Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 17 Issue 2(2025), Pages 10-25
https://doi.org/10.54671/BMAA-2025-2-2

λ-SZASZ SCHURER WITH GENERALIZED BETA OPERATORS AND APPROXIMATIONS

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Abstract. This research work focuses on λ-Szász-Mirakjan operators coupling extended beta function. These are the linear and positive sequences of operators and have become more popular because of their special characteristics and functional organization. The kernel functions used in Szász operators often possess even or odd symmetry. This symmetry influences the behavior of the operator in terms of approximation and convergence properties. The convergence properties such as uniform convergence and pointwise convergence are studied in view of Korovkin theorem, modulus of continuity, and Peetre's K-functional of these sequences of positive linear operators in depth. Further, we extend our research work for bivariate case of these sequences of operators. Their uniform rate of approximation and order of approximation are investigated in Lebesgue measurable spaces of function. The graphical depiction and error analysis in terms of convergence behavior of these operators are studied.

1. Introduction

Szász [1] presented a generalization of Bernstein polynomials [2] to investigate approximation properties on unbounded interval, i.e., $[0,\infty)$ as follows:

$$P_s(\hbar;y)=\sum_{k=0}^{\infty}\hbar\left(\frac{k}{s}\right)p_{s,k}(y), \tag{1.1}$$

where $y\in[0,\infty), s\in N$ and $p_{s,k}(y)=e^{-sy}\frac{(sy)^k}{k!}$.

These operators are introduced in the $1950's$ and have been extensively studied by mathematicians over the years to achieve the flexibility in the approximation properties. The symmetry of the kernel affects how well Szász operators can approximate functions. Symmetric kernels tend to preserve certain functional forms or properties of functions being approximated, leading to specific convergence behaviors. Many mathematicians constructed various sequences of operators based on the classical Szász-Mirakjan operators given by (1.1). Recently, various scientists are working in the other branches of sciences like medical science, robotics, computer

2000 *Mathematics Subject Classification.* 41A25, $41A27$, $41A35$, 41A36, $41A45$.
Key words and phrases. Rate of convergence; Szász operators; Korovkin theorem; Lebesgue spaces; order of approximation.

Submitted January 1, 2025. Accepted April 14, 2025. Published April 15, 2025.
Communicated by M. Mursaleen.

science and others in terms of these type of the sequences of linear positve operators. In the recent past, several mathematicians contributed a healthy literature in approximation theory via linear positive operators viz.

Braha et al. ([3, 4]), Özger et al. ([7, 8]), Rao et al. ([9]-[12]), Ayman-Mursaleen et al. ([21], [22]), Turhan et al. [5], Mohiuddine et al. [13], Nasiruzzaman et al. [14]. In continuation, Qi et al. [23] presented Szász-Mirakjan operators based on shape parameter $\lambda \in [0,1]$ as follows:

$$S_{s,\lambda}(\hbar; y) = \sum_{k=0}^{\infty} \tilde{t}_{s,k}(\lambda; y)\hbar\left(\frac{k}{s}\right), \tag{1.2}$$

where

$$\begin{aligned}
\tilde{t}_{s,0}(\lambda; y) &= p_{s,0}(y) - \frac{\lambda}{s+1} t_{s+1,1}(y), \qquad (1.3)\\
\tilde{t}_{s,k}(\lambda; y) &= p_{s,k}(y) + \lambda\left(\frac{s-2k+1}{s^2-1} p_{s+1,k}(y) - \frac{s-2k-1}{s^2-1} p_{s+1,k+1}(y)\right), 1 \leq k \leq \infty.
\end{aligned}$$

Many generalizations are investigated for the operators (1.2) viz. Özger et al. [24] constructed a sequence of Kantorovich variant of λ-Schurer operators to approximate Lebesgue measurable class. Aslan ([25], [26]) constructed Stancu-Kantorovich type $\lambda-$Szász-Schurer-Mirakjan operators based on the shape parameters λ.

For $s \in \mathbb{N}, l > 0$ and $\nu > 0$, the functional (see [20]), $C^{\nu}_{s+l,k} : C[0, 1+l] \to \mathbb{R}$, is given by

$$\begin{aligned}
C^{\nu}_{s+l,k}(t) &= \int_0^1 D^{\nu}_{s+l,k}(t)\hbar(t)dt \quad (k = 1, 2, \ldots, s-1),\\
C^{\nu}_{s+l,0}(t) &= \hbar(0), \quad C^{\nu}_{s+l,s+l}(t) = \hbar(1), \qquad (1.4)
\end{aligned}$$

where

$$D^{\nu}_{s+l,k}(t) = \frac{t^{k\nu-1}(1-t)^{(s+l-k)\nu-1}}{B(k\nu, (s+l-k)\nu)}, \tag{1.5}$$

and

$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \quad (a, b > 0)$.

Rao et al. [12] introduced a sequence of classical Szász operators coupling extended beta function as follows:

$$S^{\nu}_s(\hbar; y) = \sum_{k=0}^{\infty} C^{\nu}_{s,k}(t) p_{s,k}(y)\hbar(t). \tag{1.6}$$

Motivated with the above development of the literature, we construct a new sequence of $\lambda-$Szász-Schurer operators coupling generalized beta function

$$\Pi^{\nu}_{s+l,\lambda}(\hbar; y) = \sum_{k=0}^{\infty} C^{\nu}_{s+l,k}(t)\tilde{t}_{s+l,k}(\lambda; y)\hbar(t), \tag{1.7}$$

where $\tilde{t}_{s+l,k}(y)$ and $C^{\nu}_{s+l,k}(t)$ are defined in equations (1.3) and (1.4) replacing s by $s+l$ respectively.

Now, to derive the Lemmas for the approximation results of sequences of operators given in (1.7), we consider test functions and the central moments as $e_i(t) = t^i$ and $\eta_i(t) = (t-y)^i$, $i \in \{0, 1, 2\}$.

To present this research work, it is divided into some sections. Section one and two, hold for introductory and preliminary of this research work. In section three and four, approximation theorems and graphical analysis are investigated. In the last two sections, we study bivariate version of the operators given in (1.7) and their numerical graphical analysis are discussed.

2. Some Estimates and Approximation Results

Lemma 2.1. [12]. *For the operators $S_s^{\nu}(.;.)$ given by* (1.6). *Then, we have*

$$\begin{aligned}
S_s^{\nu} e_0; y) &= 1,\\
S_s^{\nu}(e_1; y) &= y,\\
S_s^{\nu}(e_2; y) &= \left(\frac{s\nu}{s\nu+1}\right) y^2 + \left(\frac{\nu+1}{s\nu+1}\right) y,\\
S_s^{\nu}(e_3; y) &= \left(\frac{s^2\nu^2}{s^2\nu^2+3s\nu+2}\right) y^3 + \left(\frac{3s\nu(\nu+1)}{s^2\nu^2+3s\nu+2}\right) y^2 + \left(\frac{\nu^2+3\nu+2}{s^2\nu^2+3s\nu+2}\right) y,\\
S_s^{\nu}(e_4; y) &= \left(\frac{s^3\nu^3}{s^3\nu^3+6s^2\nu^2+11s\nu+6}\right) y^4 + \left(\frac{6s^2\nu^2(\nu+1)}{s^3\nu^3+6s^2\nu^2+11s\nu+6}\right) y^3\\
&+ \left(\frac{7s\nu^3+18s\nu^2+11s\nu}{s^3\nu^3+6s^2\nu^2+11s\nu+6}\right) y^2 + \left(\frac{\nu^3+6\nu^2+11\nu+6}{s^3\nu^3+6s^2\nu^2+11s\nu+6}\right) y.
\end{aligned}$$

Lemma 2.2. *Let $\Pi_{s+l,\lambda}^{\nu}(.;.)$ be given by* (1.7). *we have*

$$\begin{aligned}
\Pi_{s+l,\lambda}^{\nu}(1; y) &= 1,\\
\Pi_{s+l,\lambda}^{\nu}(t; y) &= y + \lambda\left[\frac{1-e^{-(s+l+1)y}}{(s+l)((s+l)^2-1)} - \frac{2y}{(s+l)(s+l-1)}\right] = F_{s+l},\\
\Pi_{s+l,\lambda}^{\nu}(t^2; y) &= \left(\frac{(s+l)\nu}{(s+l)\nu+1}\right) y^2 + \left(\frac{\nu+1}{(s+l)\nu+1}\right) y\\
&+ \lambda\left(\frac{(s+l)\nu}{(s+l)\nu+1}\right)\left[\frac{2}{(s+l)((s+l)-1)} y - \frac{4((s+l)+1)}{(s+l)^2((s+l)-1)} y^2\right.\\
&+ \left.\frac{e^{-((s+l)+1)y}-1}{(s+l)^2((s+l)-1)}\right] + \frac{(s+l)}{(s+l)\nu+1}\left[\frac{1-e^{-((s+l)+1)y}}{(s+l)((s+l)-1)}\right.\\
&- \left.\frac{2y}{(s+l)((s+l)-1)}\right].
\end{aligned}$$

Proof. For $i=0$, then

$$\begin{aligned}
(i)\ \Pi_{s+l,\lambda}^{\nu}(1; y) &= \sum_{k=0}^{\infty} C_{s+l,k}^{\nu}(h)\tilde{t}_{s+l,k}(\lambda, y)\\
&= \sum_{k=0}^{\infty} \frac{\tilde{t}_{s+l,k}(\lambda, y)}{B(k\nu,(s+l-k)\nu)} \times B(k\nu,(s+l-k)\nu)\\
&= \sum_{k=0}^{\infty} \tilde{t}_{s+l,k}(\lambda, y)\\
&= 1.
\end{aligned}$$

For $i = 1$, then

$$\begin{aligned}
(ii)\ \Pi^{\nu}_{s+l,\lambda}(t;y) &= \sum_{k=0}^{\infty} \frac{\tilde{t}_{s+l,k}(\lambda,y)}{B(k\nu,(s+l-k)\nu)} \times B(k\nu+1,(s+l-k)\nu) \\
&= S^{\nu}_{s+l}(t;y) + \lambda\Bigg[\sum_{k=0}^{\infty} \frac{k}{s+l}\left(\frac{s+l-2k+1}{(s+l)^2-1}\right) p_{s+l+1,k}(y) \\
&- \sum_{k=0}^{\infty} \frac{k}{s+l}\left(\frac{(s+1)-2k-1}{(s+1)^2-1}\right) p_{s+1,k+1}(y)\Bigg] \\
&= y + \lambda\Bigg[\frac{1-e^{-(s+l)y}}{(s+l)((s+l)^2-1)} - \frac{2y}{(s+l)((s+l)-1)}\Bigg].
\end{aligned}$$

If $i = 2$, then

$$\begin{aligned}
(iii)\ \Pi^{\nu}_{s+l,\lambda}(t^2;y) &= \sum_{k=0}^{\infty} \frac{\tilde{t}_{s+1+l,k}(\lambda,y)}{B(k\nu,((s+l)-k)\nu)} \times B(k\nu+2,((s+l)-k)\nu) \\
&= S^{\nu}_{s+l}(t^2;y) \\
&+ \lambda\Bigg[\left(\frac{(s+l)\nu}{(s+l)\nu+1}\right)\sum_{k=0}^{\infty} \frac{k^2}{(s+l)^2}\left(\frac{(s+l)-2k+1}{(s+l)^2-1}\right) p_{s+1,k}(y) \\
&- \sum_{i=0}^{\infty} \frac{i^2}{(s+l)^2}\left(\frac{(s+l)-2i-1}{(s+l)^2-1}\right) p_{s+1,k+1}(y)\Bigg] \\
&+ \lambda\Bigg[\left(\frac{1}{(s+l)\nu+1}\right)\sum_{k=0}^{\infty} \frac{k}{(s+l)+l}\left(\frac{(s+l)-2k+1}{(s+l)^2-1}\right) p_{(s+l)+1,k}(y) \\
&- \sum_{k=0}^{\infty} \frac{k}{(s+l)+l}\left(\frac{(s+l)-2k-1}{(s+l)^2-1}\right) p_{(s+l)+1,k+1}(y)\Bigg] \\
&= \left(\frac{s\nu}{(s+l)\nu+1}\right) y^2 + \left(\frac{\nu+1}{(s+l)\nu+1}\right) y \\
&+ \lambda\left(\frac{(s+l)\nu}{(s+l)\nu+1}\right)\Bigg[\frac{2}{(s+l)((s+l)-1)} y \\
&- \frac{4((s+l)+1)}{(s+l)^2((s+l)-1)} y^2 + \frac{e^{-((s+l)+1)y}-1}{(s+l)^2((s+l)-1)}\Bigg] \\
&+ \frac{(s+l)}{(s+l)\nu+1}\Bigg[\frac{1-e^{-((s+l)+1)y}}{(s+l)((s+l)-1)} - \frac{2y}{(s+l)((s+l)-1)}\Bigg].
\end{aligned}$$

□

Lemma 2.3. *Using Lemma 2.2, one can easily calculate the central moments of Szász-Mirakjan coupling Extended Beta operators as follows:*

$$\begin{aligned}
\Pi^{\nu}_{s+l,\lambda}(1;y) &= 1,\\
\Pi^{\nu}_{s+l,\lambda}((t-y);y) &= \lambda\left[\frac{1-e^{-((s+l)+1)y}-2y}{(s+l)((s+l)-1)}\right] = A^{\nu}_{s+l},\\
\Pi^{\nu}_{s+l,\lambda}((t-y)^2;y) &= \left(\frac{(s+l)\nu}{(s+l)\nu+1}-1\right)y^2+\left(\frac{\nu+1}{(s+l)\nu+1}\right)y\\
&+ \lambda\left[\frac{(1+2mu)e^{-((s+l)+1)y}-(1-2(s+l))y-4y^2}{(s+l)^2((s+l)-1)}\right] = B^{\nu}_{s+l,\lambda}.
\end{aligned}$$

Definition 2.1. [6]. *Let $\omega(\hbar;\phi)$ be the modulus of continuity. Then, for continuous function $\hbar$ defined on closed interval $[0,b], b<\infty$, we have*

$$\omega(\hbar;\phi) = \sup_{|y_1-y_2|\leq\phi}|\hbar(y_1)-\hbar(y_2)|, y_1, y_2\in[0,\infty).$$

For $\hbar\in[0,b]$ $b<\infty$ and $\hbar\in C[0,\infty)$ and $\phi>0$, we obtain

$$|\hbar(y_1)-\hbar(y_2)|\leq\left(1+\frac{|y_1-y_2|}{\phi}\right)\omega(\hbar;\phi). \tag{2.1}$$

Theorem 2.4. *For $\Pi^{\nu}_{s+l,\lambda}(.;.)$ the operators defined by (1.7) and for every $\hbar\in C[0,\infty)\cap\left\{\hbar : y\geq 0, \frac{\hbar(y)}{1+y^2}\text{ is convergent as } y\longrightarrow\infty\right\}$, then $\Pi^{\nu}_{s+l,\lambda}(\hbar;y)\rightrightarrows\hbar$, where $\rightrightarrows$ denotes the uniform convergence.*

Proof. By Korovkin-type property (iv) of Theorem 4.1.4 in [15], it is enough to show that $\Pi^{y}_{s+l,\lambda}(e_j;v)\longrightarrow e_j$, for $j\in\{1,2,3\}$. By Lemma 2.2, it is clear $\Pi^{y}_{s+l,\lambda}(e_0;y)\longrightarrow e_0(y)$ as $s\longrightarrow\infty$ and for $j=1$

$$\lim_{s}\Pi^{\nu}_{s+l,\lambda}(e_1;y) = lim_{s\longrightarrow\infty}\left(y+\lambda\left[\frac{1-e^{-(s+l+l)y}}{(s+l)((s+l)^2-1)}-\frac{2y}{(s+l)((s+l)-1)}\right]\right) = e_1(y).$$

Similarly, for $j=2$, $\Pi^{\nu}_{s+l,\lambda}(e_2;y)\longrightarrow e_2(y)$. Hence, We arrived the desired proof of Theorem 2.4. □

Theorem 2.5. *For $\hbar\in C_B[0,\infty)$ and $\Pi^{\nu}_{s+l,\lambda}(.;.)$ given by (1.7), we have*

$$|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)|\leq 2\omega(\hbar;\phi),\ \textit{where } \phi=\sqrt{\Pi^{\nu}_{s+l,\lambda}(B^{\nu}_{s+l,\lambda};y)}.$$

Proof. In the direction of (2.1), we get

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right|\leq\left\{1+\phi^{-1}\sqrt{B^{\nu}_{s+l,\lambda};y}\right\}\omega(\hbar;\phi).$$

On choosing $\phi=\sqrt{\Pi^{\nu}_{s+l,\lambda}(B^{\nu}_{s+l,\lambda};y)}$ which completes the proof of Theorem 2.5. □

3. Graphical and numerical analysis

In this section, we inspect the convergence of the operator given by (1.7) for the function $\hbar(y)=\frac{1}{4}e^{-15y}y$. For the operator (1.7), we discuss numerical behavior in Table 1 for different values of $s+l$, namely 10, 15, and 25, by using $\nu=0.3,\ \lambda=0.5$ and using error formula $E_{s+l,\lambda}(\hbar;y)=|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-(\hbar y)|$. Furthermore, Figure 1 and Figure 2 present graphical representations of the convergence and error of the operator (1.7), respectively using $\hbar(y)=\frac{1}{4}e^{-15y}y$. and $s+l=10,15,25$.
Error approximation table 2.

y	$E^{\nu}_{10,\lambda}(\hbar; y)$	$E^{\nu}_{15,\lambda}(\hbar; y)$	$E^{\nu}_{25,\lambda}(\hbar; y)$
0.1	0.00855716	0.00868693	0.00888307
0.2	0.00331262	0.000933993	0.000881886
0.3	0.010011	0.00441485	0.0014342
0.4	0.0103584	0.00364727	0.00106068
0.5	0.00776931	0.00208241	0.000478021
0.6	0.00488011	0.000981387	0.000172209
0.7	0.00272745	0.00040907	0.000054154
0.8	0.00140172	0.000156346	0.0000155106
0.9	0.000676062	0.0000559933	$4.14678*10^{-6}$

TABLE 1. Error Approximation Table

.

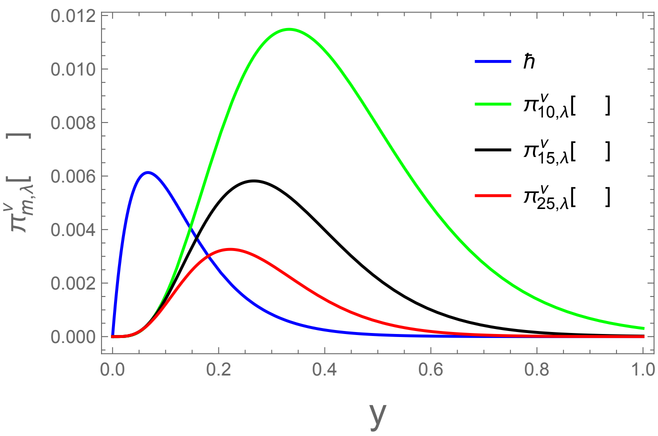

FIGURE 1. Convergence of Operator $\Pi^{\nu}_{s+l,\lambda}(\hbar; y)$ for $s + l = 10, 15, 25$

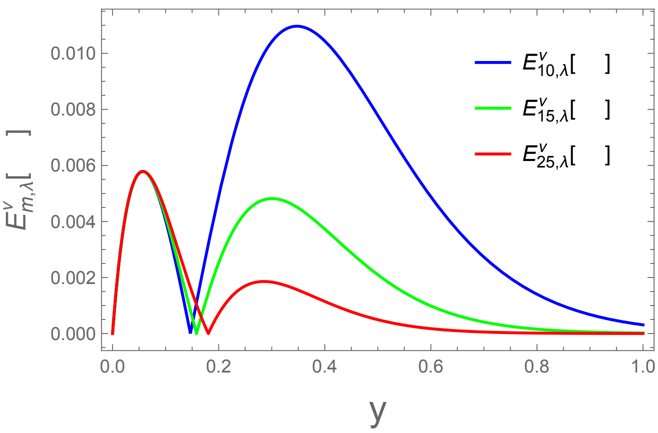

FIGURE 2. Error Approximation

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4. Local approximation

In this section, we discus direct approximation results for $\hbar \in C_B[0,\infty)$, endowed with the norm. For any $\hbar \in C_B[0,\infty)$ $||\hbar|| = sup_{0\leq y<\infty}|\hbar(y)|$. For any $\hbar \in C_B[0,\infty)$ and $\phi > 0$, Peetre's K-functional is given as:

$$K_2(\hbar;\phi) = inf\left\{||\hbar - g|| + \phi||g^{''}|| : g \in C_B^2[0,\infty)\right\},$$

where $C_B^2[0,\infty) = \left\{g \in C_B[0;\infty) : g^{'}, g^{''} \in C_B[0,\infty)\right\}$.

By DeVore and Lorentz ([?] p.177, Theorem 2.4), there exist $C > 0$ such that

$$K_2(\hbar,\phi) \leq C(\hbar, \sqrt{\phi}). \tag{4.1}$$

Second order modulus of continuity $\omega_2(\hbar;\phi)$ and is given as:

$$\omega_2(\hbar;\sqrt{\phi}) = sup_{0<\hbar\leq\sqrt{\phi}} sup_{y\in[0,\infty)}|\hbar(y+2t) - 2\hbar(y+t) + \hbar(y)|.$$

Now, we consider the auxiliary operator $\widehat{\Pi}^{\nu}_{s+l,\lambda}(.;.)$ as:

$$\widehat{\Pi}^{\nu}_{s+l,\lambda}(\hbar;y) = \Pi^{\nu}_{s+l,\lambda}(\hbar;y) + \hbar(y) - \hbar\left(y + \lambda\left[\frac{1-e^{-(s+l+1)y}}{(s+l)((s+l)^2-1)} - \frac{2y}{(s+l)((s+l)-1)}\right]\right). \tag{4.2}$$

Lemma 4.1. *Let $\hbar \in C_B^2[0,\infty)$ and $y \geq 0$. Then, we get*

$$|\widehat{\Pi}^{\nu}_{s+l,\lambda}(\hbar;y) - \hbar(y)| \leq \xi^{\nu}_{s+l,\lambda}(y)||f^{''}||,$$

where

$$\begin{aligned}
\xi^{\nu}_{s+l,\lambda}(y) &= \left(\frac{(s+l)\nu}{(s+l)\nu+1}\right)y^2 + \left(\frac{\nu+1}{(s+l)\nu+1}\right)y \\
&+ \lambda\left(\frac{(s+l)\nu}{(s+l)\nu+1}\right)\left[\frac{2}{(s+l)((s+l)-1)}y\right. \\
&- \left.\frac{4((s+l)+1)}{(s+l)^2((s+l)-1)}y^2 + \frac{e^{-((s+l)+1)y}-1}{(s+l)^2((s+l)-1)}\right] \\
&+ \frac{(s+l)}{(s+l)\nu+1}\left[\frac{1-e^{-((s+l)+1)y}}{(s+l)((s+l)-1)} - \frac{2y}{(s+l)((s+l)-1)}\right].
\end{aligned}$$

Proof. In the light of Auxiliary operators defined in (4.2), we yield

$$\widehat{\Pi}^{\nu}_{s+l,\lambda}(1;y) = 1,\ \widehat{\Pi}^{\nu}_{s+l,\lambda}(A^{\nu}_{s+l};y) = 0 \text{ and } |\widehat{\Pi}^{\nu}_{s+l,\lambda}(\hbar;y)| \leq 3||\hbar||. \tag{4.3}$$

In view of Taylor's series expansion, for $\hbar \in C_B^2[0,\infty)$, we get

$$g(t) = \hbar(y) + (t-y)\hbar^{'}(y) + \int_y^t (t-v)\hbar^{''}(v)dv. \tag{4.4}$$

Apply the auxiliary operators in the above equation (4.2), we yield

$$\widehat{\Pi}^{\nu}_{s+l,\lambda}(\hbar;y) - \hbar(y) = \hbar^{'}(y)\widehat{\Pi}^{\nu}_{s+l,\lambda}(t-y;y) + \widehat{\Pi}^{\nu}_{s+l,\lambda}\left(\int_y^1 (t-v)\hbar^{''}(v)dv;y\right).$$

On account of (4.2) and (4.3), we get

$$\begin{aligned}\widehat{\Pi}^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y) &= \widehat{\Pi}^{\nu}_{s+l,\lambda}\left(\int_y^1 (t-v)(\hbar)^{''}(v)dv; y\right) \\ &= \Pi^{\nu}_{s+l,\lambda}\left(\int_y^1 (t-y)\hbar^{''}(y)dv; y\right) - \int_y^{F_{s+l}} (F_{s+l}-v)\, h^{''}(v)dv,\end{aligned} \tag{4.5}$$

where $F_{s+l} = y + \lambda\left[\frac{1-e^{-((s+l)+1)y}}{(s+l)((s+l)^2-1)} - \frac{2y}{(s+l)((s+l)-1)}\right]$.
Since,

$$\left|\int_y^1 (t-v)\hbar^{''}(v)dv\right| \leq (t-y)^2 ||\hbar^{''}||. \tag{4.6}$$

Therefore, we yield

$$\left|\int_y^{F_{s+l}} (F_{s+l}-v)\,\hbar^{''}(v)dv\right| \leq (F_{s+l}-y)^2\,||\hbar^{''}||. \tag{4.7}$$

Applying (4.6) and (4.7) in (4.5), we obtain

$$\begin{aligned}\left|\Pi^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)\right| &\leq \left\{\Pi^{\nu}_{s+l,\lambda}((t-y)^2; y) + (F_{s+l})\right\} ||\hbar^{''}|| \\ &= \xi^{\nu}_{s+l,\lambda}(y) ||^{\hbar''}||,\end{aligned}$$

Hence, we arrived at our desired result. □

Theorem 4.2. *Let $\hbar \in C_B^2[0,\infty)$. Then, we have*

$$|\Pi^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)| \leq C\omega_2(\hbar; \sqrt{\xi^{\nu}_{s+l,\lambda}(y)}) + \omega(\hbar; \Pi^{\nu}_{s+l,\lambda}(\xi^{\nu}_{s+l,\lambda}(y); y)),$$

where $\xi^{\nu}_{s+l,\lambda}(y)$ is found in Lemma 4.1 and $C > 0$.

Proof. For $g \in C_B^2[0,\infty)$, $\hbar \in C_B[0,\infty)$ and the auxiliary operator $\widehat{\Pi}^{\nu}_{s+l,\lambda}(.;.)$, we have

$$\begin{aligned}|\Pi^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)| &\leq |\widehat{\Pi}^{\nu}_{s+l,\lambda}(\hbar - g; y)| + |(\hbar - g)(y)| + |\widehat{\Pi}^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)| \\ &+ \left|\hbar\left(F_{s+l}\right) - \hbar(y)\right|.\end{aligned}$$

From Lemma 4.1 and equation (4.3), we yield

$$\begin{aligned}\left|\Pi^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)\right| &\leq 4||\hbar - g|| + |\Pi^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)| \\ &+ \left|\hbar\left(F_{s+l}\right) - \hbar(y)\right| \\ &\leq 4||\hbar - g|| + \xi^{\nu}_{s+l,\lambda}(y)||g^{''}|| + \omega(\hbar; \Pi^{\nu}_{s+l,\lambda}(\xi^{\nu}_{s+l,\lambda}(y); y)).\end{aligned}$$

Using Peetre's K-functional, we have

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)\right| \leq C\omega_2\left(\hbar; \sqrt{\xi^{\nu}_{s+l,\lambda}(y)}\right) + \omega(\hbar; \Pi^{\nu}_{s+l,\lambda}(\xi^{\nu}_{s+l,\lambda}(y); y)).$$

Hence, we completes the proof of Theorem 4.2. We recall Lipschitz-type space here [17] as:
$Lip_M^{\rho_1\rho_2}(\gamma) := \left\{\hbar \in C_B[0,\infty) : |\hbar(t) - \hbar(y)| \leq M\frac{|t-y|^{\gamma}}{(t + \rho_1 y + \rho_2 y^2)^{\gamma/2}} : y, t \in (0,\infty)\right\}$, where $M > 0$ is a fixed constant and $0 < \gamma \leq 1$. $\rho_1 > 0$, $\rho_2 > 0$, are two real values. □

Theorem 4.3. *For $y \in (0,\infty)$, $\hbar \in Lip_M^{\rho_1,\rho_2}(\gamma)$ and sequence of operators defined by* (1.7), *one get*

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right| \leq M\left(\frac{\phi^{\nu}_{s+l,\lambda}(y)}{\rho_1 y+\rho_2 y^2}\right)^{\dfrac{\gamma}{2}}, \tag{4.8}$$

where $\gamma \in (0,1]$ and $\phi^{\nu}_{s+l,\lambda}(y)=\Pi^{\nu}_{s+l,\lambda}(\xi^2_{s+l,\lambda};y)$.

Proof. First, we consider $y \in (0,\infty)$ and $\gamma=1$ we yield

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right| \leq \Pi^{\nu}_{s+l,\lambda}(|\hbar(t)-\hbar(y);y) \leq M\Pi^{\nu}_{s+l,\lambda}\left(\frac{|t-y|}{(t+\rho_1 y+\rho_2 y^2)^{1/2}};y\right).$$

It is obvious that

$$\frac{1}{(\rho_1 y+\rho_2 y^2)} > \frac{1}{t+\rho_1 y+\rho_2 y^2}.$$

Therefore $y \in (0;\infty)$, one has

$$\begin{aligned}\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right| &\leq \frac{M}{(\rho_1 y+\rho_2 y^2)^{1/2}}\left(\Pi^{\nu}_{s+l,\lambda}(t-y)^2;y\right)^{1/2}\\ &\leq M\left(\frac{\phi_{(s)}(y)}{\rho_1 y+\rho_2 y^2}\right)^{1/2}.\end{aligned}$$

In the light of Hölder's inequality, Theorem 4.3, holds good for $\gamma=1$, with $\rho_1=2/\gamma$ and $\rho_2=2/2-\gamma$, we yield

$$\begin{aligned}\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right| &\leq \left(\Pi^{\nu}_{s+l,\lambda}(|\hbar(t)-\hbar(y)|^{\gamma/2};y)\right)^{\gamma/2}\\ &\leq M(\Pi^{\nu}_{s+l,\lambda})\left(\frac{|t-y|^2}{t+\rho_1 y+\rho_2 y^2};y\right)^{\gamma/2}.\end{aligned}$$

Since $\dfrac{1}{t+\rho_1 y+\rho_2 y^2} < \dfrac{1}{\rho_1 y+\rho_2 y^2}$ we yield

$$|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)| \leq M\left(\frac{\Pi^{\nu}_{s+l,\lambda}(|t-y|^2;y)}{\rho_1 y+\rho_2 y^2}\right)^{\gamma/2} \leq M\left(\frac{\phi^{\nu}_{s,\lambda}(y)}{\rho_1 y+\rho_2 y^2}\right)^2.$$

Hence, we arrived at our desired result. Now, we recall γ^{th} term order Lipschitz-type maximal function suggested by Lenze [18] as:

$$\widetilde{\omega}(\hbar;y)=sup_{t\neq y,t\in(0,\infty)}\frac{|\hbar(t)-\hbar(y)|}{|t-y|^{\gamma}}, y\in[0;\infty), \tag{4.9}$$

and $r \in (0,1]$. □

Theorem 4.4. *Let $\hbar \in C_B[0,\infty)$ and $r \in (0,1]$. Then, for all $y \in (0,\infty)$, one has*

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right| \leq \widetilde{\omega}_{\gamma}(\hbar;y)(\phi^{\nu}_{s,\lambda}(y))^{\gamma/2}.$$

Proof. We have

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right| \leq \Pi^{\nu}_{s+l,\lambda}\left(|\hbar(v)-\hbar(y)|;y\right).$$

In the direction equation (4.9), we have

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar;y)-\hbar(y)\right| \leq \widetilde{\omega_r}\left(\Pi^{\nu}_{s+l,\lambda}(|t-y|^{\gamma};y)\right).$$

Using Hölder's inequality with $\rho_1 = 2/\gamma$ and $\rho_2 = 2/2-\gamma$, we have

$$\left|\Pi^{\nu}_{s+l,\lambda}(\hbar; y) - \hbar(y)\right| \le \widetilde{\omega_\gamma}(h; y)\left(\Pi^{\nu}_{s+l,\lambda}|v-y|^2; y\right)^{\gamma/2},$$

we arrived at the desired result. □

5. Bivariate Extension of Extended Beta type Szász-Schurer-Mirakjan Operators

Take $T^2 = \{(y,v) : 0 \le y < \infty, 0 \le v < \infty\}$ and $C(T^2)$ represents class of continuous function over T^2 influenced with norm $||\hbar||_{C(T^2)} = \sup_{(y,v)\in T^2} |\hbar(y,v)|$. Then, for all $\hbar \in C(T^2)$ and $s_1, s_2 \in \mathbb{N}$, we introduced a bivariate extension as:

$$\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\hbar; y, v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} C^{\nu}_{s_1+l,s_2+l,j,k}(t_1,t_2)\tilde{t}_{s_1+l,s_2+l,j,k}(\lambda; y, v)\hbar(t_1,t_2), \tag{5.1}$$

where

$$C^{\nu}_{S_1+l,S_2+l,j,k}(t_1,t_2) = C^{\nu}_{s_1+l,j}(t_1)C^{\nu}_{s_2+l,k}(t_2),$$

and

$$C^{\nu}_{s_i+l,j,k}(t_i) = \int_0^1 D^{\nu}_{s_i+l,j,k}(t_i)dt_i,$$

for $i = 0,1,2$ and $\tilde{t}_{s_1+l,s_2+lj,k}(y,v) = \tilde{t}_{s_1+l,j}(\lambda; y)\tilde{t}_{s_2+l,k}(\lambda; v)$.

Lemma 5.1. *Suppose $e_{m,n} = y^m v^n$ represents the two dimensional test function, then for the operator* (5.1)*, we get*

$$\begin{aligned}
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{0,0}; y, v) &= 1,\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{1,0}; y, v) &= y + \lambda\left[\frac{1-e^{-((s_1+l)+1)y}}{(s_1+l)((s_1+l)^2-1)} - \frac{2y}{(s_1+l)((s_1+l)-1)}\right],\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{0,1}; y, v) &= v + \lambda\left[\frac{1-e^{-(s_2+l+1)v}}{(s_1+l)(s_2+l^2-1)} - \frac{2v}{(s_1+l)(s_2+l-1)}\right],\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{2,0}; y, v) &= \left(\frac{(s_1+l)\nu}{(s_1+l)\nu+1}\right)y^2 + \left(\frac{\nu+1}{(s_1+l)\nu+1}\right)y\\
&+ \lambda\left(\frac{(s_1+l)\nu}{(s_1+l)\nu+1}\right)\left[\frac{2}{(s_1+l)((s_1+l)-1)}y - \frac{4((s_1+l)+1)}{(s_1+l)^2((s_1+l)-1)}y^2\right.\\
&+ \left.\frac{e^{-((s_1+l)+1)y}-1}{(s_1+l)^2((s_1+l)-1)}\right]\\
&+ \frac{(s_1+l)}{(s_1+l)\nu+1}\left[\frac{1-e^{-((s_1+l)+1)y}}{(s_1+l)(s_2+l-1)} - \frac{2y}{(s_1+l)((s_1+l)-1)}\right].
\end{aligned}$$

$$\begin{aligned}
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{0,2};y,v) &= \left(\frac{s_2+l\nu}{s_2+l\nu+1}\right)v^2+\left(\frac{\nu+1}{s_2+l\nu+1}\right)v\\
&+ \lambda\left(\frac{s_2+l\nu}{s_2+l\nu+1}\right)\left[\frac{2}{s_2+l(s_2+l-1)}v-\frac{4(s_2+l+1)}{s_2+l^2(s_2+l-1)}v^2\right.\\
&+ \left.\frac{e^{-(s_2+l+1)v}-1}{s_2+l^2(s_2+l-1)}\right]\\
&+ \frac{s_2+l}{s_2+l\nu+1}\left[\frac{1-e^{-(s_2+l+1)v}}{s_2+l(s_2+l-1)}-\frac{2v}{s(s_2+l-1)}\right].
\end{aligned}$$

Proof. In the direction linearity property and (2.2) we have

$$\begin{aligned}
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{0,0};y,v) &= \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_0;y,v)\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_0;y,v),\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{1,0};y,v) &= \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_1;y,v)\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_0;y,v),\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{0,1};y,v) &= \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_0;y,v)\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_1;y,v),\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{2,0};y,v) &= \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_2;y,v)\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_0;y,v),\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{0,2};y,v) &= \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_0;y,v)\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_2;y,v).
\end{aligned}$$

□

Lemma 5.2. *For $\eta_{m,n}(t_1,t_1)=(t_1-y)^m(t_2-v)^n$ for $m,n=0,1,2$, then we have following equalities:*

$$\begin{aligned}
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\eta_{0,0};y,v) &= 1,\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\eta_{1,0};y,v) &= \lambda\left[\frac{1-e^{-((s_1+l)+1)y}-2y}{(s_1+l)((s_1+l)-1)}\right],\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\eta_{0,1};y,v) &= \lambda\left[\frac{1-e^{-(s_2+l+1)v}-2v}{s_2+l(s_2+l-1)}\right],\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\eta_{2,0};y,v) &= \left(\frac{(s_1+l)\nu}{(s_1+l)\nu+1}-1\right)y^2+\left(\frac{\nu+1}{(s_1+l)\nu+1}\right)y\\
&+ \lambda\left[\frac{(1+2(s_1+l)y)e^{-((s_1+l)+1)y}-(1-2(s_1+l))y-4y^2}{(s_1+l)^2((s_1+l)-1)}\right]\\
\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\eta_{0,2};y,v) &= \left(\frac{(s_2+l)\nu}{(s_2+l)\nu+1}-1\right)v^2+\left(\frac{\nu+1}{(s_2+l)\nu+1}\right)v\\
&+ \lambda\left[\frac{(1+2(s_2+l)v)e^{-(s_2+1)v}-(1-2(s_2+l))v-4v^2}{(s_2+l)^2(s_2+l-1)}\right].
\end{aligned}$$

Proof. In the light of Lemma 5.1 and linearity property, one can easily prove the required result. □

Definition 5.1. *Consider $T_1=[0,\infty)$, $T_2=[0,\infty)\subset\mathbb{R}$ as given intervals and $B(T_1\times T_2)=\{\hbar:T_1\times T_2\to\mathbb{R}:\hbar\ is\ defined\ and\ bounded\ on\ T_1\times T_2\}$. Then, for $g\in B(T_1\times T_2)$ present the total modulus of continuity is defined as: $\omega_{total}(\hbar;\cdot,*):C(T^2)\to\mathbb{R}$ provided that $(\phi_1,\phi_2)\in T_1\times T_2$ and defined by*

$$\omega_{total}(\hbar;\phi_1,\phi_2)=\sup_{|x_1-x_1'|\le\phi_1,|y_1-y_1'|\le\phi_2}\{|\hbar(x_1,y_1)-\hbar(x_1',y_1')|:(x_1,y_1),$$

$(x_1',y_1')\in T_1\times T_2\}$, is termed as the total modulus of continuity corresponding to the function $\hbar$.

Here, we discuss the convergence rate of the operators given by (4.1). To discuss convergence rate, we revisit the following result presented by the Volkov [27]:

Theorem 5.3. *Let $L_{s_1,s_2} : C(T^2) \to C(T^2), (s_1, s_2) \in \mathbb{N} \times \mathbb{N}$ be linear positive operators. If*

$$\lim_{s_1,s_2\to\infty} L_{s_1,s_2}(e_{mn}) = e_{y,v}, (m,n) \in \{(0,0),(1,0),(0,1)\},$$

If

$$\lim_{s_1,s_2\to\infty} L_{s_1,s_2}(e_{mn}) = e_{y,v}, (m,n) \in \{(0,0),(1,0),(0,1)\},$$

and

$$\lim_{s_1,s_2\to\infty} L_{s_1,s_2}(e_{20}+e_{02}) = e_{20}+e_{02},$$

uniformly on T^2, then the sequence $(L_{s_1,s_2}\hbar)$ converges to $\hbar$ uniformly on T^2 for any $\hbar \in C(T^2)$.

Theorem 5.4. *Let $e_{mn}(y,v) = y^m v^n$ $(0 \le m+n \le 2, m,n \in \mathbb{N})$ be the test functions restricted on T^2. If*

$$\lim_{(s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{mn}; y,v) = e_{mn}(y,v),$$

and

$$\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{20}+e_{02}; y,v) = e_{20}(y,v)+e_{02}(y,v),$$

uniformly on T^2, then

$$\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\hbar; y,v) = \hbar(y,v),$$

uniformly for all $\hbar \in C(T^2)$.

Proof. In view of Lemma 5.1, it is evident for $m = n = 0$

$$\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{00}; y,v) = e_{00}(y,v).$$

For $m = 1$, $n = 0$, we get

$$\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(p_{10}; y,v) = y,$$
$$\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(p_{10}; y,v) = p_{10}(y,v).$$

Similarly

$$\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{01}; y,v) = v,$$
$$\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{01}; y,v) = e_{01}(y,v),$$

and in the light Lemma (5.1), we get

$$\begin{aligned}\lim_{s_1,s_2\to\infty} \Pi^{\nu}_{s_1+l,s_2+l,\lambda}(e_{20}+e_{02}; y,v) &= y^2+v^2,\\ &= e_{20}(y,v)+e_{02}(y,v).\end{aligned}$$

In the direction Theorem 5.3, Theorem 5.4 is easily proved. □

In the last result, we deal approximation order of the sequence of operators $\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(.;.)$ given by (4.1) as:

Theorem 5.5. [28] *Let $L : C(T^2) \to B(T^2)$ be a linear positive operator. For any $\hbar \in C(T^2)$, any $(z_1, z_2) \in T^2$ and any $\phi_1, \phi_2 > 0$, the following inequality*

$$\begin{aligned} |(L\hbar)(z_1,z_2) - \hbar(z_1,z_2)| &\leq |Le_{0,0}(z_1,z_2) - 1||\hbar(z_1,z_2)| + \Big[Le_{0,0}(z_1,z_2) \\ &+ \phi_1^{-1}\sqrt{Le_{0,0}(z_1,z_2)(L(\cdot - z_1))^2(z_1,z_2)} \\ &+ \phi_2^{-1}\sqrt{Le_{0,0}(z_1,z_2)(L(* - z_2))^2(z_1,z_2)} \end{aligned}$$

$$\begin{aligned} &+ \phi_1^{-1}\phi_2^{-1}\sqrt{(Le_{0,0})^2(z_1,z_2)(L(\cdot - z_1))^2(z_1,z_2)(L(* - z_2))^2(z_1,z_2)}\Big] \\ &\times \omega_{total}(\hbar;\phi_1,\phi_2), \end{aligned}$$

holds.

Theorem 5.6. *For $\hbar \in C(T^2)$ and $(y, v) \in T^2$, $((s_1+l), s_2) \in \mathbb{N}\times\mathbb{N}$ and $\phi_1, \phi_2 > 0$, one has*

$$|\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\hbar; y, v) - \hbar(y, v)| \leq 4\omega_{total}(\hbar; \phi_1, \phi_2),$$

where $\phi_1 = \sqrt{\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_1 - y)^2; y, v\Big)}$ *and* $\phi_2 = \sqrt{\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_2 - v)^2; y, v)\Big)}$.

Proof. From Theorem 5.5, we have

$$\begin{aligned} . \quad & |\Pi^{\nu}_{s_1+l,s_2+l,\lambda}(\hbar; y, v) - \hbar(y, v)| \\ \leq \quad & \Bigg[1 + +\phi_1^{-1}\sqrt{\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_1 - y)^2; y, v\Big)} \\ + \quad & \phi_2^{-1}\sqrt{\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_2 - v)^2; y, v\Big)} \\ + \quad & \phi_1^{-1}\phi_2^{-1}\sqrt{\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_1 - y)^2; y, v\Big)\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_2 - v)^2; y, v\Big)}\ \Bigg] \\ \times \quad & \omega_{total}(\hbar; \phi_1, \phi_2). \end{aligned}$$

Selecting $\phi_1 = \sqrt{\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_1 - y)^2; y, v\Big)}$

and $\phi_2 = \sqrt{\Pi^{\nu}_{s_1+l,s_2+l,\lambda}\Big((t_2 - v)^2; y, v)\Big)}$, we arrive at the required result. □

Example 5.1. *In this section we inspect different values of parameters $\lambda = 0.5$ and $\nu = 0.3$ through the table and figure presented in the example below. The operators $\pi^{\nu}_{(s_1+l),s_2+l,\lambda}(\hbar; y, v)(.; .)$ converges uniformly to the function $\hbar(y, v) = \frac{1}{4}e^{-15(y+v)}$ (Block) and the different values $(s_1 + l) = s_2 + l = 10$ (Blue) $(s_1 + l) = s_2 + l = 15$ (Green) and $(s_1 + l) = s_2 + l = 25$ (Red) increases which is shown in the table and Figure 3.*

y, v	$\Pi^{\nu}_{10,10,\lambda}(.;.)$	$\Pi^{\nu}_{15,15,\lambda}(.;.)$	$\Pi^{\nu}_{25,25,\lambda}(.;.)$
0.1 0.1	0.000124468	$2.3644*10^{-}6$	$1.46835*10^{-}6$
0.2 0.2	0.000054595	0.0000251022	0.0000102038
0.3 0.3	0.000126461	0.0000319147	$7.12176*10^{-}6$
0.4 0.4	0.000114291	0.0000158371	$1.94043*10^{-}6$
0.5 0.5	0.0000616724	$4.69277*10^{-}6$	$3.15716*10^{-}7$
0.6 0.6	0.0000240142	$1.00341*10^{-}6$	$3.70668*10^{-}8$
0.7 0.7	$7.46513*10^{-}6$	$1.71277*10^{-}7$	$3.47392*10^{-}9$
0.8 0.8	$1.96793*10^{-}6$	$2.47905*10^{-}8$	$2.76057*10^{-}10$
0.9 0.9	$4.574*10^{-}7$	$3.1634*10^{-}9$	$1.9339*10^{-}11$

TABLE 2. Error Approximation Table

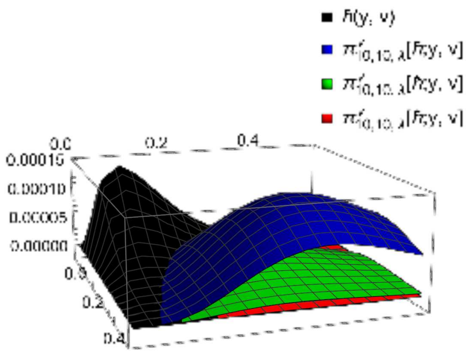

FIGURE 3. $\pi_{(s_1+l),s_2+l,\lambda}(\hbar; y, v)(.;.)$ converges to $\hbar(y, v) = \dfrac{1}{4}e^{-15(y+v)}$

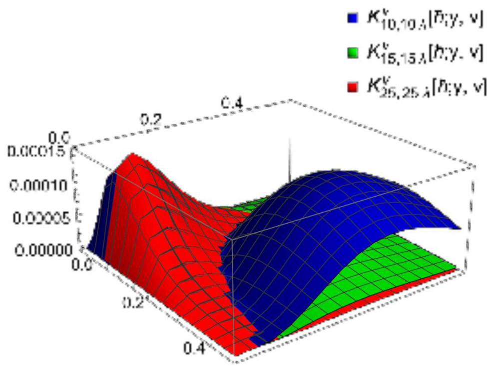

FIGURE 4. Error Approximation

6. Conclusion

In this study, we thoroughly examine how well the λ-Szsz-Mirakjan operators, which are based on the extended beta function, can approximate functions. We explore several key aspects, including:
1. Convergence: How closely these operators can approximate a given function as certain parameters change. Speed of Convergence.
2.How fast the approximation approaches the actual function. Performance on Specific Functions
3. How well the operators work for particular types of functions, like polynomials or exponential functions. Overall Approximation Ability.
4. How well the operators approximate functions across a wide range of values. Graphical Representations.
5. We provide visual examples to help illustrate how the operators behave in different situations. This approach helps us better understand how these operators perform and where they are most effective.

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