Bulletin of Mathematical analysis and Applications
ISSN:1821-1291, URL: http://www.bmathaa.org
Volume 1, Issue 1, (2009) Pages 55-61

On L_1-convergence of certain cosine sums *

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Abstract

In this paper a criterion for L_1- convergence of a certain cosine sums with quasi semi-convex coefficients is obtained. Also a necessary and sufficient condition for L_1-convergence of the cosine series is deduced as a corollary.

1 Introduction

It is well known that if a trigonometric series converges in L_1-metric to a function $f \in L_1$, then it is the Fourier series of the function f. Riesz [2] gave a counter example showing that in a metric space L_1 we cannot expect the converse of the above said result to hold true. This motivated the various authors to study L_1-convergence of the trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L_1-metric to the sum of the trigonometric series whereas the classical series itself may not. In this contest we will introduce new modified cosine series given by relation

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2},$$

and for this modified cosine series we will prove L_1-convergence, under conditions that coefficients (a_n) are quasi semi-convex.

2 Preliminaries

In what follows we will denote by

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1}$$

* *Mathematics Subject Classifications:* 42A20, 42A32.
Key words: cosine sums, L_1-convergence, quasi semi-convex null sequences.

Submitted November, 2009. Published March, 2009.

with partial sums defined by

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx, \tag{2}$$

and

$$g(x) = \lim_{n\to\infty} S_n(x). \tag{3}$$

In the sequel we will mention some notations which are useful for the further work. First let us denote

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

and

$$\widetilde{D}_n(t) = \sum_{k=1}^{n} \cos kt.$$

For all other notations see [11].

Definition 2.1 *A sequence of scalars (a_n) is said to be semi-convex if $a_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, (a_0 = 0), \tag{4}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$.

Definition 2.2 *A sequence of scalars (a_n) is said to be quasi-convex if $a_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty, (a_0 = 0), \tag{5}$$

Definition 2.3 *A sequence of scalars (a_n) is said to be quasi semi-convex if $a_n \to 0$ as $n \to \infty$, and*

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} - \Delta^2 a_n| < \infty, (a_0 = 0), \tag{6}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$.

Kolmogorov in [5], proved the following theorem:

Theorem 2.4 *If (a_n) is a quasi-convex null sequence, then for the L_1-convergence of the cosine series (1), it is necessary and sufficient that $\lim_{n\to\infty} a_n \cdot \log n = 0$.*

The case in which sequence (a_n) is convex, of this theorem was established by Young (see [10]). That is why, sometimes, this theorem is known as Young-Kolmogorov Theorem.

Remark 2.5 *If (a_n) is a quasi-convex null scalar sequence, then it is quasi semi-convex scalars sequence too.*

Bala and Ram in [1] have proved that Theorem 2.4 holds true for cosine series with semi-convex null sequences in the following form:

Theorem 2.6 *If (a_n) is a semi-convex null sequence, then for the convergence of the cosine series (1) in the metric space L, it is necessary and sufficient that $a_{k-1}\log k = 0(1), k \to \infty$.*

Garret and Stanojevic in [3], have introduced modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j)\cos kx. \tag{7}$$

The same authors (see [4]), Ram in [8] and Singh and Sharma in [9] studied the L_1-convergence of this cosine sum under different sets of conditions on the coefficients (a_n). Kumari and Ram in [7], introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)\cos kx \tag{8}$$

and

$$G_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_j}{j}\right)\sin kx, \tag{9}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) belong to different classes of sequences. Later one, Kulwinder in [6], introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx, \tag{10}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) are semi-convex null.

3 Results

In this paper we introduce the following modified cosine sums

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}. \tag{11}$$

The aim of this paper is to study the L_1-convergence of this modified cosine sums with quasi semi-convex coefficients and to give necessary and sufficient condition for L_1-convergence of the cosine series defined by relation (1).

Theorem 3.1 *Let (a_n) a the quasi semi-convex null sequence, then $N_n(x)$ converges to $g(x)$ in L_1 norm.*

Proof We have

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cdot \cos kx = \frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n} a_k \cdot \cos kx \cdot \left(2\sin\frac{x}{2}\right)^2$$

$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n} a_k[\cdot \cos(k+1)x - 2\cos kx + cos(k-1)x]$$

$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n} (a_{k-1} - 2a_k + a_{k+1}) \cdot \cos kx - \frac{a_0 \cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2} \Rightarrow$$

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n} \Delta^2 a_{k-1} \cos kx - \frac{a_0 \cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Applying Abel's transformation, we have

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{\Delta^2 a_{n-1} \cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_0 \cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Since $\widetilde{D}_n(x)$ is uniformly bounded on every segment $[\epsilon, \pi - \epsilon]$, for every $\epsilon > 0$,

$$g(x) = \lim_{n\to\infty} S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{\infty} (\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}$$

Also

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}$$

respectively

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \sum_{k=1}^{n} \Delta^2 a_{k-1}\cos kx + \frac{\Delta^2 a_n \cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

Now applying Abel's transformation we get the following relation:

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}-\Delta^2 a_k)\widetilde{D}_k(x)+\frac{\Delta^2 a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}+\frac{\Delta^2 a_n\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}$$
$$+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}$$

From above relation we will have:

$$g(x)-N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=n+1}^{\infty}(\Delta^2 a_{k-1}-\Delta^2 a_k)\widetilde{D}_k(x)-\frac{\Delta^2 a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\Delta^2 a_n\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}.$$

Thus, we have

$$\int_0^{\pi}|g(x)-N_n(x)|dx\to 0,$$

for $n\to\infty$, and definition 1.3.

Corollary 3.2 *Let (a_n) be a quasi-convex null sequence, then $N_n(x)$ converges to $g(x)$ in L_1 norm.*

Proof Proof of the corollary follows directly from Theorem 3.1 and Remark 2.5.

Corollary 3.3 *If (a_n) is a quasi semi-convex null sequence of scalars, then the necessary and sufficient condition for L_1-convergence of the cosine series (1) is* $\lim_{n\to\infty} a_n\log n = 0$.

Proof Let us start from this estimation:

$$||S_n(x)-g(x)||_{L_1}\le ||S_n(x)-N_n(x)||_{L_1}+||N_n(x)-g(x)||_{L_1} = ||N_n(x)-g(x)||_{L_1}+$$
$$\left\|\frac{a_n\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^2}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\Delta^2 a_n\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}\right\|$$

On the other hand

$$\left\|\frac{a_n\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^2}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\Delta^2 a_n\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}\right\| = \tag{12}$$
$$||N_n(x)-S_n(x)||\le ||N_n(x)-g(x)||+||g(x)-S_n(x)||,$$

and

$$\Delta^2 a_n = \sum_{k=n}^{\infty}(\Delta^2 a_k-\Delta^2 a_{k+1}) = \sum_{k=n}^{\infty}\frac{k}{k}(\Delta^2 a_k-\Delta^2 a_{k+1})\le$$
$$\frac{1}{n}\sum_{k=n}^{\infty}(\Delta^2 a_k-\Delta^2 a_{k+1}) = o\left(\frac{1}{n}\right).$$

Since

$$\int_0^\pi \frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} = O(n),$$

therefore

$$\Delta^2 a_n \cdot \int_0^\pi \frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} = o(1).$$

For the rest of the expression (11) we have this estimation:

$$\int_0^\pi \left| \frac{a_n \cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2} \right| \le \int_0^\pi a_n \left| \frac{\cos{(n+1)x}}{\left(2\sin\frac{x}{2}\right)^2} - \frac{\cos nx}{\left(2\sin\frac{x}{2}\right)^2} \right| =$$

$$= \int_0^\pi a_n \left| \widetilde{D}_n(x) - \frac{1}{2} \right| dx \sim (a_n \log n).$$

From Theorem 3.1 it follows that

$$||N_n(x) - g(x)|| = o(1), n \to \infty.$$

Finally we get this estimation

$$\lim_{n\to\infty} \int_0^\pi |g(x) - S_n(x)| = o(1),$$

if and only if

$$\lim_{n\to\infty} a_n \log n = 0,$$

with which was proved corollary.

Corollary 3.4 *If (a_n) is a quasi-convex null sequence of scalars, then the necessary and sufficient condition for L_1-convergence of the cosine series (1) is* $\lim_{n\to\infty} a_n \log n = 0$.

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