On 3-dimensional generalized (κ, μ)-contact metric manifolds

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Abstract. In the present study, we considered 3-dimensional generalized (κ, μ)-contact metric manifolds. We proved that a 3-dimensional generalized (κ, μ)-contact metric manifold is not locally ϕ-symmetric in the sense of Takahashi. However such a manifold is locally ϕ-symmetric provided that κ and μ are constants. Also it is shown that if a 3-dimensional generalized (κ, μ) -contact metric manifold is Ricci-symmetric, then it is a (κ, μ)-contact metric manifold. Further we investigated certain conditions under which a generalized (κ, μ)-contact metric manifold reduces to a (κ, μ)-contact metric manifold. Then we obtain several necessary and sufficient conditions for the Ricci tensor of a generalized (κ, μ)-contact metric manifold to be η-parallel. Finally, we studied Ricci-semisymmetric generalized (κ, μ)-contact metric manifolds and it is proved that such a manifold is either flat or a Sasakian manifold.

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Recently Blair, Koufogiorgos and Papantoniou [2] introduced the notion of (κ, μ)-contact metric manifolds with several examples. Then a full classification of such a manifold is given by E. Boeckx [5]. Assuming κ, μ as smooth functions, in 2000 Koufogiorgos and Tschlias [8] defined the notion of generalized (κ, μ)-contact metric manifolds and proved its existence for 3-dimensional case whereas for greater than 3-dimension, such a manifold does not exist. The $3-$dimensional generalized (κ, μ)-contact metric manifolds are also studied in [1], [8], [9], [10] and [11].

The present paper deals with a study of $3-$dimensional generalized (κ, μ)- contact metric manifolds. In 1977, Takahashi [15] introduced the notion of ϕ-symmetric Sasakian manifolds. After preliminaries, in Section 3 of the paper it is proved that a 3-dimensional generalized (κ, μ)-contact metric manifold is not locally ϕ-symmetric in the sense of Takahashi. However such a manifold is locally ϕ-symmetric provided that κ and μ are constants. Also it is shown that if a 3-dimensional generalized (κ, μ)

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-contact metric manifold is Ricci-symmetric, then it is a (κ,μ)-contact metric manifold. In the last section we investigate certain conditions under which a generalized (κ,μ)-contact metric manifold reduces to a (κ,μ)-contact metric manifold. Then we obtain several necessary and sufficient conditions for the Ricci tensor of a generalized (κ,μ)-contact metric manifold to be η-parallel. The notion of Ricci η-parallelity was first introduced by M. Kon [12] in a Sasakian manifold. Among others, it is shown that a generalized (κ,μ)-contact metric manifold with η-parallel Ricci tensor is either Sasakian, flat or of constant $\xi-$sectional curvature $\kappa<1$ and constant ϕ-sectional curvature $-\kappa$. Finally, we studied Ricci-semisymmetric generalized (κ,μ)-contact metric manifolds and it is proved that such a manifold is either flat or a Sasakian manifold.

1 (κ,μ)-contact manifolds

In this section, we collect some basic facts about contact metric manifolds. We refer to [4] for a more detailed treatment. A $(2n+1)$-dimensional differentiable manifold M^{2n+1} is called a *contact manifold* if there exists a globally defined 1-form η such that $(d\eta)^n\wedge\eta\neq 0$. On a contact manifold there exists a unique global vector field ξ satisfying

$$d\eta(\xi,X)=0,\quad \eta(\xi)=1, \tag{1.1}$$

for all $X\in TM^{2n+1}$.

Moreover it is well-known that there exist a $(1,1)$-tensor field ϕ, a Riemannian metric g which satisfy

$$\phi^2=-I+\eta\otimes\xi, \tag{1.2}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad g(\xi,X)=\eta(X), \tag{1.3}$$

$$d\eta(X,Y)=g(X,\phi Y), \tag{1.4}$$

for all $X,Y\in TM^{2n+1}$. As a consequence of the above relations we have

$$\phi\xi=0,\ \ \eta o\phi=0. \tag{1.5}$$

The structure (ϕ,ξ,η,g) is called *contact metric structure* and the manifold M^{2n+1} with a contact metric structure is said to be a *contact metric manifold*. Following [4], we define on M^{2n+1} the $(1,1)$-tensor field h by

$$h=\frac{1}{2}\left(\mathcal{L}_\xi\phi\right), \tag{1.6}$$

where $\mathcal{L}_\xi$ is the Lie differentiation in the direction of ξ.

The tensor field h is self adjoint and satisfy

$$h\xi = 0,\quad \mathrm{tr}h=0,\quad \mathrm{tr}\phi h=0,\quad h\phi+\phi h=0, \tag{1.7}$$

$$\nabla_X\xi = -\phi X-\phi hX,\quad (\nabla_X\eta)(Y)=-g(\phi X+\phi hX,Y) \tag{1.8}$$

where ∇ is the Levi-Civita connection of g.

A generalized (κ,μ)-manifold is defined as a contact metric manifold satisfying

$$R(X,Y)\xi = (\kappa I + \mu h)\,(\eta(Y)X - \eta(X)Y)\,, \tag{1.9}$$

for some smooth functions κ and μ on M^{2n+1} independent of the choice of vector fields X and Y.Then such a manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is called a generalized (κ, μ)-contact metric manifold [8]. In particular if κ, μ are constants then the manifold will be simply called a (κ, μ)-contact metric manifold. However, a generalized (κ, μ)-contact metric manifold does not exist for dimension greater than three whereas several examples in 3-dimensional cases has been given in [8] and [9]. Hence we confined ourselves on the study of 3-dimensional generalized (κ, μ)- contact metric manifolds.

On any generalized (κ, μ)-contact metric manifold, the following relations hold [8], [9]:

$$h^2 = (\kappa - 1)\phi^2, \qquad \kappa \leq 1 \tag{1.10}$$

$$(a)\ \xi(\ \kappa) = 0, \qquad (b)\ \xi(\ r) = 0, \qquad (c)\ hgrad\mu = grad\kappa \tag{1.11}$$

where r is the scalar curvature of the manifold. Also from (1.9), it follows that on any 3-dimensional generalized (κ, μ)-contact metric manifold, we have

$$S(X,\xi) = 2\kappa\eta(X) \tag{1.12}$$

where S is the Ricci tensor of type $(0,2)$.

Due to [2], on any generalized (κ, μ)$-$contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ we have the following:

$$\begin{aligned} (\nabla_X h)Y &= ((1-\kappa)g(X,\phi Y) - g(X,\phi hY))\xi \\ &- \eta(Y)((1-\kappa)\phi X + \phi hX) - \mu\eta(X)\phi hY, \end{aligned} \tag{1.13}$$

$$(\nabla_X \phi)Y = (g(X,Y) + g(X,hY))\xi - \eta(Y)(X + hX), \tag{1.14}$$

$$Q\phi - \phi Q = 2(2(n-1) + \mu)h\phi \tag{1.15}$$

Lemma 1. *[3] Let M^3 be a contact metric manifold on which $Q\phi = \phi Q$. Then M^3 is either Sasakain, flat or of constant ξ-sectional curvature $\kappa < 1$ and constant ϕ-sectional curvature $-\ \kappa$.*

By definition the *Weyl conformal curvature tensor* C is given by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\left[\begin{array}{c} g(Y,Z)QX - g(X,Z)QY \\ +S(Y,Z)X - S(X,Z)Y \end{array}\right] \\ &+ \frac{r}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right] \end{aligned} \tag{1.16}$$

and

$$D(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - \frac{1}{2(n-2)}\left[X(r)g(Y,Z) - Y(r)g(X,Z)\right] \tag{1.17}$$

where Q denotes the Ricci operator, i.e. $S(X,Y) = g(QX,Y)$ and r is scalar curvature [7]. The following is a well-known theorem of Weyl [16].

Theorem 2. *[16] A necessary and sufficient condition for a Riemannian manifold M to be conformally flat is that $C = 0$ for $n > 3$ and $D = 0$ for $n = 3$.*

It should be noted that if M is conformally flat and of dimension $n > 3$, then $C = 0$ implies $D = 0$.

For every 3-dimensional Riemannian manifold $C = 0$. So, the curvature tensor R of 3-dimensional Riemannian manifolds can be written the following formula:

$$
\begin{aligned}
R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\
&\quad -\frac{r}{2}\left(g(Y,Z)X - g(X,Z)Y\right).
\end{aligned} \tag{1.18}
$$

Substituting $Y = Z = \xi$ to (1.18), and using (1.9) on M^3 we obtain

$$Q = \frac{1}{2}\left(r - 2\kappa\right) I + \frac{1}{2}\left(6\kappa - r\right)\eta \otimes \xi + \mu h. \tag{1.19}$$

We see that on M^3, the scalar curvature r is equal to

$$r = 2(\kappa - \mu). \tag{1.20}$$

Using (1.19)and (1.20) in (1.18) we obtain

$$
\begin{aligned}
R(X,Y)Z &= -(\kappa+\mu)[g(Y,Z)X - g(X,Z)Y] + (2\kappa+\mu)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\
&\quad +\eta(Y)\xi\eta(Z)X - \eta(X)\eta(Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY \\
&\quad +g(hY,Z)X - g(hX,Z)Y].
\end{aligned} \tag{1.21}
$$

2 Generalized (κ, μ)- contact metric manifolds

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a generalized (κ,μ)-contact metric manifold. Then, from (1.21), it follows of (1.13), (1.10), (1.8), (1.5) and (1.3) that

$$
\begin{aligned}
(\nabla_W R)(X,Y)Z \quad = \quad & -(W\kappa + W\mu)[g(Y,Z)X - g(X,Z)Y] \\
& +(2W\kappa + W\mu)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\
& +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\
& +(2\kappa + \mu)[\{g(Y,Z)g(W+hW,\phi X)\text{-}g(X,Z)g(W+hW,\phi Y)\}\,\xi \\
& +\{\eta(Y)X - \eta(X)Y\}g(W+hW,\phi Z) \\
& -\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}(\phi W + \phi hW) \\
& +\{g(W+hW,\phi Y)X - g(W+hW,\phi X)Y\}\eta(Z)] \\
& +(W\mu)[g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y] \\
& +\mu[-\{(1-k)g(W,\phi X) \\
& +g(W,h\phi X)\}\eta(Z)Y - \eta(X)g(h\phi W,Z)Y \\
& +(1-k)\eta(X)g(\phi W,Z)Y \\
& +\mu\eta(W)g(\phi hX,Z)Y + \{(1-k)g(W,\phi Y) \\
& +g(W,h\phi Y)\}\eta(Z)X + +\eta(Y)g(h\phi W,Z)X \\
& -(1-k)\eta(Y)g(\phi W,Z)X - \mu\eta(W)g(\phi hY,Z)X \\
& +\{(1-k)g(W,\phi X) + g(W,h\phi X)\}g(Y,Z)\xi \\
& +g(Y,Z)\eta(X)h\phi W - (1-k)g(Y,Z)\eta(X)\phi W \\
& -\mu g(Y,Z)\eta(W)\phi hX - \{(1-k)g(W,\phi Y) \\
& +g(W,h\phi Y)\}g(X,Z)\xi - g(X,Z)\eta(Y)h\phi W \\
& +(1-k)g(X,Z)\eta(Y)\phi W + \mu g(X,Z)\eta(W)\phi hY.
\end{aligned} \tag{2.1}
$$

Taking W,X,Y,Z orthogonal to ξ and then using (1.2), (1.3), we obtain from (1.5) that

$$
\begin{aligned}
\phi^2((\nabla_W R)(X,Y)Z) \quad = \quad & (W\kappa + W\mu)[g(Y,Z)X - g(X,Z)Y] - \\
& -(W\mu)[g(Y,Z)hX - g(X,Z)hY + \\
& +g(hY,Z)X - g(hX,Z)Y].
\end{aligned} \tag{2.2}
$$

Definition 3. *A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be locally ϕ- symmetric in sense of Takahashi if it satisfies*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.3}$$

for all vector fields X,Y,Z,W orthogonal to ξ.

Definition 4. *If Ricci tensor of M is parallel, then M is called Ricci-symmetric.*

Hence in view of (2.2) and (2.3) , we state the following:

Theorem 5. *A 3- dimensional generalized (κ,μ)-contact metric manifold M^3 (ϕ,ξ,η,g) is not locally ϕ-symmetric in the sense of Takahashi.*

Corollary 6. *If κ and μ are constants, a 3-dimensional generalized (κ,μ)-contact metric manifold is locally ϕ-symmetric in the sense of Takahashi.*

Theorem 7. *A 3-dimensional Ricci-symmetric generalized (κ, μ)-contact metric manifold is a 3-dimensional (κ, μ) manifold.*

Proof. From (1.20) we get by virtue of (1.11) (a), (b) that

$$\xi(\mu) = 0. \tag{2.4}$$

From (1.19)we have

$$S(X,Y) = -\mu g(X,Y) + \mu g(hX,Y) + (2\kappa + \mu)\eta(X)\eta(Y). \tag{2.5}$$

By virtue of (1.13) and (1.8), we obtain from(2.5) that

$$\begin{aligned} (\nabla_Z S)(X,Y) &= Z\mu\{g(hX,Y) - g(X,Y)\} + (2(Z\kappa) + Z\mu)\eta(X)\eta(Y) + \\ &\quad +(2\kappa+\mu)[g(Z,\phi X)\eta(Y) + g(hZ,\phi X)\eta(Y) + \\ &\quad +g(Z,\phi Y)\eta(X) + g(hZ,\phi Y)\eta(X)] + \mu(1-\kappa)g(Z,\phi Y)\eta(X) \\ &\quad +\mu^2 g(hX,\phi Y)\eta(Z) + \mu(1-\kappa)[g(Z,\phi X)\eta(Y) + g(hZ,\phi X)\eta(Y)] \\ &\quad +\mu g(\phi Z, hY)\eta(X). \end{aligned} \tag{2.6}$$

From (1.20) we have

$$dr(Z) = 2[(Z\kappa) - (Z\mu)]. \tag{2.7}$$

Since the manifold M^3 under consideration is Ricci-symmetric, we have

$$dr(Z) = 0. \tag{2.8}$$

Setting $X = Y = \xi$ in (2.6) and again using parallel of Ricci tensor S we obtain

$$(Z\kappa) = 0, \tag{2.9}$$

for all Z.i.e.,κ is a constant. Hence (2.7) , (2.8) and (2.9) yield

$$(Z\mu) = 0, \tag{2.10}$$

i.e., μ is a constant. Thus one says generalized (κ, μ)-contact metric manifold is (κ, μ)-contact metric manifold. □

Again, in view of (2.9),and (2.10) we obtain from and (2.2) that

$$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ. Hence we have the following :

Corollary 8. *A 3-dimensional Ricci-symmetric generalized (κ, μ)-contact metric manifold is locally ϕ-symmetric in the sense of Takahashi.*

3 Generalized (κ, μ)-contact metric manifolds

This section deals with a 3-dimensional generalized (κ, μ)-contact metric manifold satisfying some conditions.

Definition 9. *The Ricci tensor S of a Riemannian manifold M is to be cyclic-parallel if*

$$(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0, \tag{3.1}$$

for all vector fields X, Y, Z.

Theorem 10. *If in a 3-dimensional generalized (κ, μ)-contact metric manifold M if the ricci tensor is cyclic-parallel then it is a 3-dimensional (κ, μ)-contact metric manifold.*

Proof. From (3.1), it follows that $dr(Z) = 0$ and hence (2.7)yields

$$Z(\kappa) = Z(\mu), \tag{3.2}$$

for all Z.

If the Ricci tensor S of M is cyclic parallel then replacing X and Y with ξ in (3.1), we can write

$$(\nabla_Z S)(\xi,\xi) + (\nabla_\xi S)(\xi,Z) + (\nabla_\xi S)(Z,\xi) = 0. \tag{3.3}$$

From (2.6) and using (1.11)we obtain

$$(\nabla_Z S)(\xi,\xi) = 2Z(\kappa), \quad (\nabla_\xi S)(\xi,Z) = 0 = (\nabla_\xi S)(Z,\xi). \tag{3.4}$$

Substituting (3.4) in (3.3) we get

$$Z(\kappa) = 0,$$

for all Z. i.e.,κ is a constant.Hence (3.2) yields

$$(Z\mu) = 0, \tag{3.5}$$

i.e., μ is a constant. This completes proof of theorem. □

For the case M is non-Sasakian and $n > 1$ C. Özgür proved the following result.

Theorem 11 ([6]). *Let (M^{2n+1}, g) be a non-Sasakian (κ, μ)-contact metric manifold. If the Ricci tensor S of M is cyclic parallel then M is either κ-contact or $\kappa = -\frac{1}{4}(\frac{\mu^2+4n\mu}{n})$.*

Hence, we have the following corollary,

Corollary 12. *If in a 3-dimensional generalized (κ, μ)-contact metric manifold M the Ricci tensor is cyclic-parallel then it is locally ϕ-symmetric in the sense of Takahashi.*

Definition 13. *The Ricci tensor of a contact metric manifold is said to be η-parallel if it satisfies*

$$(\nabla_Z S)(\phi X, \phi Y) = 0 \tag{3.6}$$

for all vector fields X, Y, Z.

This notion of Ricci-η-parallelity was first introduced by M. Kon [12] in a Sasakian manifold.

Theorem 14. *In a 3-dimensional generalized (κ, μ)-contact metric manifold M^3 (ϕ,ξ,η,g), the Ricci tensor is η-parallel if and only if the following relation holds :*

$$(Z\mu)[g(X, +hX, Y) - \eta(X)\eta(Y)] - \mu^2 g(\phi hX, Y)\eta(Z) = 0. \tag{3.7}$$

Proof. From (2.5) we get

$$S(\phi X, \phi Y) = -\mu[g(X, Y) + g(hX, Y) - \eta(X)\eta(Y)]. \tag{3.8}$$

In view of (2.5), (3.8) can be written as

$$S(\phi X, \phi Y) = S(X, Y) - 2\mu g(hX, Y) - 2\kappa\eta(X)\eta(Y)]. \tag{3.9}$$

From (1.14) we have

$$\nabla_X \phi Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) + \phi(\nabla_X Y). \tag{3.10}$$

Again we have

$$(\nabla_Z S)(\phi X, \phi Y) = \nabla_Z S(\phi X, \phi Y) - S(\nabla_Z \phi X, \phi Y) - S(\phi X, \nabla_Z \phi Y). \tag{3.11}$$

Using (3.8), (3.10), (1.8) and (1.13) in (3.11),we obtain by straightforward calculation

$$\begin{aligned} (\nabla_Z S)(\phi X, \phi Y) &= -(Z\mu)[g(X + hX, Y) - \eta(X)\eta(Y)] \\ &\quad -\kappa\mu[g(X, \phi Z)\eta(Y) + g(Y, \phi Z)\eta(X)] + \mu^2 g(\phi hX, Y)\eta(Z) \\ &\quad +S(Z, \phi Y)\eta(X) + S(hZ, \phi Y)\eta(X) + S(\phi X, Z)\eta(Y) \\ &\quad +S(\phi X, hZ)\eta(Y) - S(\phi X, hZ). \end{aligned} \tag{3.12}$$

From (2.5) we get

$$S(Z, \phi Y) = -\mu g(Z, \phi Y) + \mu g(hZ, \phi Y), \tag{3.13}$$

$$S(hZ, \phi Y) = \mu g(\phi hZ, Y) + \mu(1 - \kappa)g(Z, \phi Y), \tag{3.14}$$

$$S(\phi X, Z) = -\mu g(\phi X, Z) + \mu g(\phi hX, Z), \tag{3.15}$$

$$S(\phi hX, Z) = \mu g(\phi hZ, X) + \mu(1 - \kappa)g(Z, \phi X). \tag{3.16}$$

Using (3.13)-(3.16) in (3.12) we obtain our relation. □

Again, by virtue of (3.9) and (3.10) we can easily obtain from (3.11) that

$$
\begin{aligned}
(\nabla_Z S)(\phi X, \phi Y) &= (\nabla_Z S)(X,Y) - 2(Z\mu)g(hX,Y) - 2(Z\kappa)\eta(X)\eta(Y) - \\
&\quad -2\mu[(1-\kappa)\{g(Z,\phi X)\eta(Y) + g(Z,\phi Y)\eta(X)\} + \\
&\quad +g(h\phi Z, Y)\eta(X) - \mu g(\phi hX, Y)\eta(Z)] + \\
&\quad +2\kappa[g(\phi Z + \phi hZ, X)\eta(Y) + g(\phi Z + \phi hZ, Y)\eta(X)].
\end{aligned}
\tag{3.17}
$$

Thus, we have the following result:

Theorem 15. *In a 3-dimensional generalized (κ,μ)-contact metric manifold M^3 (ϕ,ξ,η,g), the Ricci tensor is η-parallel if and only if the following relation holds :*

$$
\begin{aligned}
(\nabla_Z S)(X,Y) &= 2(Z\mu)g(hX,Y) + 2(Z\kappa)\eta(X)\eta(Y) \\
&\quad +2\mu[(1-\kappa)\{g(Z,\phi X)\eta(Y) + g(Z,\phi Y)\eta(X)\} \\
&\quad +g(Z,h\phi X)\eta(Y) + g(h\phi Z, Y)\eta(X) - \mu g(\phi hX, Y)\eta(Z)] \\
&\quad -2\kappa[g(\phi Z + \phi hZ, X)\eta(Y) + (\phi Z + \phi hZ, Y)\eta(X)].
\end{aligned}
\tag{3.18}
$$

We prove the following Theorem:

Theorem 16. *If the Ricci tensor of a 3-dimensional generalized (κ,μ)-contact metric manifold M^3 (ϕ,ξ,η,g) is η-parallel then it is a (κ,μ)-contact metric manifold.*

Proof. Let $\{\ e_i : i = 1,2,3\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then setting $X = Y = e_i$ in (3.18) and taking summation over i, $1 \leq i \leq 3$, we get

$$(Zr) = 2(Z\kappa). \tag{3.19}$$

From (2.7) and (3.19), it follows that

$$(Z\mu) = 0 \quad \text{for all } Z, \tag{3.20}$$

and hence μ is constant.

Again putting $Y = Z = e_i$ in (3.18) and taking summation over i, $1 \leq i \leq 3$, we get

$$dr(X) = 4(\xi\kappa)\eta(X),$$

which yields by virtue of (1.11) (a) that

$$dr(X) = 0 \ \text{ for all } X. \tag{3.21}$$

From (3.19) and (3.21) we have

$$(Z\kappa) = 0 \quad \text{for all } Z. \tag{3.22}$$

Thus κ is constant. This completes proof of theorem. □

Using (3.20) and (3.22) in (2.2), we can state the following :

Theorem 17. *If the Ricci tensor of a 3-dimensional generalized (κ,μ)-contact metric manifold M^3 (ϕ,ξ,η,g) is η-parallel then it is locally ϕ-symmetric in the sense of Takahashi.*

Again in view of (3.20) and (3.22) we obtain from (3.18) that

$$\nabla_Z |Q|^2 = 2\sum_{i=1}^{3} g((\nabla_Z Q)e_i, Qe_i) = 0$$

which implies that

$$|Q|^2 = \text{constant}. \tag{3.23}$$

By virtue of (3.21) and (3.23), we can state the following :

Theorem 18. *Let M^3 (ϕ,ξ,η,g) be a 3-dimensional generalized (κ,μ)-contact metric manifold with η-parallel Ricci tensor. Then we have the following:*

(i)The scalar curvature r of M is constant,

(ii)The square of the length of the Ricci operator Q of M is constant, that is, $|Q|^2 =$ constant.

The above Theorem 16 generalized the corresponding results of M. Kon [[12]] in a Sasakian manifold.

Next, using(3.20) in (3.7) we obtain by virtue of (1.10)that either $\mu = 0$ or $\kappa = 1$. If $\kappa = 1$, then the manifold is Sasakian. If $\mu = 0$, then (1.15) yields (for $n = 1$) $Q\phi = \phi Q$. Consequently by virtue of Lemma 1, we can state the following :

Theorem 19. *Let M^3 (ϕ,ξ,η,g) be a 3-dimensional generalized (κ,μ)-contact metric manifold with η-parallel Ricci tensor. Then M^3 is either Sasakian, flat or of constant ξ-sectional curvature $\kappa < 1$ and constant ϕ-sectional curvature -κ.*

Theorem 20. *[13] Let $M^{2n+1}(\phi,\xi,\eta,g)$ be contact Riemannian manifold such that*

(i)$R(X,\xi).S = 0$,and

(ii)$R(X,Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), (\kappa,\mu) \in R^2$.

Then the manifold is either

(i) locally isometric to $E^{n+1}(0) \times S^{n+1}$,or

(ii) an Einstein-Sasakian manifold, or

(iii) an $\eta-$Einstein manifold if $\kappa^2+\mu^2(\kappa-1) \neq 0$.

Theorem 21. *Let M^3 (ϕ,ξ,η,g) be a 3-dimensional generalized (κ,μ)-contact metric manifold satisfying the relation $R(\xi,X).S = 0$. Then the manifold is either flat or Sasakian.*

Proof.

$$\begin{aligned} 0 &= (R(\xi,X).S)(Y,Z) = R(\xi,X).S(Y,Z) - \\ &\quad -S(R(\xi,X)Y,Z) - S(Y,R(\xi,X)Z) \end{aligned} \tag{3.24}$$

from which

$$S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z) = 0. \tag{3.25}$$

From this equation, setting $Z = \xi$ we get

$$S(R(\xi,X)Y,\xi) + S(Y,R(\xi,X)\xi) = 0. \tag{3.26}$$

Using (1.9) and (2.5) in (3.26) we obtain

$$\begin{aligned} &[2\kappa^2 + \mu\kappa + \mu^2(\kappa - 1)]g(X,Y) + (\mu\kappa + \mu^2)g(hX,Y) - \\ &-[2\kappa^2 + \mu\kappa + \mu^2(\kappa - 1)]\eta(X)\eta(Y) = 0, \end{aligned} \tag{3.27}$$

which yields

$$2\kappa^2 + \mu\kappa + \mu^2(\kappa - 1) = 0, \tag{3.28}$$

$$\mu(\kappa + \mu) = 0. \tag{3.29}$$

If $h = 0$, then (1.10) implies that $\kappa = 1$ and hence the manifold is Sasakian. From (3.29), we have either $\mu = 0$, or $\kappa = -\mu$.If $\mu = 0$, then (3.28) implies that $\kappa = 0$.

Again $\kappa = -\mu$, then also (3.28) gives $\kappa = \mu = 0$. Thus we have either $\kappa = \mu = 0$ or $\kappa = 1$.If $\kappa = \mu = 0$, then (1.21) implies that manifold is flat. If $\kappa = 1$,then manifold is again Sasakian. This completes proof of the Theorem. □

Theorem 22. *[14] Let M^{2n+1} (ϕ,ξ,η,g) be contact metric manifold with harmonic curvature tensor and ξ belonging to the (κ,μ)- nullity distribution. Then M is either*

(i)an Einstein-Sasakian manifold, or

(ii)an $\eta-$Einstein manifold, or

(iii) locally isometric to the product of a flat$(n+1)-$dimensional manifold and an n-dimensional manifold of positive constant curvature equal to 4, including a flat contact metric structure for $n = 1$.

Theorem 23. *A 3-dimensional conformally flat generalized (κ,μ)-contact metric manifold is either Sasakian or flat contact metric manifold.*

Proof. From (2.6) and after some calculations we obtain

$$\begin{aligned} &\xi(\mu)[g(hX,Y) - g(X,Y)] + \xi(\mu)\eta(X)\eta(Y) \\ &-2X(\kappa)\eta(Y) - [2(\kappa + \mu) - \mu\kappa]g(X,\phi Y) \\ &+(\mu^2 - 2\kappa)g(\phi X,hY) \\ = \; &\frac{1}{2}[\xi(r)g(X,Y) - X(r)\eta(Y)]. \end{aligned} \tag{3.30}$$

Setting $Y = \xi$ in (3.30)and using (1.7) we have

$$X(\kappa) = 0. \tag{3.31}$$

This equation says that κ is constant. Now, using κ is constant and (1.2)(c) we get

$$hgrad\mu = 0 \tag{3.32}$$

Suppose that X is different from ξ. From (3.32) we have

$$0 = g(hgrad\mu, X) = g(hX, grad\mu). \tag{3.33}$$

Setting $X = hX$ (3.33) and using (1.10) and some calculations, we get

$$(\kappa - 1)[X(\mu) + \eta(X)\xi(\mu)] = 0. \tag{3.34}$$

From(1.11) and (1.20) we obtain

$$\xi(\mu) = 0. \tag{3.35}$$

Therefore, (3.34) reduces to

$$(\kappa - 1)X(\mu) = 0. \tag{3.36}$$

So either $\kappa = 1$ or $X(\mu) = 0$. For the first case M is Sasakian. From (3.35) we can deduce that μ is constant for the second case. So, M becomes (κ, μ) contact metric manifold. From [14] M is flat. Our theorem is thus proved. □

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