectively. Further, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exist gauges γ and δ defined on $[a,c]$ and $[c,b]$, respectively, such that for any $P_1 \in \mathcal{P}_\gamma[a,c]$ and $P_2 \in \mathcal{P}_\delta[c,b]$, we have

$$|S_\phi(P_1,f) - A| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \quad \text{and} \quad |S_\phi(P_2,f) - B| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$, and let γ and δ be gauges on $[a,c]$ and $[c,b]$, respectively, satisfying the above condition. Define a gauge β on $[a,b]$ as follows:

$$\beta(x) = \begin{cases} \min\{\gamma(x), c-x\}, & \text{if } a \leq x < c \\ \min\{\gamma(c), \delta(c)\}, & \text{if } x = c \\ \min\{\delta(x), x-c\}, & \text{if } c < x \leq b. \end{cases}$$

Note that for any $P = \{(I_i;\xi_i) : i = 1,2,\ldots,n\} \in \mathcal{P}_\beta[a,b]$, $c = \xi_j$ for some $j = 1,2,\ldots,n$. Further, let $I_j = [u,v]$, $P_1 = \{(I_i;\xi_i) \in P : \xi_i < c\} \cup \{([u,c];c)\}$, and $P_2 = \{(I_i;\xi_i) \in P : \xi_i > c\} \cup \{([c,v];c)\}$. Note that $P_1 \in \mathcal{P}_\gamma[a,c]$ and $P_2 \in \mathcal{P}_\delta[c,b]$. As ϕ satisfies (N3), we obtain $\phi(c,[u,v]) = \phi(c,[u,c]) + \phi(c,[c,v])$. Consequently, $S_\phi(P,f) = S_\phi(P_1,f) + S_\phi(P_2,f)$. Hence,

$$|S_\phi(P,f) - (A+B)| \leq |S_\phi(P_1,f) - A| + |S_\phi(P_2,f) - B| \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

It follows that f is ϕ-integrable on $[a,b]$ and

$$(\phi)\int_a^b f = (\phi)\int_a^c f + (\phi)\int_c^b f. \qquad \square$$

As a consequence, assuming (N3), we have that every step function is ϕ-integrable.

Theorem 2.11. *Let $\phi : X \times \mathcal{I}[a,b] \to X$ satisfy (N3). Every step function $f : [a,b] \to X$ is ϕ-integrable on $[a,b]$. Moreover, if*

$$f = \sum_{i=1}^{n} s_i \chi_{I_i}$$

for some $s_1, s_2, \ldots, s_n \in X$ and pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in \mathcal{I}[a,b]$ whose union is $[a,b]$, then

$$(\phi)\int_a^b f = \sum_{i=1}^{n} \phi(s_i, I_i).$$

Proof. Suppose that

$$f = \sum_{i=1}^{n} s_i \chi_{I_i}$$

for some $s_1, s_2, \ldots, s_n \in X$ and pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in \mathcal{I}[a,b]$ whose union is $[a,b]$. For any $i = 1,2,\ldots,n$, let $\overline{I_i} = [u_i, v_i]$. As ϕ satisfies (N3), we obtain that f is ϕ-integrable on $\overline{I_i} = [u_i, v_i]$ and

$$(\phi)\int_{u_i}^{v_i} f = \phi(s_i, I_i).$$

By Theorem 2.10, we obtain that f is also ϕ-integrable on $[a,b]$ and

$$(\phi)\int_a^b f = \sum_{i=1}^{n}(\phi)\int_{u_i}^{v_i} f = \sum_{i=1}^{n}\phi(s_i, I_i). \qquad \square$$

Theorem 2.12. *Let $\phi : X \times \mathcal{I}[a,b] \to X$ satisfy (N5) and $f : [a,b] \to X$ be a bounded and ϕ-integrable function on $[a,b]$. If $g : [a,b] \to X$ is bounded and $f = g$ almost everywhere on $[a,b]$, then g is ϕ-integrable on $[a,b]$ and*

$$(\phi)\int_a^b g = (\phi)\int_a^b f.$$

Proof. Suppose that A is ϕ-integral value of f on $[a,b]$. Further, let (a_{ij}) be a (D)-sequence in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gauge δ on $[a,b]$, such that for any $P \in \mathcal{P}_\delta[a,b]$, we have

$$|S_\phi(P,f) - A| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

As f and g are bounded on $[a,b]$ and ϕ satisfies (N5), there exists a (D)-sequence (b_{ij}) in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $\mu > 0$ such that

$$\left|\sum_{i=1}^{n}\phi(f(x_i), I_i)\right| \leq \bigvee_{i=1}^{\infty} b_{i\varphi(i)} \quad \text{and} \quad \left|\sum_{i=1}^{n}\phi(g(x_i), I_i)\right| \leq \bigvee_{i=1}^{\infty} b_{i\varphi(i)},$$

for every $x_1, x_2, \ldots, x_n \in [a,b]$ and pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in \mathcal{I}[a,b]$ with a total length of less than μ.

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$ and let $\mu > 0$ and δ be a gauge on $[a,b]$ satisfying the above conditions. Let $N \subseteq [a,b]$ with a Lebesgue measure 0 such that $f = g$ on $[a,b]\backslash N$. Then there exists a countable collection of open intervals $\{O_j \in \mathcal{I}[a-\mu, b+\mu] : j \in \mathbb{N}\}$ with a total length less than μ and satisfying $N \subseteq \bigcup_{j=1}^{\infty} O_j$. Define a gauge γ on $[a,b]$ as follows:

$$\gamma(x) = \begin{cases} \inf\{|x-y| : y \in [a,b]\backslash O_j\}, & \text{if } x \in O_j \text{ for some } j \in \mathbb{N} \\ 1, & \text{otherwise.} \end{cases}$$

For any $P = \{(I_i;\xi_i) : i = 1,2,\ldots,n\} \in \mathcal{P}_\gamma[a,b]$, define M as the set of all indices $i = 1,2,\ldots,n$ such that $\xi_i \in O_j$ for some $j \in \mathbb{N}$. Observe that for any $i = 1,2,\ldots,n$,

- $*$ if $i \in M$, then $\xi_i \in O_j$ for some $j \in \mathbb{N}$. Hence, $I_i \subseteq O_j$.
- $*$ if $i \notin M$, then $\xi_i \notin N$. Hence, $\phi(f(\xi_i), I_i) = \phi(g(\xi_i), I_i)$.

Consequently,

$$\begin{aligned} |S_\phi(P,f) - S_\phi(P,g)| &= \left|\sum_{i\in M}\phi(f(\xi_i), I_i) - \sum_{i\in M}\phi(g(\xi_i), I_i)\right| \\ &\leq \left|\sum_{i\in M}\phi(f(\xi_i), I_i)\right| + \left|\sum_{i\in M}\phi(g(\xi_i), I_i)\right| \leq \bigvee_{i=1}^{\infty} 2b_{i\varphi(i)} \end{aligned}$$

since the total length of $I_1, I_2, \ldots, I_n$ is less than μ. Hence, for any $P \in \mathcal{P}_{\min\{\gamma,\delta\}}$,

$$|S_\phi(P,g) - A| \leq |S_\phi(P,f) - A| + |S_\phi(P,f) - S_\phi(P,g)| \leq \bigvee_{i=1}^{\infty}(a_{i\varphi(i)} + 2b_{i\varphi(i)}).$$

It follows that g is ϕ-integrable on $[a,b]$ and

$$(\phi)\int_a^b g = (\phi)\int_a^b f. \qquad \square$$

We say a function $f : [a,b] \to X$ is (D)-continuous on $[a,b]$ if there is a (D)-sequence (a_{ij}) in X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there is $\delta > 0$ such that for every $x, y \in [a,b]$ with $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Note that if $f : [a,b] \to X$ is (D)-continuous on $[a,b]$, then f is bounded. Assuming ϕ to satisfy (N3) and (N4), we show that every (D)-continuous function is ϕ-integrable.

Theorem 2.13. *Let $\phi : X \times \mathcal{I}[a,b] \to X$ satisfy (N3) and (N4). Every (D)-continuous function $f : [a,b] \to X$ is ϕ-integrable on $[a,b]$.*

Proof. Let f be a (D)-continuous function on $[a,b]$. Since f is bounded, by (N4) there exists a (D)-sequence (a_{ij}) in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ and (D)-sequence (b_{ij}) in X, there exists $\psi \in \mathbb{N}^{\mathbb{N}}$ such that

$$\left|\sum_{i=1}^{n} \phi(f(x_i), I_i) - \sum_{i=1}^{n} \phi(f(y_i), I_i)\right| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for any $x_i, y_i \in [a,b]$ with $|f(x_i) - f(y_i)| \leq \bigvee_{i=1}^{\infty} b_{i\psi(i)}$ for each $i = 1, 2, \ldots, n$ and pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in \mathcal{I}[a,b]$. Further, since f is (D)-continuous on $[a,b]$, there is a (D)-sequence (b_{ij}) in X such that for every $\psi \in \mathbb{N}^{\mathbb{N}}$, there is $\delta > 0$ such that for every $x, y \in [a,b]$ with $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq \bigvee_{i=1}^{\infty} b_{i\psi(i)}.$$

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$, and let $\psi \in \mathbb{N}^{\mathbb{N}}$ and δ satisfy the above conditions for the (D)-sequence (b_{ij}). Define a gauge γ on $[a,b]$ by $\gamma(x) = \delta$ for every $x \in [a,b]$.

Fix $P_1 = \{(I_i; \xi_i) : i = 1, 2, \ldots, n\}, P_2 \in \mathcal{P}_\gamma[a,b]$. Let $P = \{(J_j; \zeta_j) : j = 1, 2, \ldots, m\} \in \mathcal{P}_\gamma[a,b]$ be finer than P_1 and P_2. For any $i = 1, 2, \ldots, n$, define M_i as the set of all indices $j = 1, 2, \ldots, m$ such that $J_j \subseteq I_i$. Note that for any $j \in M_i$, $|\xi_i - \zeta_j| < \delta$. As ϕ satisfies (N3), we obtain

$$\begin{aligned} |S_\phi(P_1, f) - S_\phi(P, f)| &= \left|\sum_{j=1}^{m} \phi(f(\zeta_j), J_j) - \sum_{i=1}^{n} \phi(f(\xi_i), I_i)\right| \\ &= \left|\sum_{j=1}^{m} \phi(f(\zeta_j), J_j) - \sum_{i=1}^{n} \sum_{j \in M_i} \phi(f(\xi_i), J_j)\right| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}. \end{aligned}$$

Similarly, we obtain

$$|S_\phi(P_2, f) - S_\phi(P, f)| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Hence,

$$|S_\phi(P_1, f) - S_\phi(P_2, f)| \le \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

By the Cauchy criterion, f is ϕ-integrable on $[a, b]$. □

We end this paper by investigating properties of the primitive of ϕ-integrable functions.

Definition 2.14. Let $\phi : X \times \mathcal{I}[a, b] \to X$ and $f : [a, b] \to X$ be a ϕ-integrable function on $[a, b]$. The function $F_\phi : [a, b] \to X$ on $[a, b]$, defined by

$$F_\phi(t) = (\phi) \int_a^t f$$

for every $t \in [a, b]$, is called the primitive of the ϕ-integrable function f on $[a, b]$.

Using the following two lemmas, we show that the Saks-Henstock Lemma remains valid for the nonlinear integral of Riesz-valued functions.

Lemma 2.15. *Let $\phi : X \times \mathcal{I}[a, b] \to X$ and $f : [a, b] \to X$ be a ϕ-integrable function on $[a, b]$ with primitive F_ϕ. Suppose that (a_{ij}) is a (D)-sequence in X and $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that there is a gauge δ on $[a, b]$ satisfying the condition that for every $P \in \mathcal{P}_\delta[a, b]$,*

$$\left| S_\phi(P, f) - (\phi) \int_a^b f \right| \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

If $P^* = \{([u_i, v_i]; \xi_i) : i = 1, 2, 3, \ldots, p\}$ is a partial δ-fine partition on $[a, b]$, then

$$\left| S_\phi(P^*, f) - \sum_{i=1}^{p} (F_\phi(v_i) - F_\phi(u_i)) \right| \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Proof. Let $P^* = \{([u_i, v_i], \xi_i) : i = 1, 2, 3, \ldots, p\}$ be a partial δ-fine partition on $[a, b]$. Let $\{[a_t, b_t] : t = 1, 2, 3, \ldots, n\}$ be a collection of non-overlapping intervals such that $n \le p + 1$, $(u_i, v_i) \cap (a_t, b_t) = \varnothing$, and

$$\left(\bigcup_{i=1}^{p} [u_i, v_i] \right) \cup \left(\bigcup_{t=1}^{n} [a_t, b_t] \right) = [a, b].$$

Now, since f is ϕ-integrable on each $[a_t, b_t]$, we can find a (D)-sequence (b_{ij}) in X such that for every $\psi \in \mathbb{N}^{\mathbb{N}}$, there is a gauge δ^* on $[a, b]$ so that $\delta^* \le \delta$ and for every $t = 1, 2, \ldots, n$ and $P_t \in \mathcal{P}_{\delta^*}[a_t, b_t]$, we have

$$\left| S_\phi(P_t, f) - (\phi) \int_{a_t}^{b_t} f \right| \le \frac{1}{p+1} \bigvee_{i=1}^{\infty} b_{i\psi(i)}.$$

Fix a (D)-sequence (b_{ij}) and $\psi \in \mathbb{N}^{\mathbb{N}}$, and let δ^* be a gauge on $[a, b]$ satisfying the above condition. For each $t = 1, 2, \ldots, n$, fix $P_t \in \mathcal{P}_{\delta^*}[a_t, b_t]$. Define $P = P^* \cup (\bigcup_{t=1}^{n} P_t)$. Clearly, $P \in \mathcal{P}_\delta[a, b]$ and

$$S_\phi(P, f) = S_\phi(P^*, f) + \sum_{t=1}^{n} S_\phi(P_t, f).$$

Therefore,

$$\left|S_\phi(P^*,f)-\sum_{i=1}^{p}(\phi)\int_{u_i}^{v_i} f\right| \le \left|S_\phi(P,f)-(\phi)\int_a^b f\right| + \sum_{t=1}^{n}\left|S_\phi(P_t,f)-(\phi)\int_{a_t}^{b_t} f\right|$$
$$\le \bigvee_{i=1}^{\infty} a_{i\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i\psi(i)}.$$

As the inequality holds for every $\psi\in\mathbb{N}^{\mathbb{N}}$, we have the desired result. □

Lemma 2.16. *Let K be a finite subset of X and $z\in X$. If*

$$\left|\sum_{x\in S} x\right| \le z \quad \forall S\subseteq K,$$

then

$$\sum_{x\in K}|x| \le 2z.$$

Proof. Let $K=\{x_1,x_2,\dots,x_n\}$ and $C=\{(c_i)=(c_1,c_2,c_3,\dots,c_n): c_i\in\{-1,1\}\}$. Observe that

$$\sum_{x\in K}|x| = \bigvee_{(c_i)\in C} c_1x_1+c_2x_2+\cdots+c_nx_n.$$

Now, take $(c_i)\in C$. Let $S=\{k: c_k=1\}$ and $T=\{\ell: c_\ell=-1\}$. Then we have

$$|c_1x_1+c_2x_2+\cdots+c_nx_n| = \left|\sum_{k\in S}x_k-\sum_{\ell\in T}x_\ell\right| \le \left|\sum_{k\in S}x_k\right| + \left|\sum_{\ell\in T}x_\ell\right| \le 2z,$$

and hence, the conclusion follows. □

Theorem 2.17 (Saks-Henstock Lemma)**.** *Let $\phi: X\times\mathcal{I}[a,b]\to X$ and $f:[a,b]\to X$ be a ϕ-integrable function on $[a,b]$ with primitive F_ϕ. Then there is a (D)-sequence (a_{ij}) in X such that for every $\varphi\in\mathbb{N}^{\mathbb{N}}$, there is a gauge δ on $[a,b]$ such that for any partial δ-fine partition $P=\{([u_i,v_i];\xi_i): i=1,2,3,\dots,p\}$ on $[a,b]$,*

$$\sum_{i=1}^{p}|\phi(f(\xi_i),[u_i,v_i])-(F_\phi(v_i)-F_\phi(u_i))| \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Proof. Applying Lemma 2.16 to Lemma 2.15 we obtain the desired result. □

As a consequence of the Saks-Henstock Lemma, we can show that the primitive of a bounded ϕ-integrable function is absolutely (D)-continuous and hence, is of bounded variation. We say that a function $f:[a,b]\to X$ is absolutely (D)-continuous on $[a,b]$ if there is a (D)-sequence (a_{ij}) in X such that for every $\varphi\in\mathbb{N}^{\mathbb{N}}$, there is $\delta>0$ such that for every pairwise disjoint intervals $[x_1,y_1],[x_2,y_2],\dots,[x_n,y_n]\in\mathcal{I}[a,b]$ with $\sum_{i=1}^{n}(y_i-x_i)<\delta$, we have

$$\sum_{i=1}^{n}|f(x_i)-f(y_i)| \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

A function $f:[a,b]\to X$ is said to be of bounded variation on $[a,b]$ if supremum of

$$\sum_{i=1}^{n}|f(x_i)-f(y_i)|,$$

taken over all partition $P = \{[x_i, y_i] : i = 1, 2, \ldots, n\}$ on $[a, b]$, exists in X. It is easy to prove that any absolutely (D)-continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Theorem 2.18. *Let $\phi : X \times \mathcal{I}[a, b] \to X$ satisfy (N5). If $f : [a, b] \to X$ is bounded and ϕ-integrable on $[a, b]$ with primitive F_ϕ, then F_ϕ is absolutely (D)-continuous on $[a, b]$.*

Proof. As f is bounded on $[a, b]$ and ϕ satisfies (N5), there exists a (D)-sequence (a_{ij}) in X such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $\mu > 0$ such that

$$\left| \sum_{i=1}^{n} \phi(f(x_i), I_i) \right| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)},$$

for every $x_1, x_2, \ldots, x_n \in [a, b]$ and pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in \mathcal{I}[a, b]$ with a total length less than μ. Further, by Lemma 2.16, it also holds that

$$\sum_{i=1}^{n} |\phi(f(x_i), I_i)| \leq \bigvee_{i=1}^{\infty} 2a_{i\varphi(i)}.$$

On the other hand, by the Saks-Henstock Lemma, since f is ϕ-integrable on $[a, b]$, there is a (D)-sequence (b_{ij}) in X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there is a gauge δ on $[a, b]$ such that for any partial δ-fine partition $P = \{([u_i, v_i]; \xi_i) : i = 1, 2, 3, \ldots, p\}$ on $[a, b]$,

$$\sum_{i=1}^{p} |\phi(f(\xi_i), [u_i, v_i]) - (F_\phi(v_i) - F_\phi(u_i))| \leq \bigvee_{i=1}^{\infty} b_{i\varphi(i)}.$$

Fix $\varphi \in \mathbb{N}^{\mathbb{N}}$, and let $\mu > 0$ and the gauge δ satisfy the above conditions.

Let $[u_1, v_1], [u_2, v_2], \ldots, [u_n, v_n] \in \mathcal{I}[a, b]$ be pairwise disjoint intervals with a total length less than μ. Let $P = P_1 \cup P_2 \cup \ldots \cup P_n = \{([p_j, q_j]; \xi_j) : j = 1, 2, \ldots, m\}$ be a partial δ-fine partition on $[a, b]$ where $P_i \in \mathcal{P}_\delta[u_i, v_i]$ for each $i = 1, 2, \ldots, n$. Consequently,

$$\begin{aligned}
\sum_{i=1}^{n} |F_\phi(v_i) - F_\phi(u_i)| &\leq \sum_{j=1}^{m} |F_\phi(q_j) - F_\phi(p_j)| \\
&\leq \sum_{j=1}^{m} |\phi(f(\xi_j), [p_j, q_j])| + \sum_{j=1}^{m} |\phi(f(\xi_j), [p_j, q_j]) - (F_\phi(q_j) - F_\phi(p_j))| \\
&\leq \bigvee_{i=1}^{\infty} (2a_{i\varphi(i)} + b_{i\varphi(i)}).
\end{aligned}$$

The conclusion follows. □

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