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On the local properties of factored Fourier series *

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Abstract

In the present paper, a theorem on local property of $|\bar{N}, p_n, \theta_n|_k$ summability of factored Fourier series which generalizes a result of Bor [3] has been proved.

1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n the n-th (C,1) mean of the sequence (na_n). A series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [6],[8])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n \mid^k < \infty. \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k, k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{1.4}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{1.5}$$

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In the special case $p_n = 1$ for all values of n, $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability. Also,if we take $k = 1$ and $p_n = 1/(n+1)$, then summability $\mid \bar{N}, p_n \mid_k$ is equivalent to the summability $\mid R, \log n, 1 \mid$. Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty. \tag{1.6}$$

If we take $\theta_n = \frac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $\mid C, 1 \mid_k$ summability. Furthermore, if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ (see [4]) summability. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{1.7}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{1.8}$$

2 Known result

Mohanty [11] has demonstrated that the summability $\mid R, \log n, 1 \mid$ of

$$\sum A_n(t)/\log(n+1), \tag{2.1}$$

at $t = x$, is a local property of the generating function of $\sum A_n(t)$. Later on Matsumoto [9] improved this result by replacing the series (2.1) by

$$\sum A_n(t)/\log\log(n+1)^{1+\epsilon}, \epsilon > 0. \tag{2.2}$$

Generalizing the above result Bhatt [1] proved the following theorem.

Theorem A. If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $\mid R, \log n, 1 \mid$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.

Also, Mishra [10] has proved the following most general theorem on this matter.

Theorem B. If (p_n) is a sequence such that

$$P_n = O(np_n) \tag{2.3}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{2.4}$$

then the summability $|\ \bar{N}, p_n\ |$ of the series

$$\sum_{n=1}^{\infty} A_n(t) \lambda_n P_n / n p_n \tag{2.5}$$

at a point can be ensured by local property, where (λ_n) is as in Theorem A. On the other hand Bor [3] has generalized Theorem B for $|\ \bar{N}, p_n\ |_k$ summability in the following form.

Theorem C. Let $k \geq 1$ and (p_n) be a sequence such that the conditions (2.3) and (2.4) of Theorem B are satisfied. Then the summability $|\ \bar{N}, p_n\ |_k$ of the series (2.5) at a point can be ensured by local property , where (λ_n) is as in Theorem A.

3 Main result

The aim of this paper is to generalize Theorem C for $|\ \bar{N}, p_n, \theta_n\ |_k$ summability. We shall prove the following theorem.

Theorem. Let $k \geq 1$ and (p_n) be a sequence such that the conditions (2.3)-(2.4) of Theorem B are satisfied. If (θ_n) is any sequence of positive constants such that

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{v} (\lambda_v)^k = O(1) \tag{3.1}$$

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v} \right)^{k-1} \Delta \lambda_v = O(1) \tag{3.2}$$

$$\sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{v} (\lambda_{v+1})^k = O(1) \tag{3.3}$$

and

$$\sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left\{ \left(\frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{P_v} \right\}, \tag{3.4}$$

then the summability $|\ \bar{N}, p_n, \theta_n\ |_k$ of the series (2.5) at a point can be ensured by local property, where (λ_n) is as in Theorem A.

It should be noted that if we take $\theta_n = \frac{P_n}{p_n}$, then we get Theorem C. In this case the conditions (3.1)-(3.3) are obvious and the condition (3.4) reduces to

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O\left(\frac{1}{P_v} \right),$$

which always holds.

We need the following lemmas for the proof of our theorem.

Lemma 1 ([10]). If the sequence (p_n) is such that the conditions (2.3) and (2.4) of Theorem B are satisfied, then

$$\Delta(P_n/np_n) = O(1/n). \tag{3.5}$$

Lemma 2 ([5]). If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then (λ_n) is non-negative and decreasing, and $n\Delta\lambda_n \to 0$ as $n \to \infty$.

Lemma 3. Let $k \geq 1$ .If (s_n) is bounded and all conditions of the Theorem are satisfied, then the series

$$\sum_{n=1}^{\infty} A_n \lambda_n P_n/np_n \tag{3.6}$$

is summable $\mid \bar{N}, p_n, \theta_n \mid_k$, where (λ_n) is as in Theorem A.

Remark. Since (λ_n) is a convex sequence, therefore $(\lambda_n)^k$ is also convex sequence and $\sum(1/n)(\lambda_n)^k < \infty$.

Proof of Lemma 3. Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series (3.6). Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r P_r/rp_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v P_v/vp_v.$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}P_v\frac{a_v\lambda_v}{vp_v}, \quad n \geq 1, \quad (P_{-1} = 0).$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_vP_vs_v\lambda_v\frac{1}{vp_v} \\ &+ \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vs_vP_v\Delta\lambda_v\frac{1}{vp_v} \\ &+ \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\lambda_{v+1}\Delta(P_v/vp_v)s_v + s_n\lambda_n\frac{1}{n} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say. \end{aligned}$$

To prove the lemma, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{3.7}$$

Now, applying Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,1} \mid^k &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \mid s_v \mid^k p_v \left(\frac{\lambda_v P_v}{v p_v}\right)^k \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} p_v \left(\frac{P_v}{p_v}\right)^k (\lambda_v)^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} (\lambda_v)^k \frac{1}{v^k} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{k-1} (\lambda_v)^k \frac{1}{v^k} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v} (\lambda_v)^k = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem. Since

$$\sum_{v=1}^{n-1} P_v \Delta\lambda_v \leq P_{n-1} \sum_{v=1}^{n-1} \Delta\lambda_v \Rightarrow \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v \leq \sum_{v=1}^{n-1} \Delta\lambda_v = O(1),$$

by Lemma 2, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,2} \mid^k &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{v p_v}\right)^k P_v \Delta\lambda_v \mid s_v \mid^k \\
&\times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{v p_v}\right)^k \frac{1}{v^k} P_v \Delta\lambda_v \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{v p_v}\right)^k \frac{1}{v^k} \Delta\lambda_v \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \Delta\lambda_v \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

in view of the hypotheses of the Theorem and Lemma 2.

Using the fact that $\Delta(P_v/vp_v) = O(1/v)$ by Lemma 1, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,3} \mid^k &= \sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \mid \sum_{v=1}^{n-1} P_v \lambda_{v+1} \Delta(P_v/vp_v) s_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \lambda_{v+1} \frac{1}{v} \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v (\lambda_{v+1})^k \frac{1}{v^k} \\
&\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v (\lambda_{v+1})^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} (\lambda_{v+1})^k \frac{1}{v^k} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{k-1} (\lambda_{v+1})^k \frac{1}{v^k} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{v} (\lambda_{v+1})^k = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem. Finally, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,4} \mid^k &= \sum_{n=1}^{m} \theta_n^{k-1} (\lambda_n)^k \frac{1}{n^k} \\
&= O(1) \sum_{n=1}^{m} \theta_n^{k-1} (\lambda_n)^k \mid s_n \mid^k \frac{1}{n^{k-1}} \frac{1}{n} \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{1}{n} (\lambda_n)^k \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

in view of the hypotheses of the Theorem. Therefore we get that

$$\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,r} \mid^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2, 3, 4.$$

which completes the proof of the Lemma 3.

Remark. If we take $k = 1$, then we get a result due to Mishra [10].

4 Proof of the Theorem

Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighborhood of this point only, hence the truth of the Theorem is necessary consequence of Lemma 3.

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