Almost Kenmotsu f-manifolds

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Abstract. A class of manifolds which admit an f-structure with s-dimensional parallelizable kernel is introduced and studied. Such manifolds are called almost Kenmotsu $f.pk$-manifolds. If $s = 1$, one obtains almost Kenmotsu manifolds and, if $s = 2$, they carry a locally conformal almost Kähler structure. Several foliations canonically associated with an almost Kenmotsu $f.pk$-manifold are studied. Locally conformal almost Kenmotsu $f.pk$-manifolds are characterized. If $s \geq 2$, they set up a class which is disjoint from that of locally conformal almost $\mathcal{C}$-manifolds.

M.S.C. 2000: 53C15, 53D15, 53C25
Key words: f-structure, almost Kenmotsu manifold, almost Kähler manifold, conformal change of an f-structure.

Introduction

An f-structure on a C^∞ m-dimensional manifold M is defined by a non-vanishing tensor field φ of type (1,1) which satisfies $\varphi^3 + \varphi = 0$ and has constant rank r. It is known that, in this case, r is even, $r = 2n$. Moreover, TM splits into two complementary subbundles $Im\,\varphi$ and $Ker\,\varphi$ and the restriction of φ to $Im\,\varphi$ determines a complex structure on such subbundle. It is also known that the existence of an f-structure on M is equivalent to a reduction of the structure group to $U(n) \times O(s)$, where $s = m - 2n$ ([2]). An interesting case occurs when the subbundle $Ker\,\varphi$ is parallelizable, for which the reduced structure group is $U(n) \times \{I_s\}$, and we have an f-structure with parallelizable kernel, briefly denoted by $f.pk$-structure, the respective manifold being called an $f.pk$-manifold or a globally framed manifold ([8]). Then, there exists a global frame $\{\xi_i\}$ for the subbundle $Ker\ \varphi$ with dual 1-forms η^i, $1 \leq i \leq s$, satisfying $\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i$. It follows that $\varphi\xi_i = 0, \quad \eta^i \circ \varphi = 0$. From now on we will omit the sum symbol for repeated indexes varying in $\{1, \ldots, s\}$. It is well known that one can consider compatible Riemannian metrics g on M such that for any tangent vector fields X, Y, one has $g(X, Y) = g(\varphi X, \varphi Y) + \eta^i(X)\eta^i(Y)$ and, fixed a compatible metric g, $(\varphi, \xi_i, \eta^i, g)$ is called a metric $f.pk$-structure. Therefore, $T(M)$ splits as complementary orthogonal sum of its subbundles $Im\ \varphi$ and $Ker\ \varphi$. We denote their respective differentiable distributions by $\mathcal{D}$ and $\mathcal{D}^\perp$.

Balkan Journal of Geometry and Its Applications, Vol.12, No.1, 2007, pp. 32-43.

A wide class of $f.pk$-structures was introduced in [2] by D. Blair according to the following definition. A metric $f.pk$-structure is said a $\mathcal{K}$-structure if the fundamental 2-form Φ, defined usually as $\Phi(X,Y) = g(X,\varphi Y)$, is closed and the normality condition holds, i.e. $N = [\varphi,\varphi] + 2d\eta^i \otimes \xi_i = 0$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ. Several subclasses have been studied from different points of view ([2, 3, 4]), also dropping the normality condition and, in this case, the term *almost* precedes the name of the considered structures or manifolds. If $d\eta^1 = \ldots = d\eta^s = \Phi$, the (almost) $\mathcal{K}$-structure is said an (almost) $\mathcal{S}$-structure and M an (almost) $\mathcal{S}$-manifold. If $d\eta^i = 0$ for all $i \in \{1,\ldots,s\}$, then the (almost) $\mathcal{K}$-structure is called an (almost) $\mathcal{C}$-structure and M is said an (almost) $\mathcal{C}$-manifold.

In [6], we studied normal metric $f.pk$-structures and then $f.pk$-manifolds (called Kenmotsu $f.pk$-manifolds), for which the 2-form Φ verifies the condition $d\Phi = 2\eta^i \wedge \Phi$, for some $i \in \{1,\ldots,s\}$, also proving that such an index is unique and choosing $i = 1$.

This paper deals with almost Kenmotsu $f.pk$-manifolds. Firstly, we state general properties involving the coderivative of the η^i's with respect to the Levi-Civita connection. Several foliations can be described. In particular, each leaf of the distribution $Im\,\varphi$ has an almost Kähler structure and we give conditions which are equivalent to the request that $Im\,\varphi$ has Kähler or, possibly, totally umbilical leaves. Then, we explain a procedure to construct almost Kenmotsu $f.pk$-manifolds, starting from almost Kähler manifolds. Furthermore, we prove that if the leaves of $Im\,\varphi$ in an almost Kenmotsu $f.pk$-manifold M^{2n+s} are totally umbilical, then M^{2n+s} is locally a warped product of an almost Kähler manifold and $\mathbb{R}^s$, with warping function depending on a Euclidean coordinate, only.

In section 3, we study $(2n+s)$-dimensional metric $f.pk$-manifolds admitting a structure which is locally conformal to an almost Kenmotsu one and prove that, if $s \geq 2$, each of the considered conformal changes is global. We also characterize locally conformal almost $\mathcal{C}$-manifolds and prove that an almost Kenmotsu manifold M^{2n+s}, $s \geq 2$, cannot be a locally conformal almost $\mathcal{C}$-manifold. Note that, when $s = 1$, almost Kenmotsu manifolds set up a subclass of locally conformal almost cosymplectic manifolds ([13]), whereas almost $\mathcal{C}$-manifolds coincide with almost cosymplectic manifolds.

We recall that the Levi-Civita connection ∇ of a metric $f.pk$-manifold satisfies the following formula ([2],[5]):

$$\begin{aligned} 2g((\nabla_X\varphi)Y,Z) = \quad & 3\,d\Phi(X,\varphi Y,\varphi Z) - 3d\Phi(X,Y,Z) \\ & + g(N(Y,Z),\varphi X) + N^{(2)}_j(Y,Z)\eta^j(X) \\ & + 2d\eta^j(\varphi Y,X)\eta^j(Z) \; - 2d\eta^j(\varphi Z,X)\eta^j(Y). \end{aligned}$$

Each tensor field $N^{(2)}_j$ is defined by $N^{(2)}_j(X,Y) = (\mathcal{L}_{\varphi X}\eta^j)(Y) - (\mathcal{L}_{\varphi Y}\eta^j)(X)$, and can be rewritten as $N^{(2)}_j(X,Y) = 2d\eta^j(\varphi X,Y) - 2d\eta^j(\varphi Y,X)$.

1 Almost Kenmotsu $f.pk$-manifolds

In [6], a metric $f.pk$-manifold M of dimension $2n+s$, $s \geq 1$, with $f.pk$-structure (φ,ξ_i,η^i,g), is said to be a *Kenmotsu f.pk-manifold* if it is normal, the 1-forms η^i are closed and $d\Phi = 2\eta^1 \wedge \Phi$.

Definition 1.1 A metric $f.pk$-manifold M of dimension $2n+s$, $s \geq 1$, with $f.pk$-structure $(\varphi, \xi_i, \eta^i, g)$, is said to be an almost Kenmotsu $f.pk$-manifold if the 1-forms η^i are closed and $d\Phi = 2\eta^1 \wedge \Phi$.

Obviously, a normal almost Kenmotsu $f.pk$-manifold is a Kenmotsu $f.pk$-manifold.

Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be an almost Kenmotsu $f.pk$-manifold. Since the distribution $\mathcal{D}$ is integrable, we have $\mathcal{L}_{\xi_i}\eta^j = 0$, $[\xi_i, \xi_j] \in \mathcal{D}$ and $[X, \xi_i] \in \mathcal{D}$ for any $X \in \mathcal{D}$. Then, the Levi-Civita connection is given by:

$$\begin{aligned} 2g((\nabla_X\varphi)(Y), Z) &= 2g(g(\varphi X, Y)\xi_1 - \eta^1(Y)\varphi(X), Z) \\ &\quad + g(N(Y, Z), \varphi X)\,, \end{aligned} \tag{1.1}$$

for any $X, Y, Z \in \mathcal{X}(M^{2n+s})$. Putting $X = \xi_i$ we obtain $\nabla_{\xi_i}\varphi = 0$ which implies $\nabla_{\xi_i}\xi_j \in \mathcal{D}^{\perp}$ and then $\nabla_{\xi_i}\xi_j = \nabla_{\xi_j}\xi_i$ since $[\xi_i, \xi_j] = 0$.

For each $i \in \{1, \ldots, s\}$ we put $A_i = -\nabla\xi_i$ and $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi$.

Proposition 1.1 *For any $i \in \{1, \ldots, s\}$ the tensor field A_i is a symmetric operator such that:*

1) $A_i(\xi_j) = 0$, *for any $j \in \{1, \ldots, s\}$;*

2) $A_i \circ \varphi + \varphi \circ A_i = -2\delta_i^1\varphi$.

Proof. $g(A_iX, Y) - g(X, A_iY) = -2d\eta^i(X, Y) = 0$ implies that A_i is symmetric. For any $i, j, k \in \{1, \ldots, s\}$ deriving $g(\xi_i, \xi_j) = \delta_{ij}$ with respect to ξ_k, using $\nabla_{\xi_k}\xi_i = \nabla_{\xi_i}\xi_k$, we get $2g(\xi_k, A_i(\xi_j)) = 0$. Since $\nabla_{\xi_j}\xi_i \in \mathcal{D}^{\perp}$, we conclude $A_i(\xi_j) = 0$. To prove 2), we notice that for any $Z \in \mathcal{X}(M^{2n+s})$ we have $\varphi(N(\xi_i, Z)) = (\mathcal{L}_{\xi_i}\varphi)(Z)$ and, on the other hand, since $\nabla_{\xi_i}\varphi = 0$,

$$\mathcal{L}_{\xi_i}\varphi = A_i \circ \varphi - \varphi \circ A_i.$$

Applying (1.1) with $Y = \xi_i$, we have

$$2g(\varphi A_iX, Z) = -2\eta^1(\xi_i)g(\varphi(X), Z) - g(\varphi(N(\xi_i, Z)), X),$$

which implies 2). □

Proposition 1.2 *For any $i \in \{1, \ldots, s\}$ the tensor field h_i is a symmetric operator and:*

1) $h_i(\xi_j) = 0$, *for any $j \in \{1, \ldots, s\}$;*

2) $h_i \circ \varphi + \varphi \circ h_i = 0\,.$

Proof. Equation 1) is obvious. Suppose $i \geq 2$. Then, from Proposition 1.1 we get $h_i = A_i \circ \varphi = -\varphi \circ A_i$ and for any tangent vector fields X, Y, $g(h_i(X), Y) = g(\varphi X, A_iY) = -g(X, \varphi A_iY) = g(X, h_i(Y))$. Now, we consider $i = 1$ and applying Proposition 1.1 we get $h_1 = A_1 \circ \varphi + \varphi = -\varphi \circ A_1 - \varphi$, then $g(h_1(X), Y) = g(\varphi X, A_1(Y)) + g(\varphi X, Y) = g(X, h_1(Y))$. Finally, for $i \geq 2$, $h_i \circ \varphi + \varphi \circ h_i = A_i \circ \varphi^2 - \varphi^2 \circ A_i = 0$ and

$$h_1 \circ \varphi = -\varphi \circ A_1 \circ \varphi - \varphi^2\,, \quad \varphi \circ h_1 = \varphi \circ A_1 \circ \varphi + \varphi^2$$

so $h_1 \circ \varphi + \varphi \circ h_1 = 0$. □

Proposition 1.3 *Let M^{2n+s} be an almost Kenmotsu f.pk-manifold with structure $(\varphi, \xi_i, \eta^i, g)$. For any $X \in \mathcal{X}(M^{2n+s})$, we have:*

1) $\nabla_X \xi_i = -\varphi h_i X$ *for any* $i \in \{2, \ldots, s\}$,

2) $\nabla_X \xi_1 = -\varphi^2(X) - \varphi h_1 X$,

3) $\nabla \eta^i = g \circ (\varphi \times h_i)$ *and* $\delta\eta^i = 0$ *for any* $i \in \{2, \ldots, s\}$,

4) $\nabla \eta^1 = g - \eta^k \otimes \eta^k + g \circ (\varphi \times h_1)$, $\delta\eta^1 = -2n$ *and* M^{2n+s} *cannot be compact.*

Proof. For $i \geq 2$, since $h_i = -\varphi \circ A_i$, we get $\varphi(\nabla_X \xi_i) = h_i(X)$ and applying φ, we obtain 1). Now, let $i = 1$. Then $h_1 = -\varphi \circ A_1 - \varphi$ gives $\varphi(\nabla_X \xi_1) = \varphi X + h_1(X)$ and applying φ we get 2). Finally, an easy computation gives 3) and 4). □

We obtain immediately the following result.

Corollary 1.1 *All the operators h_i vanish if and only if $\nabla \xi_1 = -\varphi^2$ and $\nabla \xi_i = 0$ for $i \in \{2, \ldots, s\}$. In such a case $\xi_2, \ldots, \xi_s$ are Killing vector fields and $\eta^2, \ldots, \eta^s$ are harmonic 1-forms.*

Proposition 1.4 *Let M^{2n+s} be an almost Kenmotsu f.pk-manifold with structure $(\varphi, \xi_i, \eta^i, g)$. Then for any $X, Y \in \mathcal{X}(M^{2n+s})$, we have:*

1) $\varphi(N(X,Y)) + N(\varphi X, Y) = 2\eta^k(X) h_k(Y)$,

2) $(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)(\varphi Y) = -\eta^1(Y)\varphi X - 2g(X, \varphi Y)\xi_1 - \eta^k(Y) h_k(X)$.

Proof. The first relation follows by direct computation, using $d\eta^i = 0$ and the definition of the h_i's. In particular, we get

$$g(N(\varphi X, Y), \xi_i) = 0\,, \quad N(Y, \xi_i) = 2\varphi h_i(Y)\,. \tag{1.2}$$

The second relation follows by (1.1) and 1). □

Finally, we consider $(2n+2)$-dimensional almost Kenmotsu $f.pk$-manifolds and compare them with locally conformal almost Kähler manifolds with parallel anti-Lee form, considered by Kashiwada in [9]. We recall that an almost Hermitian manifold (M, J, g) is locally conformal almost Kähler if and only if there exists a closed 1-form ω such that the Kähler 2-form Ω satisfies $d\Omega = 2\omega \wedge \Omega$. ω is the Lee form, $\bar{\omega} = -\omega \circ J$ the anti-Lee form and B, JB are the Lee and the anti-Lee vector fields.

We need a result essentially due to Goldberg and Yano ([7, 8]).

Theorem 1.1 *Let M be a $(2n+s)$-dimensional f.pk-manifold with structure (φ, ξ_i, η^i), and s even, $s = 2p$. The tensor field J defined by:*

$$J = \varphi + \sum_{i=1}^{p} (\eta^{2i-1} \otimes \xi_{2i} - \eta^{2i} \otimes \xi_{2i-1}) \tag{1.3}$$

is an almost complex structure on M and, if g is a φ-compatible metric, (M, J, g) is an almost Hermitian manifold with Kähler 2-form

$$\Omega = \Phi - 2\sum_{i=1}^{p} \eta^{2i-1} \wedge \eta^{2i} . \tag{1.4}$$

The previous theorem and Proposition 1.3 easily imply the following result.

Theorem 1.2 *Let M^{2n+2} be an almost Kenmotsu $f.pk$-manifold with structure $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ and let J be the tensor field defined by:*

$$J = \varphi + \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1 .$$

Then, (M^{2n+2}, J, g) is a locally conformal almost Kähler manifold with Lee 1-form η^1. The anti-Lee 1-form $\eta^2 = -\eta^1 \circ J$ is parallel if and only if $h_2 = 0$.

Theorem 1.3 *Let (M^{2n+2}, J, g) be a locally conformal almost Kähler manifold with unit Lee vector field B, anti-Lee vector field $J(B)$, Lee 1-form ω and parallel anti-Lee 1-form $\bar{\omega}$. Let φ be the tensor field defined by:*

$$\varphi = J - \omega \otimes JB + \bar{\omega} \otimes B .$$

Then $(M^{2n+2}, \varphi, B, JB, \omega, \bar{\omega}, g)$ is an almost Kenmotsu $f.pk$-manifold and the operator h_2 vanishes.

Proof. Theorem 1.1 ensures that g is a compatible metric for the $f.pk$-structure $(\varphi, B, JB, \omega, \bar{\omega})$. Note that $\omega, \bar{\omega}$ are both closed and the fundamental form is given by $\Phi = \Omega + 2\omega \wedge \bar{\omega}$, so that $d\Phi = d\Omega = 2\omega \wedge \Omega = 2\omega \wedge \Phi$. Finally, since $\nabla\bar{\omega} = 0$, we have $h_2 = 0$. $\square$

2 Distributions

We describe some distributions on an almost Kenmotsu $f.pk$-manifold of dimension $2n + s$, $s \geq 1$, with structure $(\varphi, \xi_i, \eta^i, g)$.

Proposition 2.1 *Let M^{2n+s} be an almost Kenmotsu $f.pk$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$. The integral manifolds of $\mathcal{D}$ are almost Kähler manifolds with mean curvature vector field $H = -\xi_1$. They are totally umbilical submanifolds of M^{2n+s} if and only if all the operators h_i's vanish.*

Proof. Let M' be an integral manifold of $\mathcal{D}$. The tensor fields φ and g induce an almost complex structure J and a Hermitian metric g' on M'. Then, for any $X, Y \in \mathcal{X}(M')$, we have $\Omega'(X,Y) = g'(X, JY) = g(X, \varphi Y) = \Phi(X,Y)$ and $d\Omega' = (d\Phi)_{|M'} = 0$, so M' is an almost Kähler manifold. Computing the second fundamental form, since the A_i's are the Weingarten operators in the directions ξ_i, we get, via Proposition 1.3,

$$\begin{aligned} \alpha(X,Y) &= \textstyle\sum_{i=1}^{s} g(A_i X, Y)\xi_i \\ &= g(\varphi^2(X) + \varphi h_1(X), Y)\xi_1 + \textstyle\sum_{i=2}^{s} g(\varphi h_i(X), Y)\xi_i \\ &= -g(X,Y)\xi_1 + \textstyle\sum_{i=1}^{s} g(\varphi h_i(X), Y)\xi_i . \end{aligned}$$

Fixed a local orthonormal frame $(e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n)$ in TM', applying Proposition 1.1, we obtain $trA_i = 0$ for $i \geq 2$, while $trA_1 = -2n$. Hence we get

$$H = \frac{1}{2n} \sum_{i=1}^{s} (trA_i)\xi_i = -\xi_1.$$

Finally, M' is totally umbilical if and only if $h_i = 0$ for each $i \in \{1, \ldots, s\}$. $\square$

Proposition 2.2 *In an almost Kenmotsu f.pk-manifold* $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ *the distribution* $\mathcal{D}$ *has Kähler leaves if and only if, for any* $X, Y \in \mathcal{X}(M^{2n+s})$,

$$(\nabla_X \varphi)(Y) = \sum_{i=1}^{s} \left(\eta^i(Y)\varphi A_i(X) - g(\varphi A_i(X), Y)\xi_i \right) . \tag{2.1}$$

Proof. Let M' be an integral manifold of $\mathcal{D}$ with the corresponding almost Kähler structure. By the Gauss equation $\nabla_X Y = \nabla'_X Y + \sum_{i=1}^{s} g(A_i(X), Y)\xi_i$, we have

$$(\nabla'_X J)Y = (\nabla_X \varphi)Y - \sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i \,, \tag{2.2}$$

so each integral manifold M' is Kähler if and only if

$$(\nabla_X \varphi)Y = \sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i \,,$$

for any $X, Y \in \mathcal{D}$. Therefore, if $\mathcal{D}$ has Kähler leaves, given $X, Y \in \mathcal{X}(M^{2n+s})$, the vector fields $X - \eta^j(X)\xi_j$ and $Y - \eta^j(Y)\xi_j$ belong to $\mathcal{D}$ and using $\nabla_{\xi_i}\varphi = 0$, we obtain

$$\begin{aligned} (\nabla_X \varphi)Y &= \eta^k(Y)(\nabla_X \varphi)(\xi_k) + \textstyle\sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i \\ &= -\eta^k(Y)\varphi(\nabla_X \xi_k) + \textstyle\sum_{i=1}^{s} g(A_i(X), \varphi Y)\xi_i \\ &= \textstyle\sum_{i=1}^{s}(\eta^i(Y)\varphi A_i(X) - g(\varphi A_i(X), Y)\xi_i) \,. \end{aligned}$$

Vice versa (2.1) and (2.2) imply $\nabla'_X J = 0$ on each integral manifold i.e. the Kähler condition. $\square$

Proposition 2.3 *Let* M^{2n+s} *be an almost Kenmotsu f.pk-manifold with structure* $(\varphi, \xi_i, \eta^i, g)$ *such that the integral manifolds of* $\mathcal{D}$ *are Kähler. Then* M^{2n+s} *is a Kenmotsu f.pk-manifold if and only if* $\nabla\xi_1 = -\varphi^2$ *and* $\nabla\xi_i = 0$ *for each* $i \in \{2, \ldots, s\}$.

Proof. Assuming that the structure is normal, we have $\mathcal{L}_{\xi_i}\varphi = 0$ for each $i \geq 1$, which implies $A_i \circ \varphi = \varphi \circ A_i$. Combining with Proposition 1.1 we get $A_i = 0$ and then $\nabla\xi_i = 0$ for $i \geq 2$, while $A_1 \circ \varphi = -\varphi$, so that $\nabla\xi_1 = -A_1 = -\varphi^2$. Vice versa, we notice that for $i \geq 2$, $\nabla\xi_i = 0$ implies $\mathcal{L}_{\xi_i}\varphi = 0$ and from $\nabla\xi_1 = -\varphi^2$ we get $A_1 = \varphi^2$ and $\mathcal{L}_{\xi_1}\varphi = 2A_1 \circ \varphi + 2\varphi = 0$. Hence, for any $i \in \{1, \ldots, s\}$ and $Z \in \mathcal{X}(M)$ we obtain $\varphi(N(\xi_i, Z)) = 0$ and $N(\xi_i, Z) \in \mathcal{D}^\perp$. Thus $N(\xi_i, Z) = 0$, since $g(N(\xi_i, Z), \xi_k) = 0$ for each $k \in \{1, \ldots, s\}$. Finally, $N(\xi_i, \xi_j) = 0$ is trivial and for $X, Y \in \mathcal{D}$, $N(X, Y) = 0$ since $N(X, Y) = N_J(X, Y) = 0$, the leaves of $\mathcal{D}$ being Kähler manifolds. $\square$

Proposition 2.4 *An almost Kenmotsu f.pk-manifold* M^{2+s} *such that* $\nabla\xi_1 = -\varphi^2$ *and* $\nabla\xi_i = 0$ *for* $i \geq 2$ *is a Kenmotsu f.pk-manifold.*

Proof. When $n = 1$, the integral manifolds of the distribution $\mathcal{D}$ are almost Kähler of dimension two and then they are Kähler. So we apply the previous proposition. □

Proposition 2.5 *The distribution $\mathcal{D}^\perp = <\xi_1, \ldots, \xi_s>$ is integrable, with totally geodesic flat leaves.*

Proof. Just note that $[\xi_i, \xi_j] = 0$ and $\nabla_{\xi_i}\xi_j = 0$. □

When $s \geq 2$, we can consider other distributions.

Proposition 2.6 *The distribution $\mathcal{D}' = \mathcal{D} \oplus < \xi_1 >$ is integrable. Its leaves are minimal almost Kenmotsu manifolds.*

Proof. Since $\mathcal{D}' = \{X \in \mathcal{X}(M) \mid g(X, \xi_i) = 0, i \geq 2\}$, and $d\eta^i = 0$, the distribution is clearly involutive with $(2n+1)$-dimensional integral manifolds. Let M' be an integral manifold, ∇' its Levi-Civita connection and φ' the tensor field defined by $\varphi'(X) = \varphi(X)$ for any $X \in \mathcal{X}(M')$. It is easy to verify that ${\varphi'}^2 = -I + \eta^1 \otimes \xi_1$ and $d\Phi' = 2\eta^1 \wedge \Phi'$ so M' is an almost Kenmotsu manifold. Now, for any $X, Y \in \mathcal{X}(M')$, $(\nabla'_X \varphi')(Y) = (\nabla_X \varphi)(Y) - \alpha(X, \varphi Y)$, α being the second fundamental form. Then, since for any $i \geq 2$ the Weingarten operators are $A_i = -\varphi \circ h_i$, the mean curvature vector field is given by

$$H = \frac{1}{2n+1} \sum_{i=2}^{s} \left(\sum_{k=1}^{n} (g(\varphi e_k, h_i e_k) + g(\varphi^2 e_k, h_i \varphi e_k)) + g(\varphi \xi_1, h_i \xi_1) \right) \xi_i = 0.$$

□

Proposition 2.7 *For any $i \in \{1, \ldots, s\}$, let $\mathcal{D}_i = Ker\, \eta^i$. Then:*

1) *for each $i \neq 1$, the distribution $\mathcal{D}_i = \mathcal{D} \oplus < \xi_1, \ldots, \hat{\xi_i}, \ldots \xi_s >$, where ξ_i is omitted, is integrable and the integral manifolds are minimal almost Kenmotsu $f.pk$-hypersurfaces;*

2) *the distribution $\mathcal{D}_1 = \mathcal{D} \oplus < \xi_2, \ldots, \xi_s >$ is integrable and its leaves are almost $\mathcal{C}$-manifolds with mean curvature $H = -\frac{2n}{2n+s-1}\xi_1$.*

Proof. The integrability of the described distributions follows from the condition $d\eta^i = 0$, for each $i \in \{1, \ldots, s\}$. Assume $i \neq 1$ and let M' be an integral manifold of $\mathcal{D}_i$. Then, the unique Weingarten operator is $A_{\xi_i} = A_i$, the second fundamental form is given by $\alpha(X, Y) = g(A_i X, Y)\xi_i$ and its trace vanishes since A_i anticommutes with φ and $A_i(\xi_q) = 0$ for $q \neq i$. So M' is minimal. By restriction, the structure on M determines an almost Kenmotsu $f.pk$-structure $(\varphi', \xi_1, \ldots, \hat{\xi_i}, \ldots \xi_s, \eta^1, \ldots, \hat{\eta}^i, \ldots, \eta^s, g')$ on M'. Now, suppose $i = 1$. The induced structure on each leaf of $\mathcal{D}_1$ has closed fundamental form and since $d\eta^i = 0$ for $i \geq 2$, we obtain an almost $\mathcal{C}$-manifold. The unique Weingarten operator A_1 verifies $A_1(X) = \varphi^2(X) + \varphi h_1(X)$ for any $X \in \mathcal{D}_1$. Hence $\alpha(X, Y) = -g(\varphi X, \varphi Y)\xi_1 - g(h_1 X, \varphi Y)\xi_1$ and $H = -\frac{2n}{2n+s-1}\xi_1$. □

Example 1 Let $(N^{2n}, J, \tilde{g})$, $n \geq 2$, be a strictly almost Kähler manifold and consider $\mathbb{R}^s \times N^{2n}$, with coordinates $t^1, \ldots, t^s$ on $\mathbb{R}^s$. For any $i \in \{1, \ldots, s\}$, we put $\xi_i = \frac{\partial}{\partial t^i}$, $\eta^i = dt^i$ and define the tensor field φ on $\mathbb{R}^s \times N^{2n}$ such that $\varphi X = JX$, if X is a vector field on N^{2n} and $\varphi X = 0$ if X is tangent to $\mathbb{R}^s$.

Furthermore, we consider the metric $g = g_0 + c\, e^{2t^1}\tilde{g}$, where g_0 denotes the Euclidean metric on $\mathbb{R}^s$ and $c \in \mathbb{R}^*_+$. Then, the warped product $\mathbb{R}^s \times_{f^2} N^{2n}$, $f^2 = ce^{2t^1}$, with the structure $(\varphi, \xi_i, \eta^i, g)$, is a strictly almost Kenmotsu *f.pk*-manifold. Namely, it is easy to verify that the 1-forms η^i's are dual of the ξ_i's with respect to g, $\varphi^2 = -I + \eta^i \otimes \xi_i$ and g is a compatible metric. Furthermore, we get $\Phi = ce^{2t^1} p_2^*(\tilde{\Omega})$, where p_2 is the projection on N^{2n} and $\tilde{\Omega}$ is the fundamental form of N^{2n}. Then, since $d\tilde{\Omega} = 0$, $d\Phi = 2ce^{2t^1} dt^1 \wedge p_2^*(\tilde{\Omega}) = 2dt^1 \wedge \Phi = 2\eta^1 \wedge \Phi$. Finally, since the torsion N_J does not vanish, N^{2n} being strictly almost Kähler, we obtain that the *f.pk*-structure is not normal.

Remark 1 In [14], Oguro and Sekigawa describe a strictly almost Kähler structure on the Riemannian product $\mathbb{H}^3 \times \mathbb{R}$. Thus the warped product $\mathbb{R}^s \times_{f^2} (\mathbb{H}^3 \times \mathbb{R})$, $f^2 = ce^{2t^1}$ is a $(4+s)$-dimensional strictly almost Kenmotsu *f.pk*-manifold.

Theorem 2.1 *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be an almost Kenmotsu f.pk-manifold. Assume that $h_i = 0$ for any $i \in \{1, \ldots, s\}$. Then, M^{2n+s} is locally a warped product $B^s \times_{f^2} N^{2n}$ where N^{2n} is an almost Kähler manifold, B^s is a flat manifold with coordinates $(t^1, \ldots, t^s)$ and $f^2 = ce^{2t^1}$, c a positive constant.*

Proof. We know that $T(M^{2n+s}) = Ker\,\varphi \oplus Im\,\varphi$ and the corresponding distributions $< \xi_1, \ldots, \xi_s >$ and $\mathcal{D}$ are both integrable. Their integral manifolds are totally geodesic flat manifolds and totally umbilical almost Kähler manifolds with second fundamental form $\alpha = -g \otimes \xi_1$, mean curvature $H = -\xi_1$, respectively. Thus, as a manifold, M^{2n+s} is locally a product $B \times F$ with $T(B) =< \xi_1, \ldots, \xi_s >$ and F is almost Kähler. We can choose a neighborhood with coordinates $(t^1, \ldots, t^s, x^1, \ldots, x^{2n})$ such that $\pi_*(\xi_i) = \frac{\partial}{\partial t^i}$, π denoting the projection onto B. Then $\pi : B \times F \to B$ is a C^∞-submersion with vertical distribution $\mathcal{V} = T(F)$ and horizontal distribution $\mathcal{H} = T(B)$. Moreover, the splitting $\mathcal{V} \oplus \mathcal{H}$ is orthogonal with respect to the metric g and, since, for any $p \in B \times F$, $g_p(\xi_i, \xi_j) = \delta_{ij} = g_{\pi(p)}(\pi_*\xi_i, \pi_*\xi_j)$, π is a Riemannian submersion. The horizontal distribution is integrable, so the O'Neill tensor A vanishes. Moreover $N = 2nH = -2n\xi_1$ is a basic vector field. Now, computing the trace-free part T^0 of the O'Neill tensor T, for any U, V vertical vector fields, we get:

$$\begin{aligned}
T^0_U V &= h(\nabla_U V) - \tfrac{1}{2n} g(U,V)N = \alpha(U,V) + g(U,V)\xi_1 = 0\,; \\
T^0_U \xi_1 &= T_U \xi_1 + \tfrac{1}{2n} g(N, \xi_1)U = v(\nabla_U \xi_1) - g(\xi_1, \xi_1)U = U - U = 0\,; \\
T^0_U \xi_i &= v(\nabla_U \xi_i) - g(\xi_1, \xi_i)U = 0\,, \quad i \geq 2\,.
\end{aligned}$$

Thus $T^0 = 0$ and $B \times F$, and then M^{2n+s}, is locally a warped product and $N = -2n\xi_1$ is π-related to $-\frac{2n}{f} grad_{g_0} f$, g_0 being the flat metric on B ([1], 9.104). It follows that $\frac{1}{f} grad\, f = \frac{\partial}{\partial t^1}$ which implies $f = ke^{t^1}$ and $f^2 = ce^{2t^1}$, with c a positive constant. Finally, the warped metric is locally given by $\sum_{i=1}^s dt^i \otimes dt^i + ce^{2t_1}\tilde{g}$, $\tilde{g}$ being an almost Kähler metric. $\square$

3 Conformal changes

Let M be an *f.pk*-manifold of dimension $2n+s$ with structure $(\varphi, \xi_i, \eta^i, g)$. A local conformal change of the structure is given by a family $(U_\alpha, \sigma_\alpha)_{\alpha \in A}$ where $(U_\alpha)_{\alpha \in A}$ is an open covering of M and, for any $\alpha \in A$, $\sigma_\alpha \in \mathcal{F}(U_\alpha)$. Putting

$$\varphi_\alpha = \varphi_{|U_\alpha}, \ \xi_i^\alpha = e^{\sigma_\alpha}\xi_{i|U_\alpha}, \ \eta_\alpha^i = e^{-\sigma_\alpha}\eta^i{}_{|U_\alpha}, \ g_\alpha = e^{-2\sigma_\alpha}g_{|U_\alpha}, \tag{3.1}$$

$(U_\alpha, \varphi_\alpha, \xi_i^\alpha, \eta_\alpha^i, g_\alpha)$ is an $f.pk$-manifold. Note that for $s = 1$ this is the concept of conformal change of an almost contact metric structure.

Definition 3.1 An $f.pk$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is said to be a locally conformal almost Kenmotsu $f.pk$-manifold if there exists a local conformal change $(U_\alpha, \sigma_\alpha)_{\alpha\in A}$ such that for each $\alpha \in A$, $(U_\alpha, \varphi_\alpha, \xi_i^\alpha, \eta_\alpha^i, g_\alpha)$ is an almost Kenmotsu $f.pk$-manifold.

It follows that for any $\alpha \in A$ we have $d\eta_\alpha^i = 0$ so that there exists a unique $k \in \{1, \ldots, s\}$, which a priori depends on α, such that $d\Phi_\alpha = 2\eta_\alpha^k \wedge \Phi_\alpha$, where Φ_α is defined by $\Phi_\alpha(X, Y) = g_\alpha(X, \varphi_\alpha Y) = e^{-2\sigma_\alpha}g(X, \varphi Y)$, for any vector fields X, Y on U_α. Moreover, on each U_α we easily obtain

$$2h_i^\alpha = \mathcal{L}_{\xi_i^\alpha}\varphi_\alpha = 2e^{\sigma_\alpha}h_i - (d\sigma_\alpha \circ \varphi_\alpha) \otimes \xi_i \,. \tag{3.2}$$

Definition 3.2 An $f.pk$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is said to be a globally conformal almost Kenmotsu $f.pk$-manifold if there exists a smooth function σ on M^{2n+s} such that, putting

$$\tilde{\varphi} = \varphi, \tilde{\xi}_i = e^\sigma \xi_i, \tilde{\eta}^i = e^{-\sigma}\eta^i, \tilde{g} = e^{-2\sigma}g,$$

$(M^{2n+s}, \tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}^i, \tilde{g})$ is an almost Kenmotsu $f.pk$-manifold.

Theorem 3.1 *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a locally conformal almost Kenmotsu $f.pk$-manifold and $s \geq 2$. Then, up to a rearrangement of the ξ_i's, there exists a function $\sigma \in \mathcal{F}(M^{2n+s})$ such that*

$$\begin{aligned} d\Phi &= 2(d\sigma + e^{-\sigma}\eta^1) \wedge \Phi, \\ d\eta^i &= d\sigma \wedge \eta^i \ , \quad i \in \{1, \ldots, s\} \,. \end{aligned} \tag{3.3}$$

Proof. Firstly we prove that there exists a closed 1-form ω such that $d\eta^i = \omega \wedge \eta^i$ for each $i \geq 1$. Namely, considering $\alpha \in A$, since $\eta_\alpha^i = e^{-\sigma_\alpha}\eta^i{}_{|U_\alpha}$, $d\eta_\alpha^i = 0$ implies $d\eta^i{}_{|U_\alpha} = d\sigma_\alpha \wedge \eta^i{}_{|U_\alpha}$. Thus, for $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, for any $i \in \{1, \ldots, s\}$ we get $d\sigma_\alpha \wedge \eta^i = d\sigma_\beta \wedge \eta^i$ and so $(d\sigma_\alpha - d\sigma_\beta) \wedge \eta^i = 0$. Therefore, for any vector field X and any $j \in \{1, \ldots, s\}$ we obtain

$$(d\sigma_\alpha - d\sigma_\beta)(X)\eta^i(\xi_j) = (d\sigma_\alpha - d\sigma_\beta)(\xi_j)\eta^i(X)$$

and choosing $X \in \mathcal{D}$ and $j = i$ we get $(d\sigma_\alpha - d\sigma_\beta)(X) = 0$. Furthermore, since $s \geq 2$, we can choose $X = \xi_k$ with $k \neq j$ obtaining $(d\sigma_\alpha - d\sigma_\beta)(\xi_k) = 0$. Hence, the local 1-forms $d\sigma_\alpha$ give rise to the required global 1-form ω.

Now, for any $\alpha \in A$, we have $d\Phi_\alpha = 2\eta_\alpha^t \wedge \Phi_\alpha$, and, denoting by ∇^α the Levi-Civita connection on (U_α, g_α), we have $\nabla^\alpha\xi_i^\alpha = -\delta_i^t\varphi^2 - \varphi \circ h_i^\alpha$. Let $\beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Then, $d\Phi_\beta = 2\eta_\beta^k \wedge \Phi_\beta$ and, in the intersection, $\nabla^\alpha\xi_i^\alpha = \nabla^\beta\xi_i^\beta$ implies $\delta_i^t\varphi^2 + \varphi \circ h_i^\alpha = \delta_i^k\varphi^2 + \varphi \circ h_i^\beta$. Now, assuming $t \neq k$, choosing $i = t$ and then $i = k$, we get

$$\varphi^2 + \varphi \circ h_i^\alpha = \varphi \circ h_i^\beta, \qquad \varphi \circ h_i^\alpha = \varphi^2 + \varphi \circ h_i^\beta,$$

which easily imply $\varphi^2 = 0$, so obtaining a contradiction. Thus we have $t = k$ and we can suppose that, up to a rearrangement, $d\Phi_\alpha = 2\eta_\alpha^1 \wedge \Phi_\alpha$, for each $\alpha \in A$. Finally, differentiating $\Phi_\alpha = e^{-2\sigma_\alpha}\Phi$, we get $d\Phi = 2(e^{-\sigma_\alpha}\eta^1 + d\sigma_\alpha) \wedge \Phi$, in U_α and, comparing with the analogous expression in U_β, σ_α and σ_β coincide in $U_\alpha \cap U_\beta$. Hence there exists a function $\sigma \in \mathcal{F}(M^{2n+s})$ such that $\omega = d\sigma$ and $d\Phi = 2(e^{-\sigma}\eta^1 + d\sigma) \wedge \Phi$. □

Proposition 3.1 *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $s \geq 2$, be an f.pk-manifold which admits a function $\sigma \in \mathcal{F}(M^{2n+s})$ such that (3.3) holds. Then, M^{2n+s} is a globally conformal almost Kenmotsu f.pk-manifold, with function σ.*

Proof. We put $\tilde{\varphi} = \varphi$, $\tilde{\xi}_i = e^\sigma \xi_i$, $\tilde{\eta}^i = e^{-\sigma}\eta^i$, $\tilde{g} = e^{-2\sigma}g$. Then one easily verifies that $(M^{2n+s}, \tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}, \tilde{g})$ is an $f.pk$-manifold with fundamental form $\tilde{\Phi} = e^{-2\sigma}\Phi$ and $d\tilde{\Phi} = 2\tilde{\eta}^1 \wedge \tilde{\Phi}$, $d\tilde{\eta}^i = 0$, for each $i \in \{1, \ldots, s\}$. □

Remark 2 The previous two results allow to state that an $f.pk$-manifold M^{2n+s}, with $s \geq 2$, is locally conformal almost Kenmotsu if and only if it is globally conformal almost Kenmotsu or, equivalently, if and only if there exists a function $\sigma \in \mathcal{F}(M^{2n+s})$ such that (3.3) holds. Moreover, assuming that M^{2n+s} is connected, the function σ is a constant if and only if M^{2n+s} is homothetic to an almost Kenmotsu $f.pk$-manifold. Furthermore, since the normality condition is not involved in the previous discussion, the same equivalences hold for locally (globally) conformal Kenmotsu $f.pk$-manifolds.

We remark that the hypothesis $s \geq 2$ is essential in the above results. Namely, when $s = 1$, Olszak proved that an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is locally conformal almost cosymplectic if and only if there exists a closed 1-form ω such that $d\Phi = 2\omega \wedge \Phi$ and $d\eta = \omega \wedge \eta$. Furthermore, M^{2n+1} is almost α-Kenmotsu if and only if it is locally conformal almost cosymplectic with $\omega = \alpha\eta$, α being a non-vanishing constant. This means that when $s = 1$ the almost α-Kenmotsu manifolds, with α constant, set up a subclass of the locally conformal almost cosymplectic manifolds. Now, we investigate the case $s \geq 2$ from this point of view.

We need the following characterization of locally conformal almost $\mathcal{C}$-manifolds.

Proposition 3.2 *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $s \geq 2$, be an f.pk-manifold. Then, M^{2n+s} is a locally conformal almost $\mathcal{C}$-manifold if and only if there exists a 1-form ω such that*

$$d\omega = 0, \;\; d\Phi = 2\omega \wedge \Phi, \;\; d\eta^i = \omega \wedge \eta^i, for\, each\, i \in \{1, \ldots, s\}\,. \tag{3.4}$$

Proof. Assuming that M^{2n+s} is a locally conformal almost $\mathcal{C}$-manifold, we apply the same technique as at the beginning of the proof of Theorem 3.1 and determine a closed 1-form ω such that $d\eta^i = \omega \wedge \eta^i$ for each $i \in \{1, \ldots, s\}$. The condition $d\Phi = 2\omega \wedge \Phi$ is achieved since an almost $\mathcal{C}$-manifold has closed fundamental form. Vice versa, ω being locally exact, we consider an open covering $(U_\alpha)_{\alpha \in A}$ such that, for any $\alpha \in A$, $\omega_{|U_\alpha} = d\sigma_\alpha$. Then, putting

$$\varphi_\alpha = \varphi_{|U_\alpha}\,, \;\; \xi_i^\alpha = e^{\sigma_\alpha}\xi_{i|U_\alpha}\,, \;\; \eta_\alpha^i = e^{-\sigma_\alpha}{\eta^i}_{|U_\alpha}\,, \;\; g_\alpha = e^{-2\sigma_\alpha}g_{|U_\alpha}\,,$$

it is easy to check that $(U_\alpha, \varphi_\alpha, \xi_i^\alpha, \eta_\alpha^i, g_\alpha)$ is an almost $\mathcal{C}$-manifold. □

Proposition 3.3 *The class of the almost Kenmotsu $f.pk$-manifolds of dimension $2n+s$, $s \geq 2$, is disjoint from the class of the locally conformal almost $\mathcal{C}$-manifolds.*

Proof. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $s \geq 2$, be an $f.pk$-manifold which is almost Kenmotsu and locally conformal almost $\mathcal{C}$-manifold. Then there exists a 1-form ω such that $d\Phi = 2\omega \wedge \Phi$, $d\eta^i = \omega \wedge \eta^i$ for each $i \in \{1, \ldots, s\}$. Furthermore, one has $d\Phi = 2\eta^1 \wedge \Phi$ and $d\eta^i = 0$. This implies $\omega \wedge \eta^i = 0$ and then, since $s \geq 2$, we get $\omega = 0$ and $\eta^1 \wedge \Phi = 0$. Choosing $X \in \mathcal{D}$, $\|X\| = 1$ and computing $(\eta^1 \wedge \Phi)(\xi_1, X, \varphi X)$ we get $\eta^1(\xi_1) = 0$ which is a contradiction. $\square$

Remark 3 It is also easy to verify that in dimension $2n+s$, $s \geq 2$, the locally conformal almost $\mathcal{C}$-manifolds set up a class which is disjoint from the class of locally conformal almost Kenmotsu $f.pk$-manifolds.

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