# Some Results on $\mathcal{K}$–Manifolds

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**Abstract**

Since a $\mathcal{K}$–manifold of dimension $2n+s$, with $s = 1$, is a quasi- Sasakian manifold, we extend to $\mathcal{K}$–manifolds some results due to Kanemaki. We introduce indicator tensors which allow us to characterize $\mathcal{C}$–manifolds and $\mathcal{S}$–manifolds and to state a local decomposition theorem. For some special subclasses of $\mathcal{K}$–manifolds we also state local decomposition theorems. After that, we give some results on products. Finally we define an $f$–structure on a hypersurface of a $\mathcal{K}$–manifold giving also an example of induced $\mathcal{K}$–structure.

**Mathematics Subject Classification**: 53C25, 53C15

**Key words**: $f$–structure, $\mathcal{K}$–manifold, hypersurface

## 1 Introduction and preliminaries

Let $M$ be a smooth manifold. A $f$–structure on $M$ is a non-vanishing tensor field $f$ of type (1,1) on $M$ of constant rank and such that $f^3 + f = 0$. This is a natural generalization of an almost complex structure on a manifold. In fact, if $f$ is of maximal rank, equal to the dimension of $M$, then $f$ is an almost complex structure. $f$–structures were introduced by K.Yano ([13]) and then intensively investigated. Particularly interesting are the $f$–structures with complemented frames ([2]) also called $f$–structures with parallelizable kernel (briefly $f.pk$–structures). A $f.pk$–manifold is a $(2n + s)$–dimensional manifold $M$ on which is defined a $f$–structure of rank $2n$ with complemented frames. This means that there exist on $M$ a tensor field $f$ of type (1,1) and global vector fields $\xi_1, \dots, \xi_s$ such that, if $\eta^1, \dots, \eta^s$ are the dual 1-forms then

$$f\xi_i = 0, \quad \eta^i \circ f = 0,$$

for any $i = 1, ..., s$ and

$$f^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i.$$

Balkan Journal of Geometry and Its Applications, Vol.7, No.1, 2002, pp. 43-62.

It is well known that in such conditions one can consider a Riemannian metric $g$ on $M$ such that for any $X, Y \in \mathcal{X}(M)$ the following equality holds:

$$g(X,Y) = g(fX, fY) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y).$$

Here $\mathcal{X}(M)$ denotes the module of differentiable vector fields on $M$. The metric $f.pk$–structure is called a $\mathcal{K}$–structure if the fundamental 2–form $F$, defined as usually as $F(X,Y) = g(X, fY)$, is closed and the normality condition holds, i.e. $N_f = [f,f] + \sum_{i=1}^{s} 2d\eta^i \otimes \xi_i = 0$, where $[f,f]$ denotes the Nijenhuis torsion of $f$.

If $d\eta^1 = \ldots = d\eta^s = F$, the $\mathcal{K}$–structure is called an $\mathcal{S}$–structure and $M$ an $\mathcal{S}$–manifold. Finally, if $d\eta^i = 0$ for all $i \in \{1, \ldots, s\}$, then the $\mathcal{K}$–structure is called $\mathcal{C}$–structure and $M$ is said a $\mathcal{C}$–manifold.

In section 2 we extend to $\mathcal{K}$–manifolds some results obtained by S. Kanemaki who proved that an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is a quasi–Sasakian manifold if and only if there exists a symmetric tensor field $A$ of type (1,1) commuting with $\varphi$ and verifying the condition

$$(\nabla_X \varphi)Y = -\eta(Y)AX + g(AX, Y)\xi,$$

where $\nabla$ is the Levi-Civita connection of $g$ and $X, Y$ are vector fields on $M$ (cf. [9]). Among all such $A$ there exists a unique $\overline{A}$, called the indicator, (cf. [9], page 108). Via the indicator $\overline{A}$, Kanemaki characterizes the Sasakian and cosymplectic structures and gives necessary and sufficient conditions for a quasi-Sasakian manifold to be locally a product of a Sasakian manifold and a Kähler manifold.

Our paper is organized in the following way. In the section 2 we consider a metric $f.pk$–manifold of dimension $2n + s$, $s \geq 1$, and we prove that such a manifold is a $\mathcal{K}$–manifold if and only if there exists a family of selfadjoint tensor fields $\underline{A}_1, \ldots, \underline{A}_s$ of type (1,1) commuting with $f$ and allowing a simple formula for $\nabla f$. Among all possible such families, we define the family of indicators $\overline{A}_1, \ldots, \overline{A}_s$, and we use them to give necessary and sufficient conditions for a $\mathcal{K}$–manifold to be an $\mathcal{S}$–manifold or a $\mathcal{C}$–manifold. Moreover, using the indicators, we give a necessary condition for a $\mathcal{K}$–manifold to be locally the product of a $\mathcal{K}$–manifold and a Sasakian manifold. In the section 3 we study the class of manifolds satisfying the conditions $d\eta^i = 0$ for some $i \in \{1, \ldots, s\}$ and $d\eta^i = F$ for the remaining indexes and we give a local decomposition theorem for such $\mathcal{K}$–manifolds. In the section 4 we show a construction of various structures on the product of two $\mathcal{K}$–manifolds. In section 5 we present a general way of inducing $f.pk$–structure on a hypersurface of a $\mathcal{K}$–manifold. Then we give a necessary and sufficient condition for a hypersurface to be a $\mathcal{K}$–manifold. We end with an explicit example of $\mathcal{K}$–structure on a hypersurface of $\mathbb{R}^6$.

## 2 Indicators of $\mathcal{K}$–manifolds

In the sequel we will denote by $\mathcal{D}$ the space of differentiable sections of the bundle $Imf =< \xi_1, \ldots, \xi_s >^{\perp}$ and by $\mathcal{D}^{\perp}$ the space of differentiable sections of the bundle $kerf =< \xi_1, \ldots, \xi_s >$ .

We begin with the following lemma which can be easily proved ([7]).

**Lemma 1** *Let $M$ be an $f.pk$–manifold of dimension $2n+s$ with structure $(f, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$. If $M$ is normal then we have:*

1. $[\xi_i, \xi_j] = 0$
2. $2(d\eta^j)(X, \xi_i) = -(L_{\xi_i}\eta^j)X = 0$
3. $L_{\xi_i} f = 0$
4. $d\eta^i(fX, Y) = -d\eta^i(X, fY)$

*for any $i, j \in \{1, \ldots, s\}$ and $X, Y \in \mathcal{X}(M)$*

**Theorem 1** *Let $M$ be a $f.pk$–manifold of dimension $2n+s$ with structure $(f, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$. Then $M$ is a $\mathcal{K}$–manifold if and only if:*

***a)*** $L_{\xi_i}\eta^j = 0, \quad for\ any\ i, j \in \{1, \ldots, s\}$

***b)*** *there exists a family of tensor fields of type $(1,1)$, $A_i$, $i \in \{1, \ldots, s\}$ such that*

1. $(\nabla_X f)Y = \sum_{i=1}^{s}\{g(A_iX, Y)\xi_i - \eta^i(Y)A_iX\}$
2. $A_i \circ f = f \circ A_i \quad for\ any \quad i \in \{1, \ldots, s\}$
3. $g(A_iX, Y) = g(X, A_iY) \quad for\ any \quad i \in \{1, \ldots, s\}$

**Proof**. Let us suppose that $M$ is a $\mathcal{K}$–manifold. Then, condition **a**) holds by the Lemma 1 and any $\xi_i$, $i \in \{1, \ldots, s\}$, is Killing. Moreover, the Levi-Civita connection verifies (cf. [2],[6])

$$(1) \qquad g((\nabla_X f)Y, Z) = \sum_{j=1}^{s}\{d\eta^j(fY, X)\eta^j(Z) - d\eta^j(fZ, X)\eta^j(Y)\}$$

for any $X, Y, Z \in \mathcal{X}(M)$ and from Lemma 1 we have

$$d\eta^j(fZ, \xi_i) = 0 \tag{2}$$

which also implies $d\eta^j(Z, \xi_i) = 0$. Using (1), (2), and the relation *4.* of Lemma 1, we obtain:

$$\begin{aligned} g(-f(\nabla_X \xi_i), Z) &= g((\nabla_X f)\xi_i, Z) = -\sum_{j=1}^{s} d\eta^j(fZ, X)\eta^j(\xi_i) \\ &= -d\eta^i(fX, Z) = -d\eta^i(fZ, X) \end{aligned}$$

and (1) can be written as

$$\begin{aligned} g((\nabla_X f)Y, Z) &= \sum_{j=1}^{s} \{g(f(\nabla_X \xi_j), Y)\eta^j(Z) - g(f(\nabla_X \xi_j), Z)\eta^j(Y)\} \\ &= \sum_{j=1}^{s} g(g(f(\nabla_X \xi_j), Y)\xi_j - \eta^j(Y)f(\nabla_X \xi_j), Z). \end{aligned}$$

It follows that

$$(\nabla_X f)Y = \sum_{j=1}^{s} \{g(f(\nabla_X \xi_j), Y)\xi_j - \eta^j(Y)f(\nabla_X \xi_j)\}.$$

This suggests to put, for any $i \in \{1, \ldots, s\}$, $\underline{A}_i = f \circ \nabla \xi_i$, i.e., for any vector field $X$ on $M$:

$$\underline{A}_i X = f(\nabla_X \xi_i) \tag{3}$$

so that $b.1$ is immediately verified. Since in a $\mathcal{K}$–manifold $\nabla_{\xi_i} f = 0$ (cf. [2]), we get $\underline{A}_j \xi_i = 0$ for any $i, j \in \{1, \ldots, s\}$. Now, from Lemma 1 we know that $L_{\xi_i} f = 0$. On the other hand we have

$$(L_{\xi_i} f)X = [\xi_i, fX] - f[\xi_i, X] = (\nabla_{\xi_i} f)X - \nabla_{fX} \xi_i + f(\nabla_X \xi_i).$$

Thus

$$-\nabla_{fX} \xi_i + f(\nabla_X \xi_i) = 0, \tag{4}$$

that is $\underline{A}_i(fX) = f(\underline{A}_i X)$ proving condition $b.2$.

Finally, since each $\xi_i$ is Killing, using (4) we obtain

$$\begin{aligned} g(\underline{A}_i X, Y) &= g(f(\nabla_X \xi_i), Y) = -g(\nabla_X \xi_i, fY) = g(\nabla_{fY} \xi_i, X) \\ &= g(f(\nabla_Y \xi_i), X) = g(\underline{A}_i Y, X). \end{aligned}$$

Conversely, we suppose that **a**) and **b**) hold. Then, an easy computation, using **b**.*3*, shows that

$$3dF = \sigma(\nabla_X F)(Y,Z) = -\sigma g((\nabla_X f)Y, Z) = 0,$$

where $\sigma$ denotes the cyclic sum with respect to $X, Y, Z$. Furthermore, since $f^2 = -I + \sum_{j=1}^{s} \eta^j \otimes \xi_j$, for any $X \in \mathcal{X}(M)$ we have

$$(\nabla_X f) \circ f + f \circ (\nabla_X f) = \sum_{j=1}^{s} ((\nabla_X \eta^j) \otimes \xi_j + \eta^j \otimes (\nabla_X \xi_j)),$$

and then for any $X, Y \in \mathcal{X}(M)$,

$$(\nabla_X f)(fY) + f((\nabla_X f)Y) = \sum_{j=1}^{s} \{(\nabla_X \eta^j)(Y)\xi_j + \eta^j(Y)(\nabla_X \xi_j)\}.$$

Putting $Y = \xi_i$ we obtain $f((\nabla_X f)\xi_i) = \sum_{j=1}^{s} ((\nabla_X \eta^j)\xi_i)\xi_j + \nabla_X \xi_i$. Using **b.***1* and the last equation we have

$$f\left(\sum_{j=1}^{s} \{g(A_j X, \xi_i)\xi_j - \eta^j(\xi_i) A_j X\}\right) = -\sum_{j=1}^{s} \eta^j(\nabla_X \xi_i)\xi_j + \nabla_X \xi_i,$$

which implies

$$f(A_i X) = -\nabla_X \xi_i + \sum_{j=1}^{s} \eta^j(\nabla_X \xi_i)\xi_j \tag{5}$$

Now, to prove the normality condition, using **b**.1, **b**.2 and **b**.3, we obtain, for any $X, Y \in \mathcal{X}(M)$

$$[f, f](X,Y) = \sum_{i=1}^{s} 2g(A_i(fX), Y)\xi_i$$

and since

$$2d\eta^i(X,Y) = g(Y, \nabla_X \xi_i) - g(X, \nabla_Y \xi_i), \quad \textit{for } i \in \{1, \ldots, s\},$$

we get

$$N_f(X,Y) = \sum_{i=1}^{s} \{2g(A_i f X, Y) + g(Y, \nabla_X \xi_i) - g(X, \nabla_Y \xi_i)\}\xi_i.$$

Then using (5) we can write

$$N_f(X,Y) = \sum_{i,j=1}^{s} \eta^j(\nabla_X \xi_i)\eta^j(Y)\xi_i - \eta^j(\nabla_Y \xi_i)\eta^j(X)\xi_i \tag{6}$$

which clearly gives $N_f(X,Y)=0$ for $X,Y\in\mathcal{D}$. Now, $L_{\xi_i}\eta^j=0$ implies $d\eta^j(X,\xi_i)=0$ for any $X\in\mathcal{X}(M)$, so $d\eta^j(\xi_k,\xi_i)=0$ and $\eta^j[\xi_k,\xi_i]=0$, i.e. $[\xi_k,\xi_i]\in\mathcal{D}$. Using (5) we easily get $\nabla_{\xi_k}\xi_i\in\mathcal{D}^\perp$ and consequently $[\xi_k,\xi_i]=0$ for any $k,i\in\{1,\dots,s\}$. Thus, $N_f(\xi_k,\xi_i)=-[\xi_k,\xi_i]=0$. Finally, for any $i\in\{1,\dots,s\}$ and $X\in\mathcal{D}$, (6) becomes

$$N_f(X,\xi_i)=\sum_{j,k=1}^{s}\eta^j(\nabla_X\xi_k)\eta^j(\xi_i)\xi_k=\sum_{k=1}^{s}\eta^i(\nabla_X\xi_k)\xi_k\in\mathcal{D}^\perp.$$

On the other hand from Lemma 1 we have

$$\eta^j(N_f(X,\xi_i))=-(L_{\xi_i}\eta^j)(X)=0$$

i.e. $N_f(X,\xi_i)\in\mathcal{D}$. We conclude that $N_f(X,\xi_i)=0$.

**Proposition 1** *Let $M$ be a $\mathcal{K}$–manifold and $A_k$, $k\in\{1,\dots,s\}$ a family of tensor fields as in the theorem 1. Then, for any $k\in\{1,\dots,s\}$ we have*

$$rk(\underline{A}_k)\le rk(A_k)\le rk(\underline{A}_k)+s.$$

*Moreover, the rank of each $\underline{A}_k$ is even.*

**Proof**. We observe that $\underline{A}_k$ and $A_k$ coincide on $\mathcal{D}$ and $\mathcal{D}^\perp\subset ker\underline{A}_k$. This implies $\dim kerA_k\le\dim ker\underline{A}_k\le\dim kerA_k+s$.

Now, consider $k\in\{1,\dots,s\}$ and $W_k=ker\underline{A}_k\cap\mathcal{D}$. If we put $l_k=\dim W_k$ we have that $\dim ker\underline{A}_k=l_k+s$. Since obviously $f(W_k)\subset W_k$ and the restriction $f:W_k\to W_k$ is an almost complex structure, $l_k$ is even. It follows that $rk\underline{A}_k=2n+s-(l_k+s)=2n-l_k$, that is, an even number.

**Definition 1** *Let $M$ be a $\mathcal{K}$–manifold. The family*

$$\overline{A}_k=\underline{A}_k+\eta^k\otimes\xi_k,\ \ k\in\{1,\dots,s\}\tag{7}$$

*is called the family of indicators of the $\mathcal{K}$–structure.*

It is easy to see that the family of indicators $\overline{A}_k$, $k\in\{1,\dots,s\}$ verifies **b**.1, $b.2, b.3$ of theorem 1. Moreover, we observe that

$$\overline{A}_k\xi_k=\xi_k,\ \overline{A}_k\xi_i=0\ for\ i\ne k.$$

which implies $rk\overline{A}_k=rk\underline{A}_k+1$, that is an odd number.

**Proposition 2** *Let $M$ be a $\mathcal{K}$–manifold. Then:*

***i)*** *$M$ is a $\mathcal{C}$–manifold iff $\overline{A}_k = \eta^k \otimes \xi_k$ for any $k \in \{1, \dots, s\}$.*

***ii)*** *$M$ is a $\mathcal{S}$–manifold iff for any $k \in \{1, \dots, s\}.\overline{A}_k = I - \sum_{i \neq k} \eta^i \otimes \xi_i$. In this case $rk\overline{A}_k = 2n + 1$.*

**Proof**. We observe that, for any $k \in \{1, \dots, s\}$, we have $d\eta^k(X, Y) = -g(X, \nabla_Y \xi_k)$, since any $\xi_k$ is Killing.

**i)** $M$ is a $\mathcal{C}$–manifold if and only if $d\eta^k = 0$ for any $k \in \{1, \dots, s\}$, i.e. $\nabla\xi_k = 0$. This is equivalent to $\underline{A}_k = 0$ and so to $\overline{A}_k = \eta^k \otimes \xi_k$.

**ii)** $M$ is an $\mathcal{S}$–manifold if and only if $d\eta^k = F$ for any $k \in \{1, \dots, s\}$, i.e. $\nabla\xi_k = -f$. Moreover, this is equivalent to

$$\underline{A}_k = f \circ \nabla\xi_k = -f^2 = I - \sum_{i=1}^{s} \eta^i \otimes \xi_i$$

and to

$$\overline{A}_k = \underline{A}_k + \eta^k \otimes \xi_k = I - \sum_{i \neq k} \eta^i \otimes \xi_i.$$

Finally, in this case, we observe that

$$X \in ker\overline{A}_k \Leftrightarrow X \in < \xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_s > .$$

Then $rk\overline{A}_k = 2n + s - (s - 1) = 2n + 1$.

**Theorem 2** *Let $M$ be a $\mathcal{K}$–manifold and $\overline{A}_k$, $k \in \{1, \dots, s\}$, the indicators of the structure. If there exists $i \in \{1, \dots, s\}$ such that $\overline{A}_i$ is parallel and has constant rank $2p + 1$, with $1 \leq p \leq n - 1$, then $M$ is locally the product of a $\mathcal{K}$–manifold with complemented frames $\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_s$ and a Sasakian manifold of dimension $2p + 1$.*

**Proof**. Let us suppose that $\overline{A}_i$ is parallel and has constant rank $2p + 1$ for a fixed $i \in \{1, \dots, s\}$. We note that for any $h, k \in \{1, \dots, s\}$ we have

$$g(\underline{A}_k X, \nabla_Y \xi_h) = -g(\underline{A}_k X, f(\underline{A}_h Y)) = -g(\overline{A}_k X, f(\overline{A}_h Y)). \tag{8}$$

With a straightforward calculation using (7), (8) and (3) we find that

$$\begin{aligned} (\nabla_X \overline{A}_i)Y &= \eta^i(Y)\nabla_X \xi_i + (\nabla_X \eta^i)(Y)\xi_i + (\nabla_X f)(\nabla_Y \xi_i) \\ &+ f(\nabla_X(\nabla_Y \xi_i)) - f(\nabla_{\nabla_X Y} \xi_i), \end{aligned}$$

and taking the scalar product of both sides with $\xi_i$, we obtain

$$\begin{aligned} g((\nabla_X \overline{A}_i)Y, \xi_i) &= (\nabla_X \eta^i)Y - g(\overline{A}_i X, f(\overline{A}_i Y)) \\ &= g(Y, \nabla_X \xi_i) - g(\overline{A}_i(\overline{A}_i X), fY) \\ &= -g(Y, f(\overline{A}_i X)) + g(f(\overline{A}_i^2 X), Y) \\ &= g(f(\overline{A}_i^2 X - \overline{A}_i X), Y). \end{aligned}$$

Since $\overline{A}_i$ is parallel, we obtain that $(f \circ (\overline{A}_i^2 - \overline{A}_i))X = 0$. Then $\overline{A}_i^2$ and $\overline{A}_i$ coincide on $\mathcal{D}$. On the other hand for any $k \in \{1, \ldots, s\}$ we have $\overline{A}_i^2 \xi_k = \overline{A}_i \xi_k$ and then $\overline{A}_i^2 = \overline{A}_i$. We put now $B = I - \overline{A}_i$. Obviously we have: $B^2 = B$, $\nabla B = 0$, $B$ is symmetric with respect to $g$, $B \circ f = f \circ B$ and $\overline{A}_i \circ B = B \circ \overline{A}_i = 0$. Then $B$ and $\overline{A}_i$ are the projectors of an almost product structure. Moreover $rk\overline{A}_i = 2p + 1$, and then $rkB = 2(n - p) + s - 1$. It is easy to verify that the distributions $Im\overline{A}_i$ and $ImB$ are orthogonal to each other and both are completely integrable with totally geodesic integral submanifolds. Let $N_1$ and $N_2$ be maximal integral submanifolds of the distributions $Im\overline{A}_i$ and $ImB$ respectively. We denote by $\varphi$ the tensor induced by $f$ on $N_1$. We prove that $N_1(\varphi, \xi, \eta, g_1)$, where $g_1$ is the induced metric on $N_1$, $\xi = \xi_i$, $\eta = \eta^i$, is a Sasakian manifold. Obviously $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, for any vector field $X \in \mathcal{X}(N_1)$ we have

$$\varphi^2 X = f^2 X = -X + \sum_{k=1}^{s} \eta^k(X)\xi_k = -X + \eta(X)\xi,$$

since for any $k \neq i$, $\xi_i \in ImB$ and $\eta^k(X) = g(X, \xi_k) = 0$. It follows that for any $X, Y$ tangent to $N_1$:

$$\begin{aligned} g_1(\varphi X, \varphi Y) &= g(fX, fY) = g(X, Y) - \sum_{h=1}^{s} \eta^h(X)\eta^h(Y) \\ &= g(X, Y) - \eta^i(X)\eta^i(Y) = g_1(X, Y) - \eta(X)\eta(Y). \end{aligned}$$

Now if $h \in \{1, \ldots, s\}$, $h \neq i$, $X, Y \in Im\overline{A}_i$, then $\overline{A}_h X = \underline{A}_h X = f(\nabla_X \xi_h)$. Moreover $B\xi_h = \xi_h$ and then $\nabla_X \xi_h = \nabla_X(B\xi_h) = B(\nabla_X \xi_h) \in ImB$ since $B$ is parallel. It follows that

$$g(\overline{A}_h X, Y) = g(f(\nabla_X \xi_h), Y) = g(B(f(\nabla_X \xi_h)), Y) = 0.$$

Finally we have

$$\begin{aligned} (\nabla_X \varphi)Y &= \sum_{h=1}^{s} \{g(\overline{A}_h X, Y)\xi_h - \eta^h(Y)\overline{A}_h X\} \\ &= g(X, Y)\xi_i - \eta^i(Y)X = g(X, Y)\xi - \eta(Y)X \end{aligned}$$

and $N_1$ is a Sasakian manifold.

Now let $\overline{f}$ be the restriction of $f$ to $N_2$ and $g_2$ the metric induced on $N_2$. Then $N_2(\overline{f}, \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_s, \eta^1, \ldots, \eta^{i-1}, \eta^{i+1}, \ldots, \eta^s, g_2)$ is a $\mathcal{K}$-manifold. This easily follows from theorem 1 since for all $X, Y$ tangent to $N_2$, $\eta^i(X) = 0$, $\overline{A}_i X = 0$ and

$$(\nabla_X \overline{f})Y = \sum_{k \neq i} \{g(\overline{A}_k X, Y)\xi_k - \eta^k(Y)\overline{A}_k X\}.$$

**Remark 1** *Let $\{\overline{A}_1, \overline{A}_2\}$ be the indicators of a $\mathcal{K}$–manifold $M$ of dimension $2n+2$ with structure $(f, \xi_1, \xi_2, \eta^1, \eta^2, g)$. Suppose that $\overline{A}_1$ is parallel and of constant rank $2p+1$. Then $M$ is locally the product of a Sasakian manifold and of a quasi-Sasakian manifold of dimension $2(n-p)+1$.*

## 3 Special classes of $\mathcal{K}$–manifolds

$\mathcal{C}$–manifolds and $\mathcal{S}$–manifolds represent in some sense very special cases of $\mathcal{K}$–manifolds, since the 2-forms $d\eta^i$ all vanish or all are equal to the fundamental 2-form $F$. In this section we will study the case $d\eta^i = 0$ for some $i \in \{1, \ldots, s\}$ and $d\eta^j = F$ for the other values of the index.

The first result from this point of view is due to Vaisman ([11, 12]) who proved that a generalized Hopf manifold is a $\mathcal{K}$–manifold of dimension $2n+2$ with structure $(f, \xi_1, \xi_2, \eta^1, \eta^2, g)$ where $\xi_1 = B$ is the Lee vector field and $\xi_2 = J(B)$.

**Theorem 3** *(Vaisman) Let $(M, J, g)$ be a generalized Hopf manifold with Lee form $\omega$ and unit Lee vector field $B$. If we put:*

$$\xi_1 = B,\ \xi_2 = J\xi_1,\ \eta^1 = \omega,\ \eta^2 = -\omega \circ J \text{ and } f = J + \eta^2 \otimes \xi_1 - \eta^1 \otimes \xi_2,$$

*then $(M, f, \xi_1, \xi_2, \eta^1, \eta^2, g)$ is a $\mathcal{K}$–manifold of dimension $2n+2$, such that $d\eta^1 = 0$, $d\eta^2 = F$, where $F$ is the fundamental 2-form of $f$.*

We prove that the converse is also true:

**Theorem 4** *Let $(f, \xi_1, \xi_2, \eta^1, \eta^2, g)$ be a $\mathcal{K}$–structure on a $(2n+2)$–dimensional manifold $M$ such that $d\eta^1 = 0$ and $d\eta^2 = F$. Then $M$ is a generalized Hopf manifold with Lee vector field $B = \xi_1$ and anti-Lee vector field $J(B) = \xi_2$.*

Actually the proof of the above theorem can be obtained as a corollary from the following Theorem 5, which together with Theorem 6 is essentially due to Goldberg and Yano. Namely, in [7] Goldberg and Yano proved that a globally framed $f$–manifold carries an almost complex structure in the even dimensional case and an almost contact structure in the odd dimensional case. Furthermore if the given $f$–structure is normal, then the induced structures are integrable and normal, respectively.

**Theorem 5** *Let* $(M, f, \xi_i, \eta^i, g)$*,* $i \in \{1, \ldots, s\}$ *be a* $\mathcal{K}$*–manifold of even dimension* $2n + s$*,* $s = 2p$*,* $p \geq 1$*. Then, the induced almost complex structure*

$$J = f + \sum_{i=1}^{p} (\eta^i \otimes \xi_{p+i} - \eta^{p+i} \otimes \xi_i)$$

*makes* $(M, g)$ *a Hermitian manifold. Moreover, if* $M$ *is a* $\mathcal{C}$*–manifold, then* $(M, J, g)$ *is Kähler.*

**Proof**. From Theorem 1 in [7] we know that $(M, J)$ is a complex manifold. It is easy to verify that $g$ is Hermitian and the Kähler form is given by

$$\Omega = F - \sum_{i=1}^{p} \eta^i \wedge \eta^{p+i}.$$

Then, since $dF = 0$, $d\Omega = -\sum_{i=1}^{p} d\eta^i \wedge \eta^{p+i} + \sum_{i=1}^{p} \eta^i \wedge d\eta^{p+i}$ Obviously, $d\eta^i = 0$ for each $i \in \{i, \ldots, 2p\}$ implies $d\Omega = 0$ and $(M, J, g)$ is Kähler.

**Corollary 1** *Let* $M$ *be a* $\mathcal{K}$*–manifold of dimension* $2n + 2$ *with structure* $(f, \xi_1, \xi_2, \eta^1, \eta^2, g)$ *such that* $d\eta^1 = 0$ *and* $d\eta^2 = F$*. Then* $M$ *is a generalized Hopf manifold with Lee vector field* $B = \xi_1$ *and anti-Lee vector field* $J(B) = \xi_2$*.*

**Proof**. Simple observe that the above theorem implies $\Omega = F - \eta^1 \wedge \eta^2$ and $d\Omega = \eta^1 \wedge d\eta^2 = \eta^1 \wedge F = \eta^1 \wedge \Omega$.

**Theorem 6** *Let* $(M, f, \xi_i, \eta^i, g)$*,* $i \in \{1, \ldots, s\}$*, be a* $\mathcal{K}$*–manifold of odd dimension* $2n + s$*,* $s = 2p + 1$*. Then the induced almost contact structure*

$$\overline{f} = f + \sum_{i=1}^{p} (\eta^i \otimes \xi_{p+i} - \eta^{p+i} \otimes \xi_i)$$

*makes* $(M, \overline{f}, \xi, \eta, g)$ *a normal almost contact manifold with* $\xi = \xi_{2p+1}$*,* $\eta = \eta^{2p+1}$*. Moreover, if* $d\eta^i = 0$ *for all* $i \in \{1, \ldots, 2p\}$ *we obtain a quasi-Sasakian manifold, which can not be Sasakian but turns out to be cosymplectic if* $d\eta^{2p+1} = 0$*.*

**Proof**. From Theorem 3 of [7] we know that $\overline{f}$ is a normal almost contact structure. It is easy to verify that the metric $g$ is compatible with $\overline{f}$. The fundamental 2-form $\overline{F}$ is given by

$$\overline{F} = F - \sum_{i=1}^{p} \eta^i \wedge \eta^{p+i}$$

and so, since $dF = 0$, we get

$$d\overline{F} = -\sum_{i=1}^{p} d\eta^i \wedge \eta^{p+i} + \sum_{i=1}^{p} \eta^i \wedge d\eta^{p+i}$$

which implies $d\overline{F} = 0$ if $d\eta^i$ vanishes for $i \in \{1, \dots, 2p\}$ and the induced structure is quasi-Sasakian. Obviously, $d\eta^{2p+1} = 0$ gives the cosymplectic case. Finally, to have a Sasakian manifold, we would have $d\eta^{2p+1} = \overline{F}$, i.e. $d\eta^{2p+1} = F - \sum_{i=1}^{p} \eta^i \wedge \eta^{p+i}$ which is impossible, since for $r \in \{1, \dots, p\}$ we obtain $d\eta^{2p+1}(\xi_r, \xi_{p+r}) = 0, \;\; F(\xi_r, \xi_{p+r}) = 0$ and

$$\sum_{i=1}^{p} \eta^i \wedge \eta^{p+i}(\xi_r, \xi_{p+r}) = \sum_{i=1}^{p} (\delta^i_r \delta^{p+i}_{p+r} - \delta^{p+i}_r \delta^i_{p+r}) = \sum_{i=1}^{p} \delta^i_r \delta^{p+i}_{p+r} = 1.$$

**Remark 2** *Supposing that $d\eta^i = 0$, for each $i \in \{1, \dots, s\}$, i.e. $M$ is a $\mathcal{C}$–manifold, then for any fixed $r \in \{1, \dots, s\}$ we can construct a $\overline{f}_r$ such that $(M, \overline{f}_r, \xi_r, \eta^r, g)$ is a cosymplectic manifold.*

Now we give a theorem of local decomposition.

**Theorem 7** *Let $M$ be a $\mathcal{K}$–manifold of dimension $2n + s$, $s \geq 2$, with structure $(f, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$. Suppose that $r$ 1-forms among the $\eta^i$ 's are closed, $1 \leq r \leq s$, whereas the remaining $t = s - r$ coincide with $F$. Then we have two cases:*

***a)*** *$t < r$ and $M$ is locally a Riemannian product of a $\mathcal{K}$–manifold $M_1$ of dimension $2n + 2t$ and of a flat manifold $M_2$ of dimension $r - t$;*

***b)*** *$t \geq r$ and $M$ is locally a Riemannian product of an $\mathcal{S}$–manifold $M_1$ of dimension $2n + t$ and a flat manifold $M_2$ of dimension $r$.*

**Proof**. In the first case let us put $p = r - t$, so that $s = 2t + p$. Without loss of generality we can suppose that $d\eta^1 = \ldots = d\eta^t = F$ and $d\eta^{t+1} = \ldots = d\eta^{2t+p} = 0$. Then we consider

$$\mathcal{D}_1 = \mathcal{D} \oplus < \xi_1, \dots, \xi_{2t} >, \quad \mathcal{D}_2 =< \xi_{2t+1}, \dots, \xi_{2t+p} > .$$

It is easy to verify that $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable distributions of dimension $2n + 2t$ and $p$ respectively. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ are autoparallel and totally geodesic with respect to the Levi-Civita connection. Let $M_1$ and $M_2$ be maximal integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Let $\varphi_1$ be the tensor field induced by $f$ on $M_1$ and $g_1$ the induced metric on $M_1$. Then it is easy to prove that $(M_1, \varphi_1, \xi_1, \ldots, \xi_{2t}, \eta^1, \ldots, \eta^{2t}, g_1)$ is a $\mathcal{K}$–manifold of dimension $2n + 2t$. Moreover $M_2$ is a flat manifold of dimension $p$ as required in our claim.

In the second case, supposing that $d\eta^1 = \ldots = d\eta^r = 0$, we put

$$\mathcal{D}_1 = \mathcal{D} \oplus < \xi_{r+1}, \ldots, \xi_s >; \quad \mathcal{D}_2 =< \xi_1, \ldots, \xi_r > .$$

Also in this case $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable autoparallel distributions of dimension $2n + t$ and $r$ respectively. Let $M_1$ and $M_2$ be maximal integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$. We denote by $\varphi_1$ the tensor field induced by $f$ on $M_1$ and $g_1$ the induced metric on $M_1$. Then $(M_1, \varphi_1, \xi_{r+1}, \ldots, \xi_s, \eta^{r+1}, \ldots, \eta^s, g_1)$ is an $\mathcal{S}$–manifold of dimension $2n+t$, while $M_2$ is a flat manifold of dimension $r$.

**Remark 3** *Note that in the case **a)**, the factor $M_1$ admits a Hermitian structure, via the Theorem 5, and it is a generalized Hopf manifold if $t = 1$. Moreover $M_1$ falls in the case **b)**, with $t = r$, so it is locally product of an $\mathcal{S}$–manifold of dimension $2n + t$ and a flat manifold of dimension $t$. This means that, in any case, $M$ can be viewed locally as a product of an $\mathcal{S}$–manifold and a flat manifold.*

# 4 $\mathcal{K}$–structures and products

Let $M_1$ and $M_2$ be differentiable manifolds and consider the product manifold $M = M_1 \times M_2$ with projections $p_1 : M \to M_1$, $p_2 : M \to M_2$.

**Proposition 3** *Let $(M_1, f_1, \xi_i, \eta^i, g_1)$, $i \in \{1, \ldots, s\}$ be a $\mathcal{K}$–manifold and $(M_2, g_2, J)$ a Kähler manifold of dimension 2m. Then the Riemannian product $M$ is a $\mathcal{K}$–manifold of dimension $2(n + m) + s$ with structure $(f, \overline{\xi_i}, \overline{\eta^i}, g)$ defined by $fX = f_1(p_{1*}X) + J(p_{1*}X)$ for any $X \in \mathcal{X}(M)$, $\overline{\xi_i} = (\xi_i, 0)$, $\overline{\eta^i} = p_1^* \eta^i$.*

**Proof**. We simply observe that $N_f = p_1^* N_{f_1} + p_2^*[J, J]$ and $F = p_1^* F_1 + p_2^* \Omega$.

**Proposition 4** *Let $(M_1, f_1, \xi_i, \eta^i, g_1)$, $(M_2, f_2, \zeta_i, \theta^i, g_2)$ $i \in \{1, \ldots, s\}$ be $\mathcal{K}$–manifolds of dimension 2n+s and 2m+s respectively and $M = M_1 \times M_2$ be their Riemannian product. Then the tensor field*

$$J = f_1 - \sum_{i=1}^{s} \theta^i \otimes \xi_i + f_2 + \sum_{i=1}^{s} \eta^i \otimes \zeta_i,$$

*where $f_1$, $f_2$, $\eta^i$ and $\theta^i$ stand for ${p_1}^*(f_1)$, ${p_1}^*(f_2)$, ${p_1}^*(\eta^i)$ and ${p_1}^*(\theta^i)$. makes $(M, J, g)$ a Hermitian manifold. Moreover if $M_1$ and $M_2$ are $\mathcal{C}$–manifolds, then $M$ is a Kähler manifold.*

**Proof**. We have

$$[J,J] = p_1^* N_{f_1} + p_2^* N_{f_2}, \quad \Omega = p_1^* F_1 + p_2^* F_2 + \sum_{i=1}^{s} \theta^i \wedge \eta^i,$$

which immediately give the result.

With the same meaning of symbols we have

**Proposition 5** *Let* $(M_1, f, \xi_i, \eta^i, g_1)$, $i \in \{1, \ldots, s\}$, $(M_2, f_2, \zeta_j, \theta^j, g_2)$, $j \in \{1, \ldots, t\}$ *be* $\mathcal{C}$*–manifolds of dimension* $2n+s$ *and* $2m+t$, $s < t$. *If we put on the Riemannian product* $M$ *of* $M_1$ *and* $M_2$*:*

$$f = f_1 - \sum_{j=1}^{s} \theta^j \otimes \xi_j + f_2 + \sum_{j=1}^{s} \eta^j \otimes \zeta_j$$

*then* $(M, f, \zeta_j, \theta^j, g)$, $j \in \{s+1, \ldots, t\}$, *is a* $\mathcal{C}$*–manifold of dimension* $2(n+m+s)+p$, $p = t - s$.

# 5 $f$–structures on hypersurfaces of a $\mathcal{K}$–manifold

Let $\widetilde{M}$ be a $(2n+s)$–dimensional $\mathcal{K}$–manifold with structure $(\widetilde{f}, \xi_i, \eta^i, g)$ and $M$ a hypersurface tangent to the $\xi_i$'s, i.e. for all $p \in M$, $\widetilde{\mathcal{D}}_p^{\perp} \subset T_pM$. We denote by $N$ the unit normal vector field to $M$ and put

$$\xi_{s+1} = \widetilde{f} N.$$

Then, since $\eta^i(N) = g(N, \xi_i) = 0$ for $i \in \{1, \ldots, s\}$, we have

$$g(\xi_{s+1}, \xi_{s+1}) = g(\widetilde{f}N, \widetilde{f}N) = g(N,N) - \sum_{i=1}^{s} \eta^i(N)\eta^i(N) = g(N,N) = 1$$
$$g(\xi_{s+1}, N) = g(\widetilde{f}N, N) = 0, \quad g(\xi_{s+1}, \xi_i) = \eta^i(\widetilde{f}N) = 0,$$

so that $\xi_{s+1}$ is tangent to $M$ and belongs to $\widetilde{\mathcal{D}}$, as well as $N$. We define a (1,1)-tensor field $f$ on $M$, putting for any $X \in \mathcal{X}(M)$

$$fX = \widetilde{f}X + \eta^{s+1}(X)N$$

where $\eta^{s+1}$ is the 1-form dual to $\xi_{s+1}$ on $M$ with respect to $g$. Clearly, since

$$g(\widetilde{f}X, N) = -g(X, \ \widetilde{f}N) = -g(X, \xi_{s+1}) = -\eta^{s+1}(X),$$

$fX$ represents the tangent part of $\widetilde{f}X$. Moreover it is easy to verify that

$$\widetilde{f}\xi_{s+1} = -N; \quad f\xi_i = 0, \quad \eta^i \circ f = 0, \quad for\ all \quad i \in \{1, \dots, s+1\}$$

and

$$f^2 = -I + \sum_{1=1}^{s+1} \eta^i \otimes \xi_i.$$

Finally, denoting again with $g$ the induced metric on $M$, we get

$$g(fX, fY) = g(X, Y) - \sum_{i=1}^{s+1} \eta^i(X)\eta^i(Y).$$

Thus we have just verified that $(M, f, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s'\}$, is a metric $f.pk$–manifold of dimension $2(n-1)+(s+1)$. As regards the fundamental 2-form, we get $F(X,Y) = \widetilde{F}(X,Y), \quad \forall\ X, Y \in \mathcal{X}(M)$ and consequently $dF = 0$ since $d\widetilde{F} = 0$. Now, we denote by $\alpha$ and $A_N$ the second fundamental form and the shape operator of the hypersurface $M$, respectively. Note that we have the splittings:

$$T(\widetilde{M}) = \widetilde{\mathcal{D}} \oplus <\xi_1, \dots, \xi_s> = \mathcal{D} \oplus <\xi_1, \dots, \xi_s, \xi_{s+1}> \oplus <N>$$
$$T(M) = \mathcal{D} \oplus <\xi_1, \dots, \xi_s, \xi_{s+1}>; \quad \widetilde{\mathcal{D}} = \mathcal{D} \oplus <\xi_{s+1}>$$

Now, looking for the link between the normality conditions for $f$ and $\widetilde{f}$, by a direct computation, we easily obtain, for any $X, Y \in \mathcal{D}$:

**a)** $N_f(X,Y) = N_{\widetilde{f}}(X,Y)$,

**b)** $\forall i \in \{1, \dots, s\} \quad N_f(X, \xi_i) = N_{\widetilde{f}}(X, \xi_i) - \eta^{s+1}([\widetilde{f}X, \xi_i])N$,

**c)** $N_f(X, \xi_{s+1}) = N_{\widetilde{f}}(X, \xi_{s+1}) + [\widetilde{f}X, N] - \widetilde{f}[X, N] - \eta^{s+1}([\widetilde{f}X, \xi_{s+1}])N$,

**d)** $\forall\ i \in \{1, \dots, s\} \quad N_f(\xi_{s+1}, \xi_i) = N_{\widetilde{f}}(\xi_{s+1}, \xi_i) - \widetilde{f}[N, \xi_i]$,

**e)** $\forall\ i, j \in \{1, \dots, s\} \quad N_f(\xi_i, \xi_j) = N_{\widetilde{f}}(\xi_i, \xi_j)$.

Hence, since $N_{\widetilde{f}} = 0$, we have that $f$ is a $\mathcal{K}$–structure of corank $s+1$ on $M$ if and only if

1. $\eta^{s+1}([\widetilde{f}X, \xi_i]) = 0, \quad \forall\ X \in \mathcal{D},\ \forall i \in \{1, \dots, s\}$,
2. $[\widetilde{f}X, N] - \widetilde{f}[X, N] - \eta^{s+1}([\widetilde{f}X, \xi_{s+1}])N = 0, \quad \forall\ X \in \mathcal{D}$,
3. $\widetilde{f}[N, \xi_i] = 0, \quad \forall\ i \in \{1, \dots, s\}$.

**Lemma 2** *The following properties hold:*

***i)*** $[N, \xi_i] = 0 \ \ \forall\ i \in \{1, \dots, s\}$,

***ii)*** $\eta^{s+1}([\widetilde{f}X, \xi_i]) = 0 \ \ \forall X \in \mathcal{D}, \ \forall i \in \{1, \dots, s\}$,

***iii)*** $\eta^{s+1}([\widetilde{f}X, \xi_{s+1}]) = \alpha(X, \xi_{s+1}), \quad \forall X \in \mathcal{D}$,

***iv)*** $\eta^{s+1}([fX, Y]) = \alpha(fX, fY) - \alpha(X, Y), \quad \forall \ X, Y \in \mathcal{D}$.

**Proof**. Since $\| N \| = 1$, $\xi_i$ is Killing and $\widetilde{\nabla}_{\xi_i}\xi_j = 0 \ \ \forall \ i, j \in \{1, \dots, s\}$, we have

$$g([N, \xi_i], N) = g(\widetilde{\nabla}_N \xi_i, N) - g(\widetilde{\nabla}_{\xi_i} N, N) = g(\widetilde{\nabla}_N \xi_i, N) = 0$$

$$g([N, \xi_i], \xi_j) = g(\widetilde{\nabla}_N \xi_i, \xi_j) - g(\widetilde{\nabla}_{\xi_i} N, \xi_j) = -g(\widetilde{\nabla}_{\xi_j} \xi_i, N) + g(N, \widetilde{\nabla}_{\xi_i} \xi_j) = 0.$$

On the other hand, for any $X$ orthogonal to $N$ and to the $\xi_i$ 's:

$$\begin{aligned} g([N, \xi_i], X) &= g(\widetilde{\nabla}_N \xi_i, X) - g(\widetilde{\nabla}_{\xi_i} N, X) = -g(\widetilde{\nabla}_X \xi_i, N) + g(N, \widetilde{\nabla}_{\xi_i} X) \\ &= -\alpha(X, \xi_i) + \alpha(\xi_i, X) = 0 \end{aligned}$$

and **i)** is proved. For **ii)**, since $\widetilde{f}(\widetilde{\nabla}_X \xi_i) = \widetilde{\nabla}_{\widetilde{f}X} \xi_i$, and $\widetilde{\nabla}_{\xi_i} \widetilde{f} = 0$, we have

$$\begin{aligned} g(\xi_{s+1}, [\widetilde{f}X, \xi_i]) &= g(\xi_{s+1}, \widetilde{\nabla}_{\widetilde{f}X} \xi_i) - g(\xi_{s+1}, \widetilde{\nabla}_{\xi_i} \widetilde{f}X) \\ &= g(\xi_{s+1}, \widetilde{f}(\widetilde{\nabla}_X \xi_i)) - g(\xi_{s+1}, \widetilde{f}(\widetilde{\nabla}_{\xi_i} X)) \\ &= g(N, \widetilde{\nabla}_X \xi_i) - g(N, \widetilde{\nabla}_{\xi_i} X) = 0. \end{aligned}$$

Since $\| \xi_{s+1} \| = 1$, using (1) we get

$$\begin{aligned} \eta^{s+1}([\widetilde{f}X, \xi_{s+1}]) &= g(\xi_{s+1}, \widetilde{\nabla}_{\widetilde{f}X} \xi_{s+1}) - g(\xi_{s+1}, \widetilde{\nabla}_{\xi_{s+1}} \widetilde{f}X) \\ &= -g(\xi_{s+1}, (\widetilde{\nabla}_{\xi_{s+1}} \widetilde{f})X) - g(\xi_{s+1}, \widetilde{f}(\widetilde{\nabla}_{\xi_{s+1}} X)) \\ &= g(N, \widetilde{\nabla}_{\xi_{s+1}} X) = \alpha(\xi_{s+1}, X). \end{aligned}$$

Finally, since $\widetilde{M}$ is a $\mathcal{K}$–manifold, we have that $\widetilde{f}((\widetilde{\nabla}_X \widetilde{f})Y) = 0 \ \forall X, Y \in \widetilde{\mathcal{D}}$. Then, $fX = \widetilde{f}X$, $fY = \widetilde{f}Y$ and

$$\begin{aligned} \eta^{s+1}([fX, Y]) &= g(\widetilde{\nabla}_{\widetilde{f}X} Y, \widetilde{f}N) - g(\widetilde{\nabla}_Y \widetilde{f}X, \widetilde{f}N) \\ &= -g(\widetilde{f}(\widetilde{\nabla}_{\widetilde{f}X} Y), N) + g(\widetilde{f}(\widetilde{\nabla}_Y \widetilde{f}X), N) \\ &= g(\widetilde{\nabla}_{\widetilde{f}X} \widetilde{f}Y, N) - g(\widetilde{\nabla}_Y X, N) = \alpha(fX, fY) - \alpha(Y, X). \end{aligned}$$

**Theorem 8** *The hypersurface $M$ with the structure $(f, \xi_i, \eta^i, g)$ just defined is a $\mathcal{K}$–manifold if and only if*

$$\forall \ X \in \mathcal{D}, \quad A_N(fX) = f(A_N X).$$

**Proof**. Using the lemma 2 in the relations, **a)**, **b)**, **c)**, **d)**, **e)**, we have that $M$ is a $\mathcal{K}$–manifold if and only if

$$[\widetilde{f}X, N] - \widetilde{f}[X, N] - \alpha(X, \xi_{s+1})N = 0 \tag{9}$$

for all $X \in \mathcal{D}$. Observe that $X \in \mathcal{D}$ implies $\widetilde{f}X = fX \in \mathcal{D}$, so that

$$\begin{aligned}[\widetilde{f}X, N] - \widetilde{f}[X, N] &= \widetilde{\nabla}_{\widetilde{f}X}N - \widetilde{\nabla}_N\widetilde{f}X - \widetilde{f}(\widetilde{\nabla}_X N) + \widetilde{f}(\widetilde{\nabla}_N X)\\ &= -A_N(\widetilde{f}X) - (\widetilde{\nabla}_N\widetilde{f})X + \widetilde{f}(A_N X)\end{aligned}$$

and, applying (1), $(\widetilde{\nabla}_N\widetilde{f})X \in <\xi_1, \dots, \xi_s>$ . Now,

$$(\widetilde{\nabla}_N\widetilde{f})X = \sum_{i=1}^{s}\eta^i((\widetilde{\nabla}_N\widetilde{f})X)\xi_i = \sum_{i=1}^{s}\alpha(\widetilde{f}X, \xi_i)\xi_i,$$

since $g(\xi_i, (\widetilde{\nabla}_N\widetilde{f})X) = g(\xi_i, \widetilde{\nabla}_N\widetilde{f}X) = -g(\widetilde{\nabla}_N\xi_i, \widetilde{f}X) = \alpha(\widetilde{f}X, \xi_i)$. Thus (9) is equivalent to

$$-A_N(\widetilde{f}X) + \widetilde{f}(A_N X) - \sum_{i=1}^{s}\alpha(\widetilde{f}X, \xi_i)\xi_i - \alpha(X, \xi_{s+1})N = 0$$

and to

$$- A_N(fX) + f(A_N)X - \sum_{i=1}^{s}\alpha(\widetilde{f}X, \xi_i)\xi_i - 2\alpha(X, \xi_{s+1})N = 0, \tag{10}$$

since

$$\eta^{s+1}(A_N X) = -g(\xi_{s+1}, \widetilde{\nabla}_X N) = g(\widetilde{\nabla}_X\xi_{s+1}, N) = \alpha(X, \xi_{s+1})$$

and

$$\widetilde{f}(A_N X) = f(A_N X) - \eta^{s+1}(A_N X)N = f(A_N X) - \alpha(X, \xi_{s+1})N.$$

Hence $N_f = 0$ implies **2')**, then $\alpha(X, \xi_{s+1}) = 0$ and, taking the scalar product with $\xi_h$, $h \in \{1, \dots, s\}$, $\alpha(f(X), \xi_h) = 0$ so that we obtain

$$A_N(fX) = f(A_N X) \quad \forall\ X \in \mathcal{D}.$$

Conversely, $A_N(fX) = f(A_N X)$ for any $X \in \mathcal{D}$ implies $A_N(fX) \in \mathcal{D}$. Thus $\eta^i(A_N(fX)) = \alpha(fX, \xi_i) = 0 \quad \forall\ i \in \{1, \dots, s+1\}$. Substituing $fX$ to $X$, we obtain $\alpha(X, \xi_i) = 0$ so that **2')** holds and $M$ is a $\mathcal{K}$–manifold.

**Remark 4** *The condition* $A_N(fX) = f(A_N X)$ *for any* $X \in \mathcal{D}$ *is obviously equivalent to* $\alpha(fX, Y) + \alpha(X, fY) = 0$ *for any* $X, Y \in \mathcal{D}$.

**Corollary 2** *$(M, f, \xi_i, \eta^i, g)$ $i \in \{i, \dots, s+1\}$ is a $\mathcal{K}$–manifold if and only if $\xi_{s+1}$ is a Killing vector field on $M$.*

**Proof**. Supposing that $M$ is normal, the general theory of $\mathcal{K}$–manifolds implies that $\xi_{s+1}$ is Killing on $M$. Conversely, supposing $\xi_{s+1}$ Killing, since for any $X, Y \in \mathcal{X}(M)$

$$g(\nabla_X \xi_{s+1}, Y) = g(\widetilde{\nabla}_X \xi_{s+1}, Y) - g(\alpha(X, \xi_{s+1})N, Y) = g(\widetilde{\nabla}_X \xi_{s+1}, Y)$$

we get, for each $X, Y \in \mathcal{X}(M)$

$$g(\nabla_X \xi_{s+1}, Y) + g(\nabla_Y \xi_{s+1}, X) = g(\widetilde{\nabla}_X \xi_{s+1}, Y) + g(\widetilde{\nabla}_X \xi_{s+1}, Y).$$

On the other hand, for each $X, Y \in \mathcal{D}$,

$$\begin{aligned} g(\widetilde{\nabla}_X \xi_{s+1}, Y) &= g(\widetilde{\nabla}_X \widetilde{f} N, Y) = g((\nabla_X \widetilde{f})N, Y) + g(\widetilde{f}(\widetilde{\nabla}_X N), Y) \\ &= g(\widetilde{\nabla}_X N, \widetilde{f} Y) = g(A_N X, fY) = \alpha(X, fY) \end{aligned}$$

and

$$g(\nabla_X \xi_{s+1}, Y) + g(\nabla_Y \xi_{s+1}, X) = g(\widetilde{\nabla}_X \xi_{s+1}, Y) + g(\widetilde{\nabla}_Y \xi_{s+1}, X) = \alpha(X, fY) + \alpha(Y, fX)$$

and by the Remark 4, $M$ is a $\mathcal{K}$–manifold.

We end with an example inspired by an example of Calin (cf. [4]). Consider on $\mathbb{R}^6$ with coordinates $(x^1, \dots, x^6)$ the tensor field $\widetilde{f}$ given by

$$\widetilde{f} = \sum_{i,h} \widetilde{f}_i^h dx^i \otimes \frac{\partial}{\partial x^h},$$

where

$$(\widetilde{f}_i^h) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2x^4 & 0 & 0 \end{pmatrix}.$$

We put $\xi_1 = \frac{\partial}{\partial x^6}$, $\xi_2 = \frac{\partial}{\partial x^5}$, $\eta^1 = dx^6 - 2x^4 dx^2$, $\eta^2 = dx^5 - 2x^3 dx^1$. The metric $g$ on $\mathbb{R}^6$ is given by

$$g = (g_{ij}) = \begin{pmatrix} 1 + 4(x^3)^2 & 0 & 0 & 0 & -2x^3 & 0 \\ 0 & 1 + 4(x^4)^2 & 0 & 0 & 0 & -2x^4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2x^3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2x^4 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that $(\mathbb{R}^6, \widetilde{f}, \xi_1, \xi_2, \eta^1, \eta^2, g)$ is a metric $f.pk$–manifold, with closed fundamental 2-form

$$F = -2dx^1 \wedge dx^3 + 2dx^2 \wedge dx^4$$

and satisfying the normality condition. Thus it is a $\mathcal{K}-$ manifold. Let $M$ be the hypersurface of $\mathbb{R}^6$ defined by the equations

$$x^1 = u^1, \quad x^2 = (u^3)^2, \quad x^3 = u^2, \quad x^4 = u^3, \quad x^5 = u^4, \quad x^6 = u^5.$$

Then, the local frame for $M$ is given by

$$\frac{\partial}{\partial u^1} = \frac{\partial}{\partial x^1}, \quad \frac{\partial}{\partial u^2} = \frac{\partial}{\partial x^3}, \quad \frac{\partial}{\partial u^3} = \frac{\partial}{\partial x^4} + 2u^3 \frac{\partial}{\partial x^2},$$
$$\frac{\partial}{\partial u^4} = \frac{\partial}{\partial x^5} = \xi_2, \quad \frac{\partial}{\partial u^5} = \frac{\partial}{\partial x^6} = \xi_1.$$

The unitary vector field normal to $M$ is given by $N = \widetilde{N}/\parallel \widetilde{N} \parallel$, where

$$\widetilde{N} = (\frac{\partial}{\partial x^2} + (1+4x^2)\frac{\partial}{\partial x^5}), \quad \parallel \widetilde{N} \parallel^2 = 2(1+4x^2)$$

and

$$\xi_3 = \widetilde{f}N = \left\{(1+4x^2)\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4} + 2x^4(1+4x^2)\frac{\partial}{\partial x^6}\right\} \frac{1}{\sqrt{2(1+4x^2)}}.$$

The tensor field $f$ on $M$ is given by

$$fX = \widetilde{f}X + g(X, \xi_3)N$$

and a $f$–adapted local frame is

$$\left\{E_1 = \frac{\partial}{\partial u^2}, \quad E_2 = f(E_1) = \frac{\partial}{\partial u^1} + 2u^2\xi_2, \quad \xi_1, \quad \xi_2, \quad \xi_3\right\}.$$

Now, to prove that $(M, f, \xi_1, \xi_2, \xi_3, \eta^1, \eta^2, \eta^3, g)$ is a $\mathcal{K}$–manifold, we prove that $\forall X \in \mathcal{D}, \quad A_N X = 0$ and we apply the Theorem 8. Now,

$$\begin{aligned}
\widetilde{\nabla}_{E_1}\widetilde{N} &= \sum_{i=1}^{6}\left\{\widetilde{\Gamma}^h_{32}\frac{\partial}{\partial x^h} + (1+4(x^4)^2)\widetilde{\Gamma}^h_{34}\frac{\partial}{\partial x^h}\right\} \\
\widetilde{\nabla}_{E_2}\widetilde{N} &= \sum_{i=1}^{6}\left\{\widetilde{\Gamma}^h_{12}\frac{\partial}{\partial x^h} + (1+4(x^4)^2)\widetilde{\Gamma}^h_{14}\frac{\partial}{\partial x^h} + \right. \\
&\quad \left. + 2x^3\widetilde{\Gamma}^h_{52}\frac{\partial}{\partial x^h} + 2x^3(1+4(x^4)^2)\widetilde{\Gamma}^h_{54}\frac{\partial}{\partial x^h}\right\}.
\end{aligned}$$

By a direct computation we obtain

$$\widetilde{\Gamma}^h_{32} = \widetilde{\Gamma}^h_{34} = \widetilde{\Gamma}^h_{12} = \widetilde{\Gamma}^h_{14} = \widetilde{\Gamma}^h_{52} = \widetilde{\Gamma}^h_{54} = 0,$$

so that $\widetilde{\nabla}_{E_1}N = \widetilde{\nabla}_{E_2}N = 0$ and then $A_N(E_1) = A_N(E_2) = 0$.

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