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On the local properties of factored Fourier series *

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Abstract

In this paper we have improved the result of Bor [Bull. Math. Anal. Appl.1, (2009), 15-21] on local property of $\left|\overline{N}, p_n, \theta_n\right|_k$ summability of factored Fourier series by proving under weaker conditions.

1 Introduction

Let $\sum a_n$be a given series with partial sums (s_n), and let (p_n) be a sequence of positive numbers such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty \; as \; n \to \infty$. The sequence- to- sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (T_n) of the $\left(\overline{N}, p_n\right)$ means of the sequence (s_n), generated by the sequence coefficients (p_n). The series $\sum a_n$ is summable $\left|\overline{N}, p_n, \theta_n\right|_k$ summability, $k \geq 1$, if (*see* [4])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_n - T_{n-1} |^k < \infty. \tag{1.1}$$

Let f be a function with period 2π, integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of is zero, so that

$$\int_{-\pi}^{\pi} f(t) dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} C_n(t). \tag{1.2}$$

It is well known [5] that convergence of a Fourier series at any point $t = x$ is a local property of f, i.e., for arbitrarily small $\delta > 0$, the behaviour of $(s_n(t))$, the

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n-th partial sum of the series (1.2), depends only the natura of f in the interval $(x-\delta, x+\delta)$ and is not affected by the values it takes outside the interval.A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integers n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Lemma 1 ([3]). If the sequence (p_n) satisfies the conditions

$$P_n = O(np_n) \tag{1.3}$$

and

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{1.4}$$

then

$$\Delta\ (P_n/np_n) = O(1/n). \tag{1.5}$$

Lemma 2 ([2]). If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then (λ_n) is non-negative and decreasing, and $n\Delta\lambda_n \to 0$ as $n \to \infty$.

2 Known result

Theorem A. Let $k \geq 1$ and (p_n) be a sequence such that the conditions (1.3) and (1.4) are satisfied. Let (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. If (θ_n) is any sequence of positive constants such that

$$\sum_{v=1}^{\infty} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v}\ \lambda_n^k \ < \ \infty \tag{2.1}$$

$$\sum_{v=1}^{\infty} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \Delta\lambda_v \ < \ \infty \tag{2.2}$$

$$\sum_{v=1}^{\infty} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v}\ \lambda_{v+1}^k \ < \ \infty \tag{2.3}$$

and

$$\sum_{n=v+1}^{\infty} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left(\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right) \tag{2.4}$$

then the summability of $\left|\overline{N}, p_n, \theta_n\right|_k$ of the series

$$\sum_{n=1}^{\infty} C_n(t)\, \lambda_n P_n / np_n \tag{2.5}$$

at a point can be ensured by local property of f.

3 The main result

The purpose of this paper is to improve Theorem A by proving under weaker conditions. Now, we give the following theorem.

Theorem. Let $k \geq 1$ and (p_n) be a sequence such that the condition (1.5) is satisfied. Also let (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. If (θ_n) is any sequence of positive constants such that

$$\sum_{v=1}^{\infty} \theta_v^{k-1} \left(\frac{\lambda_v}{v}\right)^k < \infty \tag{3.1}$$

$$\sum_{v=1}^{\infty} \theta_v^{k-1} \frac{P_v}{v^k p_v} \Delta\lambda_v < \infty \tag{3.2}$$

and

$$\sum_{n=v+1}^{\infty} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left\{\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v}\right\} \tag{3.3}$$

then the summability of $\left|\overline{N}, p_n, \theta_n\right|_k$ of the series

$$\sum_{n=1}^{\infty} C_n(t)\, \lambda_n P_n / n p_n \tag{3.4}$$

at a point can be ensured by local property of f.

It may be remarked that (1.3) and (1.4) $\Rightarrow$ (1.5) by Lemma 1. It is obvious that (1.3) and (2.1) $\Rightarrow$(3.1), and also (1.3) and (2.2) $\Rightarrow$ (3.2). Furthermore, since (λ_n) monotonic decreasing, the conditions (2.1) and (2.3) are the same.

Proof of the Theorem. As mentioned in the beginning, the convergence of Fourier series at a point is a local property. Therefore in order to prove the theorem it is sufficient to prove that if (s_n) is bounded, then under the conditions of our theorem, the series $\sum \lambda_n a_n P_n / n\, p_n$ is summable $\left|\overline{N}, p_n; \theta_n\right|_k$. Now, let (T_n) denote the $\left(\overline{N}, p_n\right)$ means of this series. Then we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} X_v \lambda_v a_v, \quad X_n = P_n / n p_n$$

Applying Abel's transformation to this sum we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} s_n \Delta\left(P_{v-1} X_v \lambda_v\right) + \frac{p_n s_n P_{n-1} X_n \lambda_n}{P_n P_{n-1}} \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} s_v \lambda_{v+1} \Delta\left(P_{v-1} X_v\right) + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} s_v P_{v-1} X_v \Delta\lambda_v + \frac{s_n \lambda_n}{n} \\ &= T_1 + T_2 + T_3, \quad \textit{say}. \end{aligned}$$

For the proof of the lemma, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_r|^k < \infty, \quad r = 1, 2, 3..$$

Now, since $s_n = O(1)$, It follows that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_1|^k \quad = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} \lambda_{v+1} \left| \Delta \left(P_{v-1} X_v \right) \right| \right\}^k.$$

On the other hand, in view of

$$\Delta \left(P_{v-1} X_v \right) = -p_v X_v + P_v \Delta X_v = -\frac{P_v}{v} + P_v \Delta X_v = P_v \left(-\frac{1}{v} + \Delta X_v \right),$$

it is clear that the condition $\Delta X_v = O(1/v)$ is equivalent to $\Delta \left(P_{v-1} X_v \right) = O\left(\frac{P_v}{v}\right)$. Therefore, making use of Hölder's inequality and lemma 2, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_1|^k &= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} \lambda_{v+1} \frac{P_v}{v} \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} \lambda_v X_v p_v \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} \lambda_v^k X_v^k p_v \right\} \left\{ \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \lambda_v^k X_v^k p_v \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} X_v^k |\lambda_v|^k p_v \left(\frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{P_v} \\
&= O(1) \sum_{v=1}^{m} \theta_v^{k-1} \left(\frac{\lambda_v}{v} \right)^k = O(1), \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of (3.1). Again, since $\sum_{v=1}^{n-1} P_{v-1} \Delta \lambda_v \leq P_{n-1} \sum_{v=1}^{n-1} \Delta \lambda_v = O(P_{n-1})$ by lemma 2, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_2\mid^k &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}P_{v-1}\,X_v\Delta\lambda_v\right\}^k\\ &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\sum_{v=1}^{n-1}P_{v-1}X_v^k\Delta\lambda_v\left\{\sum_{v=1}^{n-1}P_{v-1}\Delta\lambda_v\right\}^{k-1}.\\ &= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_{v-1}X_v^k\Delta\lambda_v\\ &= O(1)\sum_{v=1}^{m}P_{v-1}X_v^k\Delta\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{\theta_np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}\theta_v^{k-1}\,\frac{P_v\Delta\lambda_v}{v^kp_v}\quad = O(1), as\, m\to\infty,\end{aligned}$$

by virtue of (3.2).

Finally, it is clear that

$$\sum_{n=1}^{m}\theta_n^{k-1}\mid T_3\mid^k = O(1)\sum_{n=1}^{m}\theta_n^{k-1}\left(\frac{\lambda_n}{n}\right)^k = O(1)\ as\ \ m\to\infty,$$

by virtue of (3.1). This completes the proof.

References

[1] H. Bor, On the local properties of factored Fourier Series, *Bull. Math. Anal. Appl.***1**,(2009), 15-21.

[2] H. C. Chow, On the summability factors of infinite series, *J. London Math. Soc.* **16**,(1941), 215-220.

[3] K. N. Mishra, Multipliers for $\left|\overline{N},p_n\right|$ summability of Fourier series, *Bull. Inst. Math. Acad. Sinica* **14**, (1986), 431-438.

[4] W.T. Sulaiman, On some summability factors of infinite series, *Proc. Amer. Math. Soc.* **115**, (1992), 313-317.

[5] E. C. Tichmarsh, The Theory of Functions, *Oxford Univ. Press, London* 1961.

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