Seiberg$-$Witten equations on 8$-$dimensional manifolds with different self$-$duality

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Abstract. Seiberg$-$Witten equations, which are defined on any 4$-$manifolds, consist of two equations [3, 6, 7]. The first of these equations is called Dirac equation and the latter curvature equation. In higher dimensions, generalized self$-$duality is used to describe Seiberg$-$Witten equations [1, 6, 4]. In this paper, Seiberg$-$Witten equations were obtained by choosing different self$-$duality and by means of $Spin^c-$structure which was given in [5]. Then, non$-$trivial solutions are given by choosing different self$-$duality.

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Key words: Seiberg$-$Witten equations; spinor; Dirac operator; self-duality.

1 Introduction

On 4$-$dimensional manifolds, the self$-$duality of 2$-$forms in the sense of Hodge is meaningful. But, if the dimension is not 4, self$-$duality of 2$-$forms in the sense of Hodge is meaningless. Consequently, there is no natural generalization of the curvature equation. The purpose of this paper is to determine the Seiberg$-$Witten equation on 8$-$dimensional manifolds with different self$-$duality and to give non$-$trivial solutions to the Seiberg$-$Witten equations.

This paper is organized as follows: we begin with a section introducing some basic definitions and notations. In the following section, some basic facts concerning the Seiberg$-$Witten equations on 4$-$dimensional manifolds are given. Then, on 8$-$dimensional manifolds, the Seiberg$-$Witten equations are written, depending on a different self$-$duality. Finally, non$-$trivial solutions for these equations are given.

2 Definitions and notations

Definition 2.1. A complex vector bundle S, called spinor bundle, can be constructed by making use of a given $Spin^c$ representation, $\kappa_n : Spin^c \to Aut(\Delta_n)$. The sections

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of this complex vector bundle are called spinor fields on M. Spinor bundles split into two pieces $S = S^+ \otimes S^-$ in case of the dimension of M is even [3]. For a given linear map $\kappa_n : \mathbb{R}^n \to End(S)$, which meet the following conditions:

$$\kappa_n(v)^* + \kappa_n(v) = 0, \qquad \kappa_n(v)^*\kappa_n(v) = |v|^2\mathbb{I}$$

for every $v \in \mathbb{R}^n$,

$$\begin{aligned} \rho : \Lambda^2(T^*M) &\rightarrow End(S) \\ \eta = \sum_{i<j} \eta_{ij} e^i \wedge e^j &\rightarrow \rho(\eta) = \sum_{i<j} \eta_{ij}\kappa(e_i)\kappa(e_j) \end{aligned}$$

can be defined on the orthonormal frame $\{e_1, e_2, ..., e_n\}$ by extending the map $\kappa : TM \to End(S)$ of κ_n.

We note that ρ can be extended to complex valued $2-$forms [6], such that

$$\rho : \Lambda^2(T^*M) \otimes \mathbb{C} \to End(S).$$

The half$-$spinor bundle $S^\pm$ are invariant under $\rho(\eta)$ for all $\eta \in \Lambda^2\big(T^*(M) \otimes \mathbb{C}\big)$. Then $\rho^+(\eta) = \rho(\eta)\Big|_{S^+}$, and $\rho^-(\eta) = \rho(\eta)\Big|_{S^-}$ can be defined.

By means of a spinor covariant derivative operator ∇^A on S, the definition of Dirac operator $D_A : \Gamma(S^+) \to \Gamma(S^-)$ can be given as

$$D_A(\Psi) = \sum_{i=1}^{n} \kappa(e_i)\nabla^A_{e_i}\Psi,$$

for any local orthonormal frame $\{e_1, e_2, ..., e_n\}$ of TM.

3 Seiberg-Witten equations on $4-$manifolds

The Seiberg$-$Witten equations on $4-$dimensional $Spin^c$ manifold can be expressed as follows:

1. $D_A\Psi = 0$,
2. $\rho^+\big(F_A\big) = \big(\Psi\Psi^*\big)_0$,

where $F_A \in \Omega^2(M, i\mathbb{R}), \Psi \in \Gamma(S^+)$ and $\big(\Psi\Psi^*\big)_0$ are the tracefree parts of $\Psi.\Psi^*$ [6]. The first part of these equations is called Dirac equation and the other one is called curvature equation. For $4-$dimensional manifolds, Seiberg$-$Witten equations are examined in [3, 6, 7].

In the following section, Seiberg$-$Witten equations are constructed on $8-$dimensional manifolds by choosing a self$-$duality which is different from the self$-$duality concept given in [1, 5]. Then solutions to these equations are given.

4 Seiberg-Witten equations on 8−manifolds

On an 8−dimensional manifold M with structure group $Spin(7)$, there exists a fundamental 4−form Φ, which is nonzero everywhere. By means of this 4−form, $\Omega^2(M)$ splits up into two parts as:

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{21}(M),$$

where

$$\Omega^2_7(M) = \{\omega \in \Omega^2(M) | *(\Phi \wedge \omega) = 3\omega\}$$

and

$$\Omega^2_{21}(M) = \{\omega \in \Omega^2(M) | *(\Phi \wedge \omega) = -\omega\}.$$

In this paper $\Omega^2_{21}(M)$ is considered as the space of self− dual 2−forms. Also this decomposition can be extended to $i\mathbb{R}$ valued 2−form as follows:

$$\Omega^2(M, i\mathbb{R}) = \Omega^{2,+}(M, i\mathbb{R}) \oplus \Omega^{2,-}(M, i\mathbb{R}).$$

If $F_A \in \Omega^2(M, i\mathbb{R})$, then it can be written as $F_A = F_A^+ + F_A^-$, where $F_A^+ \in \Omega^2_{21}(M, i\mathbb{R})$ and $F_A^- \in \Omega^2_7(M, i\mathbb{R})$. The explicit form of F_A^+ is

$$F_A^+ \quad = \quad Proj_{\Omega^2_{21}(M,i\mathbb{R})} F_A.$$

Definition 4.1. Let $\kappa : TM \to End(S)$ be a Spin^c−structure on n−dimensional orientable M manifold. In this case, σ is defined as follows:

$$\begin{array}{rcl} \sigma : \Gamma(S) & \to & \Omega^2(M, i\mathbb{R}), \\ \Psi & \mapsto & \sigma(\Psi), \end{array}$$

$\forall X, Y \in \chi(M)$, $\sigma(\Psi)(X, Y) = \langle X, Y \rangle |\Psi|^2 + \langle \kappa(X)\kappa(Y)\Psi, \Psi \rangle$ is $i\mathbb{R}$ valued 2−form [3].

Then the Seiberg−Witten equations on 8−dimensional (M, g, Φ) $Spin^c$ manifolds with $Spin(7)$−structure are given as follows:

1. $D_A^+(\Psi) = 0,$
2. $F_A^+ = \frac{1}{8}\sigma(\Psi)^+,$

where F_A^+ is the self−dual part of F_A and $\sigma(\Psi)^+$ is the projection of $\sigma(\Psi)$ onto $\Omega^2_{21}(M, i\mathbb{R})$.

4.1 Some local discussions

Let M be a $Spin^c$ manifold endowed with $Spin^c$−structure which was given in [5] and let $A = \sum\limits_{i=1}^{8} A_i dx^i$ be the connection 1−form on P_{S^1}. If so, its curvature 2−form is $F_A = dA = \sum\limits_{i<j} F_{ij} dx^i \wedge dx^j \in \Omega^2(M, \mathrm{i}\mathbb{R})$, where $F_{ij} = \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)$, $1 \le i < j \le 8$.

The covariant derivative $\nabla^A\Psi$ of a spinor $\Psi \in \Gamma(S^+)$ is computed according to this formula

$$\nabla^A\Psi = d\Psi + \tfrac{1}{2}\sum_{i<j}\omega_{ij}e_ie_j\Psi + \tfrac{1}{2}A\Psi,$$

where $\omega_{ij} = g(\nabla_{e_i}, e_j)$ are forms defining the Levi−Civita connection. ω_{ij} are vanished in the case of $M = \mathbb{R}^8$. Then the local expression of the covariant derivative $\nabla^A_{e_i}\Psi$ of Ψ in the direction e_i is

$$\nabla^A_{e_i}\Psi \quad = \quad \Big(d\Psi + \tfrac{1}{2}A\Psi\Big)(e_i) \quad = \quad d\Psi(e_i) + \tfrac{1}{2}A\Psi(e_i).$$

Since

$$d\Psi(e_i) = \begin{bmatrix} d\psi_1 \\ d\psi_2 \\ \vdots \\ d\psi_8 \end{bmatrix}(e_i) \quad = \quad \begin{bmatrix} d\psi_1(e_i) \\ d\psi_2(e_i) \\ \vdots \\ d\psi_8(e_i) \end{bmatrix} = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_i} \\ \frac{\partial\psi_2}{\partial x_i} \\ \vdots \\ \frac{\partial\psi_8}{\partial x_i} \end{bmatrix}$$

and $\frac{1}{2}A\Psi(e_i) = \frac{1}{2}\sum_{i=1}^{8} A_i dx^i(e_i)\Psi = \frac{1}{2}A_i\Psi$, then

$$\nabla^A_{e_i}\Psi = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_i} \\ \frac{\partial\psi_2}{\partial x_i} \\ \vdots \\ \frac{\partial\psi_8}{\partial x_i} \end{bmatrix} + \tfrac{1}{2}A_i \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_8 \end{bmatrix} \quad = \quad \begin{bmatrix} \frac{\partial\psi_1}{\partial x_i} + \frac{1}{2}A_i\psi_1 \\ \frac{\partial\psi_2}{\partial x_i} + \frac{1}{2}A_i\psi_2 \\ \vdots \\ \frac{\partial\psi_8}{\partial x_i} + \frac{1}{2}A_i\psi_8 \end{bmatrix}.$$

According to the above data the explicit form of the first equation $D^+_A\Psi = 0$ is

$$\begin{aligned}
&-\tfrac{\partial}{\partial x_1}\psi_1 + \tfrac{\partial}{\partial x_3}\psi_2 - \tfrac{\partial}{\partial x_5}\psi_4 + \tfrac{\partial}{\partial x_7}\psi_8 + i\big(\tfrac{\partial}{\partial x_2}\psi_1 - \tfrac{\partial}{\partial x_4}\psi_2 + \tfrac{\partial}{\partial x_6}\psi_4 + \tfrac{\partial}{\partial x_8}\psi_8\big) \\
&+\tfrac{1}{2}\Big(-\psi_1A_1 + \psi_2A_3 - \psi_4A_5 + \psi_8A_7 + i\big(\psi_1A_2 - \psi_2A_4 + \psi_4A_6 + \psi_8A_8\big)\Big) = 0 \\[1ex]
&-\tfrac{\partial}{\partial x_3}\psi_1 - \tfrac{\partial}{\partial x_1}\psi_2 - \tfrac{\partial}{\partial x_5}\psi_3 + \tfrac{\partial}{\partial x_7}\psi_7 + i\big(-\tfrac{\partial}{\partial x_4}\psi_1 - \tfrac{\partial}{\partial x_2}\psi_2 + \tfrac{\partial}{\partial x_6}\psi_3 + \tfrac{\partial}{\partial x_8}\psi_7\big) \\
&+\tfrac{1}{2}\Big(-\psi_2A_1 - \psi_1A_3 - \psi_3A_5 + \psi_7A_7 + i\big(\psi_2A_2 - \psi_1A_4 + \psi_3A_6 + \psi_7A_8\big)\Big) = 0 \\[1ex]
&\tfrac{\partial}{\partial x_5}\psi_2 - \tfrac{\partial}{\partial x_1}\psi_3 + \tfrac{\partial}{\partial x_3}\psi_4 + \tfrac{\partial}{\partial x_7}\psi_6 + i\big(\tfrac{\partial}{\partial x_6}\psi_2 + \tfrac{\partial}{\partial x_2}\psi_3 + \tfrac{\partial}{\partial x_4}\psi_4 + \tfrac{\partial}{\partial x_8}\psi_6\big) \\
&+\tfrac{1}{2}\Big(-\psi_3A_1 + \psi_4A_3 + \psi_2A_5 + \psi_6A_7 + i\big(\psi_3A_2 + \psi_4A_4 + \psi_2A_6 + \psi_6A_8\big)\Big) = 0 \\[1ex]
&\tfrac{\partial}{\partial x_5}\psi_1 - \tfrac{\partial}{\partial x_3}\psi_3 - \tfrac{\partial}{\partial x_1}\psi_4 + \tfrac{\partial}{\partial x_7}\psi_5 + i\big(\tfrac{\partial}{\partial x_6}\psi_1 + \tfrac{\partial}{\partial x_4}\psi_3 - \tfrac{\partial}{\partial x_2}\psi_4 + \tfrac{\partial}{\partial x_8}\psi_5\big) \\
&\tfrac{1}{2}\Big(-\psi_4A_1 - \psi_3A_3 + \psi_1A_5 + \psi_5A_7 + i\big(-\psi_4A_2 + \psi_3A_4 + \psi_1A_6 + \psi_5A_8\big)\Big) = 0 \\[1ex]
&-\tfrac{\partial}{\partial x_7}\psi_4 - \tfrac{\partial}{\partial x_1}\psi_5 + \tfrac{\partial}{\partial x_3}\psi_6 - \tfrac{\partial}{\partial x_5}\psi_8 + i\big(\tfrac{\partial}{\partial x_8}\psi_4 + \tfrac{\partial}{\partial x_2}\psi_5 - \tfrac{\partial}{\partial x_4}\psi_6 - \tfrac{\partial}{\partial x_6}\psi_8\big) \\
&+\tfrac{1}{2}\Big(-\psi_5A_1 + \psi_6A_3 - \psi_8A_5 - \psi_4A_7 + i\big(\psi_5A_2 - \psi_6A_4 - \psi_8A_6 + \psi_4A_8\big)\Big) = 0 \\[1ex]
&-\tfrac{\partial}{\partial x_7}\psi_3 - \tfrac{\partial}{\partial x_3}\psi_5 - \tfrac{\partial}{\partial x_1}\psi_6 - \tfrac{\partial}{\partial x_5}\psi_7 + i\big(\tfrac{\partial}{\partial x_8}\psi_3 - \tfrac{\partial}{\partial x_4}\psi_5 + \tfrac{\partial}{\partial x_2}\psi_6 - \tfrac{\partial}{\partial x_6}\psi_7\big) \\
&+\tfrac{1}{2}\Big(-\psi_6A_1 - \psi_5A_3 - \psi_7A_5 - \psi_3A_7 + i\big(-\psi_6A_2 - \psi_5A_4 - \psi_7A_6 + \psi_3A_8\big)\Big) = 0 \\[1ex]
&-\tfrac{\partial}{\partial x_7}\psi_2 + \tfrac{\partial}{\partial x_5}\psi_6 - \tfrac{\partial}{\partial x_1}\psi_7 + \tfrac{\partial}{\partial x_3}\psi_8 + i\big(\tfrac{\partial}{\partial x_8}\psi_2 - \tfrac{\partial}{\partial x_6}\psi_6 + \tfrac{\partial}{\partial x_2}\psi_7 + \tfrac{\partial}{\partial x_4}\psi_8\big) \\
&+\tfrac{1}{2}\Big(-\psi_7A_1 + \psi_8A_3 + \psi_6A_5 - \psi_2A_7 + i\big(\psi_7A_2 + \psi_8A_4 - \psi_6A_6 + \psi_2A_8\big)\Big) = 0 \\[1ex]
&-\tfrac{\partial}{\partial x_7}\psi_1 + \tfrac{\partial}{\partial x_5}\psi_5 - \tfrac{\partial}{\partial x_3}\psi_7 - \tfrac{\partial}{\partial x_8}\psi_1 + i\big(\tfrac{\partial}{\partial x_8}\psi_1 - \tfrac{\partial}{\partial x_6}\psi_5 + \tfrac{\partial}{\partial x_4}\psi_7 + \tfrac{\partial}{\partial x_2}\psi_8\big) \\
&+\tfrac{1}{2}\Big(-\psi_8A_1 - \psi_7A_3 - \psi_5A_5 - \psi_1A_7 + i\big(-\psi_8A_2 + \psi_7A_4 - \psi_5A_6 + \psi_1A_8\big)\Big) = 0
\end{aligned}$$

where $A = \sum_{i=1}^{8} A_i dx^i \in \Omega^1(M, i\mathbb{R})$ and $\Psi \in \Gamma(S^+)$.

In the following, by using orthogonal basis of $\Omega^2_{21}(M, i\mathbb{R})$, the explicit form of curvature equation is obtained. Then, the non−trivial solution to these equations is given.

4.1.1 Curvature Equation on $\mathbb{R}^8$

The second part of the Seiberg−Witten equation, which is called the curvature equation and denoted by $F_A^+ = \frac{1}{8}\sigma(\Psi)^+$, is described by the orthogonal basis of $\Omega^2_{21}\big(\mathbb{R}^8, i\mathbb{R}\big)$, which are given as follows [2]:

$$\begin{aligned}
g_1 &= dx_1 \wedge dx_5 - dx_2 \wedge dx_6 - dx_3 \wedge dx_7 + dx_4 \wedge dx_8\\
g_2 &= dx_1 \wedge dx_2 - dx_3 \wedge dx_4 + dx_5 \wedge dx_6 - dx_7 \wedge dx_8\\
g_3 &= dx_1 \wedge dx_6 + dx_2 \wedge dx_5 + dx_3 \wedge dx_8 + dx_4 \wedge dx_7\\
g_4 &= dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_7 + dx_6 \wedge dx_8\\
g_5 &= dx_1 \wedge dx_7 - dx_2 \wedge dx_8 + dx_3 \wedge dx_5 - dx_4 \wedge dx_6\\
g_6 &= dx_1 \wedge dx_4 - dx_2 \wedge dx_3 + dx_5 \wedge dx_8 - dx_6 \wedge dx_7\\
g_7 &= dx_1 \wedge dx_8 + dx_2 \wedge dx_7 - dx_3 \wedge dx_6 - dx_4 \wedge dx_5
\end{aligned}$$

$$\begin{aligned}
g_8 &= dx_1 \wedge dx_5 + dx_2 \wedge dx_6 - dx_3 \wedge dx_7 - dx_4 \wedge dx_8\\
g_9 &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + dx_7 \wedge dx_8\\
g_{10} &= dx_1 \wedge dx_6 - dx_2 \wedge dx_5 + dx_3 \wedge dx_8 - dx_4 \wedge dx_7\\
g_{11} &= dx_1 \wedge dx_3 - dx_2 \wedge dx_4 + dx_5 \wedge dx_7 - dx_6 \wedge dx_8\\
g_{12} &= dx_1 \wedge dx_7 + dx_2 \wedge dx_8 + dx_3 \wedge dx_5 + dx_4 \wedge dx_6\\
g_{13} &= dx_1 \wedge dx_4 + dx_2 \wedge dx_3 + dx_5 \wedge dx_8 + dx_6 \wedge dx_7\\
g_{14} &= dx_1 \wedge dx_8 - dx_2 \wedge dx_7 - dx_3 \wedge dx_6 + dx_4 \wedge dx_5
\end{aligned}$$

$$\begin{aligned}
g_{15} &= dx_1 \wedge dx_5 - dx_2 \wedge dx_6 + dx_3 \wedge dx_7 - dx_4 \wedge dx_8\\
g_{16} &= dx_1 \wedge dx_2 - dx_3 \wedge dx_4 - dx_5 \wedge dx_6 + dx_7 \wedge dx_8\\
g_{17} &= dx_1 \wedge dx_6 + dx_2 \wedge dx_5 - dx_3 \wedge dx_8 - dx_4 \wedge dx_7\\
g_{18} &= dx_1 \wedge dx_3 + dx_2 \wedge dx_4 - dx_5 \wedge dx_7 - dx_6 \wedge dx_8\\
g_{19} &= dx_1 \wedge dx_7 - dx_2 \wedge dx_8 - dx_3 \wedge dx_5 + dx_4 \wedge dx_6\\
g_{20} &= dx_1 \wedge dx_4 - dx_2 \wedge dx_3 - dx_5 \wedge dx_8 + dx_6 \wedge dx_7\\
g_{21} &= dx_1 \wedge dx_8 + dx_2 \wedge dx_7 + dx_3 \wedge dx_6 + dx_4 \wedge dx_5.
\end{aligned}$$

According to these orthogonal basis $F_A^+ = \frac{1}{8}\sigma(\Psi)^+$ is equivalent to the equation

$$F_A^+ \quad = \quad \frac{1}{8}\sum_{i=1}^{21} \frac{\big\langle f_i, \sigma(\Psi)\big\rangle}{\big\langle f_i, f_i\big\rangle}.f_i$$

and the more explicit form of the equation $F_A^+ = \frac{1}{8}\sigma(\Psi)^+$ is

$$\begin{aligned}
F_{15}-F_{26}-F_{37}+F_{48} &= \tfrac{1}{4}(-\psi_2\bar{\psi}_3+\psi_2\bar{\psi}_8+\psi_3\bar{\psi}_2-\psi_3\bar{\psi}_5\\
&\quad +\psi_5\bar{\psi}_3-\psi_5\bar{\psi}_8-\psi_8\bar{\psi}_2+\psi_8\bar{\psi}_5)\\
F_{12}-F_{34}+F_{56}-F_{78} &= \tfrac{1}{2}i(\psi_2\bar{\psi}_2-\psi_5\bar{\psi}_5)\\
F_{16}+F_{25}+F_{38}+F_{47} &= -\tfrac{1}{4}i(\psi_2\bar{\psi}_3-\psi_2\bar{\psi}_8+\psi_3\bar{\psi}_2+\psi_3\bar{\psi}_5\\
&\quad +\psi_5\bar{\psi}_3-\psi_5\bar{\psi}_8-\psi_8\bar{\psi}_2-\psi_8\bar{\psi}_5)\\
F_{13}+F_{24}+F_{57}+F_{68} &= \tfrac{1}{4}(\psi_1\bar{\psi}_2-\psi_1\bar{\psi}_5-\psi_2\bar{\psi}_1-\psi_2\bar{\psi}_6\\
&\quad +\psi_5\bar{\psi}_1+\psi_5\bar{\psi}_6+\psi_6\bar{\psi}_2-\psi_6\bar{\psi}_5)\\
F_{17}-F_{28}+F_{35}-F_{46} &= \tfrac{1}{4}(-\psi_1\bar{\psi}_3+\psi_1\bar{\psi}_8+\psi_3\bar{\psi}_1+\psi_3\bar{\psi}_6\\
&\quad -\psi_6\bar{\psi}_3+\psi_6\bar{\psi}_8-\psi_8\bar{\psi}_1-\psi_8\bar{\psi}_6)\\
F_{14}-F_{23}+F_{58}-F_{67} &= \tfrac{1}{4}i(\psi_1\bar{\psi}_2+\psi_1\bar{\psi}_5+\psi_2\bar{\psi}_1+\psi_2\bar{\psi}_6\\
&\quad +\psi_5\bar{\psi}_1+\psi_5\bar{\psi}_6+\psi_6\bar{\psi}_2+\psi_6\bar{\psi}_5)\\
F_{18}+F_{27}-F_{36}-F_{45} &= \tfrac{1}{4}i(\psi_1\bar{\psi}_3-\psi_1\bar{\psi}_8+\psi_3\bar{\psi}_1-\psi_3\bar{\psi}_6\\
&\quad -\psi_6\bar{\psi}_3+\psi_6\bar{\psi}_8-\psi_8\bar{\psi}_1+\psi_8\bar{\psi}_6)
\end{aligned}$$

$$\begin{aligned}
F_{15}+F_{26}-F_{37}-F_{48} &= \tfrac{1}{4}(-\psi_1\bar{\psi}_4-\psi_1\bar{\psi}_7+\psi_4\bar{\psi}_1+\psi_4\bar{\psi}_6\\
&\quad -\psi_6\bar{\psi}_4-\psi_6\bar{\psi}_7+\psi_7\bar{\psi}_1+\psi_7\bar{\psi}_6)\\
F_{12}+F_{34}+F_{56}+F_{78} &= -\tfrac{1}{4}i(\psi_3\bar{\psi}_3-\psi_8\bar{\psi}_8)\\
F_{16}-F_{25}+F_{38}-F_{47} &= -\tfrac{1}{4}i(\psi_1\bar{\psi}_4+\psi_1\bar{\psi}_7+\psi_4\bar{\psi}_1-\psi_4\bar{\psi}_6\\
&\quad -\psi_6\bar{\psi}_4-\psi_6\bar{\psi}_7+\psi_7\bar{\psi}_1-\psi_7\bar{\psi}_6)\\
F_{13}-F_{24}+F_{57}-F_{68} &= \tfrac{1}{4}(\psi_3\bar{\psi}_4+\psi_3\bar{\psi}_7-\psi_4\bar{\psi}_3+\psi_4\bar{\psi}_8\\
&\quad -\psi_7\bar{\psi}_3+\psi_7\bar{\psi}_8-\psi_8\bar{\psi}_4-\psi_8\bar{\psi}_7)\\
F_{17}+F_{28}+F_{35}+F_{46} &= \tfrac{1}{4}(\psi_2\bar{\psi}_4+\psi_2\bar{\psi}_7-\psi_4\bar{\psi}_2+\psi_4\bar{\psi}_5\\
&\quad -\psi_5\bar{\psi}_4-\psi_5\bar{\psi}_7-\psi_7\bar{\psi}_2+\psi_7\bar{\psi}_5)\\
F_{14}+F_{23}+F_{58}+F_{67} &= -\tfrac{1}{4}i(\psi_3\bar{\psi}_4+\psi_3\bar{\psi}_7+\psi_4\bar{\psi}_3+\psi_4\bar{\psi}_8\\
&\quad +\psi_7\bar{\psi}_3+\psi_7\bar{\psi}_8+\psi_8\bar{\psi}_4+\psi_8\bar{\psi}_7)\\
F_{18}-F_{27}-F_{36}+F_{45} &= -\tfrac{1}{4}i(\psi_2\bar{\psi}_4+\psi_2\bar{\psi}_7+\psi_4\bar{\psi}_2+\psi_4\bar{\psi}_5\\
&\quad +\psi_5\bar{\psi}_4+\psi_5\bar{\psi}_7+\psi_7\bar{\psi}_2+\psi_7\bar{\psi}_5).
\end{aligned}$$

$$\begin{aligned}
F_{15}-F_{26}+F_{37}-F_{48} &= \frac{1}{4}(-\psi_2\bar{\psi}_3-\psi_2\bar{\psi}_8+\psi_3\bar{\psi}_2+\psi_3\bar{\psi}_5\\
&\quad -\psi_5\bar{\psi}_3-\psi_5\bar{\psi}_8+\psi_8\bar{\psi}_2+\psi_8\bar{\psi}_5)\\
F_{12}-F_{34}-F_{56}+F_{78} &= -\frac{1}{2}i(\psi_1\bar{\psi}_1-\psi_6\bar{\psi}_6)\\
F_{16}+F_{25}-F_{38}-F_{47} &= -\frac{1}{4}i(\psi_2\bar{\psi}_3+\psi_2\bar{\psi}_8+\psi_3\bar{\psi}_2-\psi_3\bar{\psi}_5\\
&\quad -\psi_5\bar{\psi}_3-\psi_5\bar{\psi}_8+\psi_8\bar{\psi}_2-\psi_8\bar{\psi}_5)\\
F_{13}+F_{24}-F_{57}-F_{68} &= \frac{1}{4}(\psi_1\bar{\psi}_2+\psi_1\bar{\psi}_5-\psi_2\bar{\psi}_1+\psi_2\bar{\psi}_6\\
&\quad -\psi_5\bar{\psi}_1+\psi_5\bar{\psi}_6-\psi_6\bar{\psi}_2-\psi_6\bar{\psi}_5)\\
F_{17}-F_{28}-F_{35}+F_{46} &= \frac{1}{4}(\psi_1\bar{\psi}_3+\psi_1\bar{\psi}_8-\psi_3\bar{\psi}_1+\psi_3\bar{\psi}_6\\
&\quad -\psi_6\bar{\psi}_3-\psi_6\bar{\psi}_8-\psi_8\bar{\psi}_1+\psi_8\bar{\psi}_6)\\
F_{14}-F_{23}-F_{58}+F_{67} &= \frac{1}{4}i(\psi_1\bar{\psi}_2-\psi_1\bar{\psi}_5+\psi_2\bar{\psi}_1-\psi_2\bar{\psi}_6\\
&\quad -\psi_5\bar{\psi}_1+\psi_5\bar{\psi}_6-\psi_6\bar{\psi}_2+\psi_6\bar{\psi}_5)\\
F_{18}+F_{27}+F_{36}+F_{45} &= -\frac{1}{4}i(\psi_1\bar{\psi}_3+\psi_1\bar{\psi}_8+\psi_3\bar{\psi}_1+\psi_3\bar{\psi}_6\\
&\quad +\psi_6\bar{\psi}_3+\psi_6\bar{\psi}_8+\psi_8\bar{\psi}_1+\psi_8\bar{\psi}_6). \qquad (4.1)
\end{aligned}$$

$$A = \sum_{j=1}^{8} 2ix_j dx^j,$$

and

$$\Psi = (0,0,0,e^{\sum_{j=1}^{8} -\frac{i}{2}x_j^2},0,0,e^{\sum_{j=1}^{8} -\frac{i}{2}x_j^2},0) \tag{4.2}$$

is the non−trivial solution of the equation. Although this solution is non−trivial, it is flat since $F_A = 0$. It is possible to give a non−flat solution for these equations. Namely, if we assume that way

$$A_1 = 0,\ A_2 = 0,\ A_3 = 0,\ A_4 = 0,\ A_5 = 2ix_1,$$
$$A_6 = 2ix_2,\ A_7 = 2ix_3,\ A_8 = 2ix_4,$$

and

$$\psi_1 = 0,\ \psi_2 = 0,\ \psi_3 = 0,\ \psi_4 = e^{-i\left(x_1x_5+x_2x_6+x_3x_7+x_4x_8\right)},$$
$$\psi_5 = 0,\ \psi_6 = 0, \psi_7 = e^{-i\left(x_1x_5+x_2x_6+x_3x_7+x_4x_8\right)},\ \psi_8 = 0.$$

Also if the curvature equation is defined by

$$\rho^+\left(F_A^+\right) = \left(\Psi\Psi^*\right)^+,$$

where $\left(\Psi\Psi^*\right)^+$ is the projection of $\Psi\Psi^*$ onto $\rho^+\Big(\Omega^2_{21}(M, i\mathbb{R})\Big)$, the same results are obtained as in the (4.1). Moreover, if we define Seiberg−Witten equations according to the $\Omega^{21}_7(M, i\mathbb{R})$, then (4.2) is the non−trivial but flat solution of [5].

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