Bulletin of Mathematical analysis and Applications
ISSN:1821-1291, URL: http://www.bmathaa.org
Volume 1, Issue 1, (2009) Pages 1-14

Some applications of differential subordination to a general class of multivalently analytic functions involving a convolution structure *

J. K. Prajapat & R. K. Raina

Abstract

By appealing to the familiar convolution structure of analytic functions, we investigate various useful properties and characteristics of a general class of multivalently analytic functions using the techniques of differential subordination. Several results are presented exhibiting relevant connections with some of the results presented here and those obtained in earlier works.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $\mathbb{U} = \{z;\ z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{P}(\gamma)$ the class of functions $\phi(z)$ of the form

$$\phi(z) = 1 + c_1 z + c_2 z^2 + ... \tag{1.2}$$

which are analytic in $\mathbb{U}$ and satisfy the following inequality:

$$\Re(\phi(z)) > \gamma \qquad (0 \leq \gamma < 1;\ z \in \mathbb{U}).$$

Let the functions f and g be analytic in $\mathbb{U}$, then we say that f is subordinate to g in $\mathbb{U}$, and write $f \prec g$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ such that $|w(z)| < 1,\ z \in \mathbb{U}$, and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. In particular,

* *Mathematics Subject Classifications:* Primary 30C45; Secondary 26A33, 33C20.
Key words: Multivalently analytic functions, Hadamard product (or convolution), Differential subordination, Hypergeometric functions, Fractional Differintegral operator, Ruscheweyh derivative operator.

Submitted: November 2008, Published January 2009

if f is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

If $f \in \mathcal{A}_p$ is given by (1.1) and $g \in \mathcal{A}_p$ is given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{1.3}$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{1.4}$$

For a given function $g(z) \in \mathcal{A}_p$ (defined by (1.3)), we define here a class $\mathcal{J}_p(g; \alpha, A, B)$ of functions belonging to the subclasses of $\mathcal{A}_p$ which consist of functions $f(z)$ of the form (1.1) and satisfying the following subordination:

$$(1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{(f*g)'(z)}{z^{p-1}} \prec \frac{1+Az}{1+Bz} \tag{1.5}$$

$$(z \in \mathbb{U};\ \alpha > 0;\ -1 \leq B < A \leq 1).$$

Since

$$\Re\left(\frac{1+Az}{1+Bz}\right) > \frac{1-A}{1-B} \qquad (z \in \mathbb{U};\ -1 \leq B < A \leq 1), \tag{1.6}$$

therefore, by choosing

$$A = 1 - \frac{2\beta}{p} \quad (0 \leq \beta < p) \qquad \text{and} \qquad B = -1,$$

in subordination relation (1.5), we obtain the following inequality

$$\Re\left((1-\alpha)\,\frac{(f*g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{(f*g)'(z)}{z^{p-1}}\right) > \frac{\beta}{p} \qquad (z \in \mathbb{U};\ \alpha > 0;\ 0 \leq \beta < p). \tag{1.7}$$

We further let $\mathcal{J}_p^*(g; \alpha, \beta)$ denote the class of functions $f \in \mathcal{A}_p$ which satisfies the inequality (1.7).

Our aim in this paper is to investigate various useful and interesting properties of the function classes $\mathcal{J}_p(g; \alpha, A, B)$ and $\mathcal{J}_p^*(g; \alpha, \beta)$ (defined above) involving the Hadamard product of two multivalently analytic functions and the pricipal of subordination. Several corollaries are deduced from the main results and their connections with known results are also pointed out. The concluding section exhibits relevant connections to some of the results presented here and those obtained in earlier works.

2 Preliminaries and Key Lemmas

We require the following lemmas to investigate the function class $\mathcal{J}_p(g; \alpha, A, B)$ (defined above).

Lemma 1 ([5, p. 71]). *Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0)=1$, and let the function $\phi(z)$ of the form (1.2) be analytic in $\mathbb{U}$. If*

$$\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z) \qquad (\Re(\gamma)\geq 0 \ \ (\gamma\neq 0);\ z\in\mathbb{U}), \tag{2.1}$$

then

$$\phi(z)\prec\psi(z):=\frac{\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}\, h(t)\, dt\prec h(z) \qquad (z\in\mathbb{U}), \tag{2.2}$$

and $\psi(z)$ is the best dominant.

Lemma 2([14, Lemma 1]). *Let $\phi(z)$ of the form (1.2) be analytic in $\mathbb{U}$. If*

$$\Re(\phi(z)+\eta\, z\phi'(z))>\beta \qquad (\Re(\eta)\geq 0 \ \ (\eta\neq 0);\ \beta<1; z\in\mathbb{U}),$$

then

$$\Re\left(\phi(z)\right)>\beta+(1-\beta)(2\,\Upsilon-1),$$

where Υ is given by

$$\Upsilon=\Upsilon_{\Re(\eta)}=\int_0^1\left(1+t^{\,\Re(\eta)}\right)^{-1}\, dt.$$

Lemma 3 ([13]). *If $\phi(z)\in\mathcal{P}(\gamma)$ $(0\leq\gamma<1)$, then*

$$\Re(\phi(z))\ \geq 2\gamma-1+\frac{2(1-\gamma)}{1+|z|} \qquad (z\in\mathbb{U}).$$

Lemma 4 ([20], see also [17, Lemma 3]). *For $0\leq\gamma_1,\ \gamma_2<1$,*

$$\mathcal{P}(\gamma_1)*\mathcal{P}(\gamma_2)\ \subset\ \mathcal{P}(\gamma_3) \qquad (\gamma_3=1-2(1-\gamma_1)(1-\gamma_2)).$$

The result γ_3 is the best possible.

Lemma 5 ([5, p. 35]). *Suppose that the function $\psi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ satisfies the condition*

$$\Re(\psi(ix,y;z))\leq\varepsilon$$

for some $\varepsilon>0$, real x, $y\leq-(1+x^2)/2$ for all $z\in\mathbb{U}$. If $\phi(z)$ given by (1.2) is analytic in $\mathbb{U}$ and

$$\Re(\psi(\phi(z),z\phi'(z);z))>\varepsilon,$$

then

$$\Re(\phi(z))>0 \qquad (z\in\mathbb{U}).$$

The generalized hypergeometric function ${}_pF_q$ is defined by(cf., e. g. [16, p. 333])

$${}_pF_q(z)\ \equiv\ {}_pF_q(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s;\ z)$$

$$=: \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{z^n}{n!} \tag{2.3}$$

$(z \in \mathbb{U};\ \alpha_j \in \mathbb{C}(j = 1, ..., q),\ \ \beta_j \in \mathbb{C}\backslash\{0, -1, -2, ...\}(j = 1, ..., s),\ q \leq s+1;\ q, s \in \mathbb{N}_0)$,

where the symbol $(\alpha)_k$ is the familiar Pochhammer symbol defined by

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1);\ k \in \mathbb{N}.$$

Each of the following identities (asserted by Lemma 6) below are quite well known (see [1, pp. 556-558]).

Lemma 6. *For real or complex numbers a, b and c $(c \neq 0, -1, -2, ...)$:*

$$\int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ (1-zt)^{-a}\ dt = \frac{\Gamma(b)\ \Gamma(c-b)}{\Gamma(c)}\ {}_2F_1\ (a, b;\ c; z) \qquad (\Re(c) > \Re(b) > 0); \tag{2.4}$$

$${}_2F_1\ (a, b;\ c; z) = (1-z)^{-a}\ {}_2F_1\ \left(a, c-b; c; \frac{z}{z-1}\right); \tag{2.5}$$

$$(b+1)\ {}_2F_1\ (1, b;\ b+1; z) = (b+1)\ +\ b\ z\ {}_2F_1\ (1, b+1;\ b+2;\ z) \tag{2.6}$$

3 Main Results

Our first main result is given by Theorem 1 below.

Theorem 1. *Let* $f(z) \in \mathcal{A}_p$,

$$\mathcal{Q}_0 f(z) \ = \ \frac{f(z)}{z^p} \qquad\qquad (z \in \mathbb{U}), \tag{3.1}$$

$$\mathcal{Q}_n^{\mu} f(z) \ = \ \frac{\mu+1}{z^{\mu+1}} \int_0^z t^{\mu} \left(\mathcal{Q}_{n-1}^{\mu} f(t)\right)\ dt \qquad (\mu > -1;\ n \in \mathbb{N}_0;\ z \in \mathbb{U}), \tag{3.2}$$

where $\mathcal{Q}_0^{\mu} f(z) = \mathcal{Q}_0 f(z)$. *If* $f(z) \in \mathcal{J}_p(g; \alpha, A, B)$, *then (for* $|z| = r < 1$*)*

$$\Re\left(\mathcal{Q}_n^{\mu}(f * g)(z)\right) \geq \rho_n(r) > \rho_n(1) \qquad (n \in \mathbb{N}), \tag{3.3}$$

where

$$0 < \rho_n(r) = 1 + (B-A)(\mu+1)^n p \sum_{k=1}^{\infty} \frac{B^{k-1}\ r^k}{(p+\alpha k)\ (k+\mu+1)^n} < 1. \tag{3.4}$$

The estimate (3.3) is sharp.

Proof. We shall prove this theorem by the principal of mathematical induction on n. Let $f(z) \in \mathcal{J}_p(g; \alpha, A, B)$, and assume that

$$\frac{(f * g)(z)}{z^p} = q(z). \tag{3.5}$$

It is clear that $q(z)$ is of the form (1.2) and is analytic in $\mathbb{U}$ with $q(0) = 1$. Differenting (3.5) with respect to z, and after some computation, we obtain

$$\begin{aligned}(1-\alpha)\ \frac{(f*g)(z)}{z^p}+\frac{\alpha}{p}\ \frac{(f*g)'(z)}{z^{p-1}} &= q(z)+\frac{\alpha}{p}\ z\,q'(z)\\ &\prec \frac{1+Az}{1+Bz} \qquad (z\in\mathbb{U}).\end{aligned}$$

Now, by using Lemma 1 for $\gamma = \frac{p}{\alpha}$, we deduce that

$$\frac{(f*g)(z)}{z^p} \prec \Omega(z) = \frac{p}{\alpha}\ z^{-\frac{p}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1}\ \frac{1+At}{1+Bt}\ dt. \tag{3.6}$$

Since $-1 \leq B < A \leq 1$ and $\alpha > 0$, it follows from (3.6) that for $|z| = r < 1:$

$$\begin{aligned}\Re(q(z)) &\geq \frac{p}{\alpha} \int_0^1 u^{\frac{p}{\alpha}-1}\ \Re\left(\frac{1+Auz}{1+Buz}\right)\ du\\ &\to \Omega(-r), \quad \text{as} \quad z\to -r.\end{aligned} \tag{3.7}$$

Using Lemma 6 in (3.7), we get

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) \geq \rho_0(r) = \begin{cases} {}_2F_1\left(1,\frac{p}{\alpha};\ \frac{p}{\alpha}+1;Br\right) - \frac{p}{p+\alpha}Ar\ {}_2F_1\left(1,\frac{p}{\alpha}+1;\ \frac{p}{\alpha}+2;Br\right) & (B\neq 0);\\ 1-\frac{p}{\alpha+p}\ Ar & (B=0)\ .\end{cases}$$

Simplifying right-hand side of the above estimate, we infer that

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) \geq \rho_0(r) = \begin{cases} 1+(B-A)p\sum_{k=1}^{\infty}\frac{B^{k-1}\ r^k}{p+\alpha k} & (B\neq 0);\\ 1-\frac{p}{\alpha+p}\ Ar & (B=0),\end{cases}$$

which implies that (3.3) holds true for $n = 0$.

For $n = 1$, we find that

$$\begin{aligned}\Re\left(\mathcal{Q}_1^{\mu}(f*g)(z)\right) &= \Re\left(\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu}\left(\frac{(f*g)(t)}{t^p}\right)\ dt\right)\\ &= \frac{\mu+1}{r^{\mu+1}}\int_0^r u^{\mu}\ \Re\left(\frac{(f*g)(ue^{i\theta})}{(ue^{i\theta})^p}\right)\ du\\ &\geq \frac{\mu+1}{r^{\mu+1}}\int_0^r u^{\mu}\ \left(1+(B-A)p\sum_{k=1}^{\infty}\frac{B^{k-1}\ u^k}{p+\alpha k}\right)\ du\\ &= 1+\frac{\mu+1}{r^{\mu+1}}(B-A)p\int_0^r\left(\sum_{k=1}^{\infty}\frac{B^{k-1}\ u^{\mu+k}}{p+\alpha k}\right)\ du.\end{aligned}$$

We note that for $|B| \leq 1$, $u < 1$ and $p + \alpha k \geq p + \alpha$ $(k \geq 1)$, the series in the right-hand side is uniformly convergent in $\mathbb{U}$, so that it can be integrated term by term. Thus, we have

$$\Re\left(\mathcal{Q}_1^{\mu}(f*g)(z)\right) \geq \rho_1(r) = 1+(B-A)(\mu+1)p\sum_{k=1}^{\infty}\frac{B^{k-1}\ r^k}{(p+\alpha k)\ (k+\mu+1)},$$

and this shows that (3.3) is also true for $n = 1$.

Next, we assume that (3.3) holds true for $n = m$. By letting $t = ue^{i\theta}$, we obtain

$$\begin{aligned}
\Re\left(\mathcal{Q}^{\mu}_{m+1}(f*g)(z)\right) &= \Re\left(\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu}\left(\mathcal{Q}^{\mu}_{m}(f*g)(t)\right)\,dt\right)\\
&= \frac{\mu+1}{r^{\mu+1}}\int_0^r u^{\mu}\,\Re\left(\mathcal{Q}^{\mu}_{m}(f*g)(ue^{i\theta})\right)\,du\\
&\geq \frac{\mu+1}{r^{\mu+1}}\int_0^r u^{\mu}\left(1+(B-A)(\mu+1)^m p\sum_{k=1}^{\infty}\frac{B^{k-1}\,u^k}{(p+\alpha k)\,(k+\mu+1)^m}\right)du\\
&= 1+\frac{\mu+1}{r^{\mu+1}}(B-A)(\mu+1)^{m+1}p\int_0^r\left(\sum_{k=1}^{\infty}\frac{B^{k-1}\,u^{\mu+k}}{(p+\alpha k)\,(k+\mu+1)^{m+1}}\right)du.
\end{aligned}$$

Noting that the integrand is uniformly convergent in $\mathbb{U}$, we deduce that

$$\Re\left(\mathcal{Q}^{\mu}_{m+1}(f*g)(z)\right)\geq\rho_{m+1}(r)=1+(B-A)(\mu+1)^{m+1}p\sum_{k=1}^{\infty}\frac{B^{k-1}\,r^k}{(p+\alpha k)\,(k+\mu+1)^{m+1}}.$$

Therefore, we conclude that

$$\Re\left(\mathcal{Q}^{\mu}_{n}(f*g)(z)\right)\geq\rho_n(r),$$

for any integer $n \in \mathbb{N}_0$.

Finally to prove the sharpness of the result (3.3), let us consider the function

$$\mathcal{G}_n(r)=1+(B-A)(\mu+1)^n p\sum_{k=1}^{\infty}\frac{B^{k-1}\,r^k}{(p+\alpha k)\,(k+\mu+1)^n}\qquad(0<r<1).$$

The series $\mathcal{G}_n(r)$ is absolutely and uniformly convergent for each $n \in \mathbb{N}_0$ and $0 < r < 1$. By suitably rearranging the terms of $\mathcal{G}_n(r)$, it is easy to see that $0 < \mathcal{G}_n(r) < 1$. Further since $\mathcal{G}_n(r) \leq \mathcal{G}_{n-1}(r)$ and

$$r^{\mu+1}\,\mathcal{G}_n(r)=(\mu+1)\int_0^r u^{\mu}\,\mathcal{G}_{n-1}(u)\,du\qquad(n\in\mathbb{N}),$$

we assert that $\mathcal{G}'_n(r) \leq 0$, and that $\mathcal{G}_n(r)$ decreases with r as $r \to 1^-$ for fixed n, and increases to 1 as $n \to \infty$ for fixed r. This implies that $\mathcal{G}_n(r) > \mathcal{G}_n(1)$. Therefore the estimate in (3.3) is sharp. This completes the proof of Theorem 1.

Setting

$$A=1-\frac{2\beta}{p}\quad(0\leq\beta<p)\qquad\text{and}\qquad B=-1$$

in Theorem 1, we obtain the following result (which also leads to the result of Owa and Patel [9, Corollary 3] when $\mu = 0$, $\alpha = 1$, $\beta = 0$ and $g(z) = \frac{z^p}{1-z}$):

Corollary 1. *If $f(z) \in \mathcal{J}_p^*(g;\alpha,\beta)$, then*

$$\Re\left(\mathcal{Q}_n^{\mu}(f*g)(z)\right) \geq \rho_n^*(r) > \rho_n^*(1) \qquad (n \in \mathbb{N}_0;\ |z| = r < 1),$$

where

$$0 < \rho_n^*(r) = 1 - 2(p-\beta)(\mu+1)^n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\ r^k}{(p+\alpha k)\ (k+\mu+1)^n} < 1.$$

The result is sharp.

Furthermore on putting

$$\alpha = n = 1 \qquad \text{and} \qquad g(z) = \frac{z^p}{1-z}$$

in Theorem 1, we get

Corollary 2. *If $f(z) \in \mathcal{A}_p$ such that*

$$\frac{f'(z)}{z^{p-1}} \prec p\,\frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

then

$$\Re\left(\frac{\mu+1}{z^{\mu+1}} \int_0^z t^{\mu-p}\ f(t)\ dt\right) \geq \rho \qquad (z \in \mathbb{U}),$$

where

$$\rho = 1 + (B-A)(\mu+1)p \sum_{k=1}^{\infty} \frac{(B)^{k-1}\ r^k}{(p+k)\ (k+\mu+1)}.$$

The result is sharp.

Theorem 2. *If $-1 \leq B_j < A_j \leq 1$ $(j = 1,2)$, and $f_j(z) \in \mathcal{J}_p(g;\alpha,A_j,B_j)$ $(j = 1,2)$, then*

$$\Xi(z) \ \in \ \mathcal{J}_p(g;\alpha,1-2\eta_0,-1), \tag{3.8}$$

where

$$\Xi(z) \ = \ ((f_1 * f_2) * g)(z) \tag{3.9}$$

and

$$\eta_0 = 1 + \frac{2(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left({}_2F_1\left(1,1;\ \frac{p}{\alpha}+1;\frac{1}{2}\right) - 2\right). \tag{3.10}$$

The result is sharp for $B_1 = B_2 = -1$.

Proof. Since $f_j(z) \in \mathcal{J}_p(g;\alpha,A_j,B_j)$, it follows that

$$\phi_i(z) \ = \ (1-\alpha)\,\frac{(f_i*g)(z)}{z^p} + \frac{\alpha}{p}\ \frac{(f_i*g)'(z)}{z^{p-1}},$$

and we can write that

$$\phi_i(z) \ \in \ \mathcal{P}(\gamma_i)$$

for $\gamma_i = (1-A_i)/(1-B_i)$ $(i=1,2)$. Integration by parts shows that

$$(f_i * g)(z) = \frac{p}{\alpha}\, z^{p-\frac{p}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1}\, \phi_i(t)\, dt \qquad (i=1,2), \tag{3.11}$$

and from (3.11), we can express

$$(\Xi * g)(z) \;=\; \frac{p}{\alpha}\, z^{p-\frac{p}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1}\, \phi_0(t)\, dt, \tag{3.12}$$

where (for convenience sake)

$$\begin{aligned}\phi_0(z) \;&=\; (1-\alpha)\,\frac{(\Xi * g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{(\Xi * g)'(z)}{z^{p-1}}\\ &= \frac{p}{\alpha}\, z^{-\frac{p}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1}\,(\phi_1 * \phi_2)(t)\, dt.\end{aligned} \tag{3.13}$$

Since $\phi_i(z) \in \mathcal{P}(\gamma_i)$ $(i=1,2)$, it follows from Lemma 4 that

$$(\phi_1 * \phi_2)(z) \in \mathcal{P}(\gamma_3) \;\;\text{for}\; \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2). \tag{3.14}$$

Using the substitution $t = uz$ in (3.13), we obtain the following equality:

$$\Re\{\phi_0(z)\} \;=\; \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\, \Re\{(\phi_1 * \phi_2)(uz)\}\, du.$$

In view of Lemma 3, the above last equality gives

$$\begin{aligned}\Re\{\phi_0(z)\} \;&\geq\; \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|}\right) du\\ &> \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right) du.\\ &= 1 + \frac{2(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left({}_2F_1\left(1,1;\,\frac{p}{\alpha}+1;\frac{1}{2}\right) - 2\right).\\ &= \eta_0,\end{aligned}$$

and this establishes (3.8).

Now, to prove the sharpness of the result, we find that (3.11) for $B_1 = B_2 = -1$, gives

$$\frac{(f_i * g)(z)}{z^p} \;=\; \frac{p}{\alpha}\, z^{-\frac{p}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1}\,\frac{1+A_i t}{1-t}\, dt \qquad (i=1,2),$$

and noting that

$$\left(\frac{1+A_1 t}{1-t}\right) * \left(\frac{1+A_2 t}{1-t}\right) = 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)'}{1-t},$$

it then follows that

$$\phi_0(z) \;=\; \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\left(1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{1-uz}\right)du.$$

If we choose z on the real axis, and let $z \to 1^-$, we obtain

$$\phi_0(z) \;\to\; 1+\frac{(A_1+1)(A_2+1)}{2}\left({}_2F_1\left(1,1;\;\frac{p}{\alpha}+1;\frac{1}{2}\right)-2\right),$$

which proves Theorem 2.

Theorem 3. *If* $f_i(z)\in\mathcal{A}_p$ $(j=1,2)$ *such that*

$$(1-\alpha)\frac{(f_i*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(f_i*g)'(z)}{z^{p-1}} \;\in\; \mathcal{P}(\eta_i) \qquad (0\le\eta_i<1;\;\; i=1,2), \quad (3.15)$$

and the function Ξ *is defined by (3.9), then*

$$\Re\left(\frac{z\,(\Xi*g)'(z)}{(\Xi*g)(z)}\right) \;>\; p-\frac{p}{\alpha}, \qquad (3.16)$$

provided that

$$(1-\eta_1)(1-\eta_2) \;<\; \frac{\alpha+2p}{2\left(\alpha\left\{{}_2F_1\left(1,1;\frac{p}{\alpha}+1;\frac{1}{2}\right)-2\right\}^2+2p\right)}. \qquad (3.17)$$

Proof. By hypothesis and Lemma 4, it follows that

$$\Re\left[\left((1-\alpha)\frac{(\Xi*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(\Xi*g)'(z)}{z^{p-1}}\right)+\frac{\alpha}{p}\,z\,\left((1-\alpha)\frac{(\Xi*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(\Xi*g)'(z)}{z^{p-1}}\right)'\right]$$
$$=\Re\left[\left((1-\alpha)\frac{(f_1*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(f_1*g)'(z)}{z^{p-1}}\right)*\left((1-\alpha)\frac{(f_2*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(f_2*g)'(z)}{z^{p-1}}\right)\right]$$
$$>\;1-2(1-\eta_1)(1-\eta_2), \qquad (3.18)$$

which in view of Lemma 2 (for $\eta=\frac{\alpha}{p}$ and $\beta=1-2(1-\eta_1)(1-\eta_2)$) yields

$$\Re\left((1-\alpha)\frac{(\Xi*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(\Xi*g)'(z)}{z^{p-1}}\right)>1+2(1-\eta_1)(1-\eta_2)\left({}_2F_1\left(1,1;\frac{p}{\alpha}+1;\frac{1}{2}\right)-2\right). \qquad (3.19)$$

Again from (3.19) and Lemma 2, we have

$$\Re\left(\frac{(\Xi*g)(z)}{z^p}\right)>1-2(1-\eta_1)(1-\eta_2)\left({}_2F_1\left(1,1;\frac{p}{\alpha}+1;\frac{1}{2}\right)-2\right)^2.$$

Now let

$$p(z)=\frac{\alpha}{p}\,\frac{z\,(\Xi*g)'(z)}{(\Xi*g)(z)}-\alpha+1 \qquad \text{and} \qquad q(z)=\frac{(\Xi*g)(z)}{z^p}.$$

Elementary computations shows that

$$\left((1-\alpha)\frac{(\Xi*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(\Xi*g)'(z)}{z^{p-1}}\right)+\frac{\alpha}{p}\ z\ \left((1-\alpha)\frac{(\Xi*g)(z)}{z^p}+\frac{\alpha}{p}\frac{(\Xi*g)'(z)}{z^{p-1}}\right)'$$

$$= q(z)\ \left(p^2(z)+\frac{\alpha}{p}\ z\ p'(z)\right)$$

$$= \psi(p(z), zp'(z); z), \tag{3.20}$$

where $\psi(u,v;z)=q(z)\left(u^2+\frac{\alpha}{p}v\right)$. Thus, by using (3.18) in (3.20), we get

$$\Re\left(\psi(p(z), zp'(z); z)\right)\ >\ 1-2(1-\eta_1)(1-\eta_2) \qquad (z\in\mathbb{U}),$$

and for all $x,\ y\leq -\frac{1}{2}(1+x^2)$, we infer that

$$\begin{aligned}\Re\{\psi(ix,y,z)\} &= \Re(q(z))\left(\frac{\alpha}{p}\ y-x^2\right)\\ &\leq \frac{1}{2p}\ \Re(q(z))(\alpha+(\alpha+2p)x^2)\\ &\leq -\frac{\alpha}{2p}\ \Re(q(z))\ \leq\ 1-2(1-\eta_1)(1-\eta_2) \qquad (z\in\mathbb{U}).\end{aligned}$$

Now applying Lemma 5, we get $\Re(p(z))>0$ in $\mathbb{U}$ which implies that

$$\Re\left(\frac{z\ (\Xi*g)'(z)}{(\Xi*g)(z)}\right)\ >\ p-\frac{p}{\alpha} \qquad (z\in\mathbb{U}),$$

and this complete the proof of Theorem 3.

Setting

$$\alpha=1 \qquad \text{and} \qquad g(z)=\frac{z^p}{1-z}$$

in Theorem 3, we get the following Corollary which in turn yields the corresponding result of Lashin [4, Theorem 1] (for $p=1$).

Corollary 3. *If* $f_i(z)\in\mathcal{A}_p$ $(i=1,2)$ *and the function* $f_i'(z)/pz^{p-1}\in\mathcal{P}(\eta_i)$ $(0\leq\eta_i<1\ (i=1,2))$, *then*

$$\Re\left(\frac{z\ (f_1*f_2)'(z)}{(f_1*f_2)(z)}\right)\ >\ 0 \qquad (z\in\mathbb{U}),$$

provided that

$$(1-\eta_1)(1-\eta_2)\ <\ \frac{2p+1}{2\left(\alpha\left\{{}_2F_1\left(1,1;p+1;\frac{1}{2}\right)-2\right\}^2+2p\right)}.$$

4 Applications and observations

In view of the subordination (1.5) which is expressed in terms of the convolution of the function (1.3) with the function defined by (1.1) (involving the arbitrary sequence b_k), and appropriately selecting the sequence b_k, the results presented in this paper would lead further to various new or known results. For example, if the sequence b_k in (1.3) and the value of parameter α in (1.5) are, respectively, chosen as follows:

$$b_k = \frac{\Gamma(k+1)\,\Gamma(p+1-\lambda)}{\Gamma(p+1)\,\Gamma(k+1-\lambda)} \text{ and } \alpha = \frac{\delta p}{p-\lambda} \; (-\infty < \lambda < p+1;\ \delta \geq 0;\ p \in \mathbb{N}), \tag{4.1}$$

and in the process making use of the identity [12, p.112, Eq. (1.10)]:

$$z\,(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda)\,\Omega_z^{(\lambda+1,p)} f(z) + \lambda\,\Omega_z^{(\lambda,p)} f(z) \qquad (z \in \mathbb{U}) \tag{4.2}$$

in (1.3), then Theorems 2 and 3 correspond to the results given recently by Patel and Mishra [12, p. 116, Theorems 1.11 and 1.12]. The operator $\Omega_z^{(\lambda,p)}$ used in (4.2) is the extended fractional differintegral operator of order λ $(-\infty < \lambda < p+1)$ ([12]) defined by

$$\Omega_z^{(\lambda,p)} f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\Gamma(n+1)\,\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+1-\lambda)}\, a_n\, z^p$$

$$= \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}\, z^{\lambda}\, D_z^{\lambda} f(z) \qquad (z \in \mathbb{U}),$$

where D_z^{λ} represents, respectively, the fractional integral of $f(z)$ of order $-\lambda$ $(-\infty < \lambda < 0)$, and the fractional derivative of $f(z)$ of order λ when $(0 < \lambda < p+1)$ (Owa [8]).

On the other hand, if we set the coefficient b_k in (1.3) and the value of parameter α in (1.5), respectively, as follows:

$$b_k = \frac{(\lambda+p)_{k-p}}{(k-p)!} \qquad \text{and} \qquad \alpha = \frac{\delta p}{\lambda+p} \qquad (\lambda > -p;\ \delta > 0;\ p \in \mathbb{N}), \tag{4.3}$$

and in the process use the identity [3, p. 124, Eq.(4)]:

$$z\left(D^{\lambda+p-1} f(z)\right)' = (\lambda+p)\, D^{\lambda+p} f(z) - \lambda\, D^{\lambda+p-1} f(z) \qquad (\lambda > -p;\ p \in \mathbb{N};\ z \in \mathbb{U}) \tag{4.4}$$

in (1.3), then Theorem 2 corresponds to the known result of Dinggong and Liu [3, p. 129, Theorem 4]. The operator $D^{\lambda+p-1}$ used in (4.4) is the generalized Ruscheweyh derivative which is defined by ([3, p. 123, Eq. (1)])

$$D^{\lambda+p-1} f(z) = \frac{z^p}{(1-z)^{\lambda+p}} * f(z) \qquad (\lambda > -p;\ f \in \mathcal{A}_p).$$

Furthermore, if we choose the coefficients b_k in (1.3) and the value of the parameter α in (1.5), respectively, as follows:

$$b_k = \left(\frac{p+1}{k+1}\right)^{\sigma} \quad \text{and} \quad \alpha = \frac{\lambda}{(p+1)} \quad (\sigma > 0;\ \lambda \geq 0;\ p \in \mathbb{N}), \tag{4.5}$$

and apply the following identity([10, p. 3, Eq. (1.17)]):

$$z\left(I^{\sigma} f(z)\right)' = (p+1) I^{\sigma-1} f(z) - I^{\sigma} f(z) \qquad (p \in \mathbb{N};\ \sigma > 0) \tag{4.6}$$

in (1.3), then the result contained in Theorem 2 also yields a recently established result due to Ozkan [10, p. 4, Theorem 2.4], where the operator $I^{\sigma} f(z)$ in (4.6) is defined by ([10])

$$\begin{aligned} I^{\sigma} f(z) &= \frac{(p+1)^{\sigma}}{z\,\Gamma(\sigma)} \int_0^z \left(log\ \frac{z}{t}\right)^{\sigma-1} f(t)dt \\ &= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+1}{k+1}\right)^{\sigma} a_k z^k \qquad (f \in \mathcal{A}_p;\ p \in \mathbb{N};\ \sigma > 0). \end{aligned}$$

We conclude this paper by observing that on specializing the coefficients b_k in (1.3) and the parameter α in (1.5) suitably, one may also obtain the results given by Ding *et al.* [2], Lashian [4], Obradovic [6], Owa [7], Owa and Patel [9], Patel *et al.* [11], Sham *et al.* [15] and Srivastava *et al.* ([18], [19]).

Acknowledgements. The authors express their sincerest thanks to the referees for some useful suggestions.

References

[1] M. Abramowitz and I. A. Stegun (Editors), Handbook of Mathematical Functions and Formulas, Graphs and Mathematical Tables, Dover Publications, New York, 1971.

[2] S. Ding, Y. Ling and G. Bao, Some properties of a class of analytic functions, *J. Math. Anal. Appl.*, **195**, (1995), 71-81.

[3] Y. Dinggong and J. L. Liu, On a class of analytic functions involving Ruscheweyh derivatives, *Bull. Korean Math. Soc.*, **39(1)**, (2002), 123-131.

[4] A. Y. Lashin, Some convolution properties of analytic functions, *Appl. Math. Lett.* **18**, (2005), 135-138.

[5] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York, 2000.

[6] M. Obradovic, On certain inequalities for some regular functions in $|z|$, *Int. J. Math. & Math. Sci.*, **8**, (1985), 671-681.

[7] S. Owa, Certain integral operators, *Proc. Japan Acad.*, **67**, (1991), 88-93.

[8] S. Owa, On the distortion theorems I, *Kyungpook Math. J.*, **18**, (1978), 53-59.

[9] S. Owa and J. Patel, Properties of certain integral operators, *Southeast Asian Bull. Math.*, **24**, (2000), 411-419.

[10] O. Ozkan, Some subordination results on multivalent functions defined by integral operator, *J. Inequal. Appl.*, **Volume 2007**, (2007), Article ID 71616, 1-8 (electronic).

[11] J. Patel, N. E. Cho and H. M. Srivastava, Certain subclasses of multivalent function associated with a family of linear operators, *Math. Comput. Modelling*, **43**, (2006), 320-328.

[12] J. Patel and A. K. Mishra, On certain subclasses of multivalent function associated with an extended fractional differintegral operator, *J. Math. Anal. Appl.*, **332**, (2007),109-122.

[13] D. Z. Pashkouleva, The starlikeness and spiral-convexity of certain subclasses of analytic functions, in: H. M. Srivastava and S. Owa (Editors), Current Topics in Anaytic Function Theory, pp. 266-273, World Scientific Publishing Company, Singapore, New Jersey, London and Hongkong, 1992.

[14] S. Poonusamy, Differential subordination and Bazilevic functions, *Proc. Indian Acad. Sci. (Math. Sci.),* **105(2)**, (1995), 169-186.

[15] S. Sham, S. R. Kulkurni and J. M. Jahangiri, Subordination properties of p-valent functions defined by integral operators, *Internat. J. Math. Math. Sci.*, **Volume 2006**, (2006), Article ID 94572, 1-3 (electronic).

[16] H. M. Srivastava and S. Owa (Eds.), Univalent functions, Fractional calculus, and Their Applications, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[17] H. M. Srivastava and J. Patel, Applications of differential subordination to certain subclasses of meromorphically multivalent functions, *J. Inequal. Pure Appl. Math.*, **6(3)**, (2005), Article 88, 15 pp.

[18] H. M. Srivastava, J. Patel and G. P. Mohapatra, Some applications of differential subordination to a general class of multivalent functions, *Adv. Stud. Contemp. Math.*, **3(1)**, (2001), 1-15.

[19] H. M. Srivastava, J. Patel and G. P. Mohapatra, A certain class of p-valently analytic functions, *Math. Comput. Modelling*, **41**, (2005), 321-324.

[20] J. Stankiewicz and Z. Stankiewicz, Some applications of Hadamard convolution in the theory of functions, *Ann. Univ. Mariae Curie-Sklodowska Sect.*, **A 40**, (1986), 251-265.

J. K. Prajapat
Departament of Mathematics
Bhartiya Institute of Engineering and Technology
Near Sanwali By-Pass Circle
Sikar-332021, Rajasthan, India.
email: jkp_0007@rediffmail.com

R. K. Raina
10/11 Ganpati Vihar, Opposite Sector 5,
Udaipur 313002, Rajasthan, India
email: rkraina_7@hotmail.com