# Invariant Submanifolds and Their Second Fundamental Forms

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**Abstract**

We study the second fundamental form of invariant submanifolds of a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature. Then we shall show that some Kon's results are special cases of ours.

**Mathematics Subject Classification:** 53C15
**Key Words:** contact metric manifold, compact minimal submanifold, Simons' type formulae.

## 1 Introduction

In [12], Simons classified compact minimal submanifolds of a sphere into three categories with respect to the length of the second fundamental form. Then he got a very important formula (Simons' type formula) with respect to minimal submanifolds (see [12], pp. 81). Kon [9] studied, by using Simons' type formula, the pinching problem for the length of the second fundamental form of a compact invariant submanifold in Sasakian manifolds of constant $\bar{\phi}$-sectional curvatures and got the condition for its invariant submanifold to be totally geodesic. Moreover, he studied an invariant submanifold with trivial normal connection.

On the other hand, the present author [8] got the curvature tensor of a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature. The purpose of this paper is to lead Simons' type formula for the second fundamantal form of invariant submanifolds of a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature and to show that some Kon's results [9] are special cases of ours.

## 2 Contact metric manifolds

Let $\bar{M}$ be an $(2r+1)$-dimensional contact metric manifold and $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be its contact metric structure. Then we have

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$$\bar\phi^2 = -I + \bar\eta\otimes\bar\xi, \quad \bar\phi\bar\eta = 0, \quad \bar\eta\circ\bar\phi = 0, \quad \bar\eta(\bar\xi) = 1,$$
$$<\bar\phi\bar X, \bar\phi\bar Y> = <\bar X, \bar Y> - \bar\eta(\bar X)\bar\eta(\bar Y), \quad <\bar X, \bar\xi> = \bar\eta(\bar X),$$
$$d\bar\eta(\bar X, \bar Y) = <\bar\phi\bar X, \bar Y>, \quad \bar X, \bar Y \in \mathcal{X}(\bar M),$$

where we denote the metric tensor field by $<,>$ instead of $\bar g$ and the Lie algebra of vector fields on $\bar M$ by $\mathcal{X}(\bar M)$. On such a manifold we define $\bar h = -\frac{1}{2}\mathcal{L}_{\bar\xi}\bar\phi$ ($\mathcal{L}$ denotes the Lie differentiation). Then $\bar h$ is symmetric, $\bar h$ anti-commutes with $\bar\phi$ (i. e., $\bar\phi\bar h + \bar h\bar\phi = 0$), $\bar h\bar\xi = 0$, $\bar\eta\circ\bar h = 0$ and $\operatorname{Tr}\bar h = 0$, where $\operatorname{Tr}\bar h$ is the trace of $\bar h$. It is well-known that the vector field $\bar\xi$ is a Killing vector field if and only if $\bar h$ vanishes. Also

$$\bar\nabla_{\bar X}\bar\xi = \bar\phi\bar X + \bar\phi\bar h\bar X \quad (\text{and thus } \bar\nabla_{\bar\xi}\bar\xi = 0), \tag{2.1}$$

where $\bar\nabla$ is the Riemannian connection of $<,>$ (e.g., [7], cf. [1]). A contact metric manifold $\bar M$ for wich $\bar\xi$ is Killing is called a $K$-contact manifold. We also recall that on a $K$-contact manifold it is valid $\bar R(\bar X, \bar\xi)\bar\xi = \bar X - \bar\eta(\bar X)\bar\xi$, $\bar X \in \mathcal{X}(\bar M)$, where $\bar R$ is the curvature tensor of $\bar M$. A contact structure on $\bar M^{2r+1}$ gives rise to an almost complex structure on the product $\bar M^{2r+1}\times\mathbf{R}$, where $\mathbf{R}$ is the real line. If this almost complex structure is integrable, the contact metric manifold is said to be *Sasakian*. Equivalently, a contact metric manifold is *Sasakian* if and only if

$$\bar R(\bar X, \bar Y)\bar\xi = \bar\eta(\bar Y)\bar X - \bar\eta(\bar X)\bar Y, \quad X, Y \in \mathcal{X}(\bar M).$$

The *k-nullity distribution* (e.g., see [13]) of a Riemannian manifold $(M, <,>)$ for a real number $k$ is the distribution

$$N(k): p \to N_p(k) = \{\bar Z \in T_p(\bar M) | \bar R(\bar X, \bar Y)\bar Z = k(<\bar Y, \bar Z>\bar X - <\bar X, \bar Z>\bar Y)\}$$

for any $\bar X, \bar Y \in T_p(\bar M)$. From now on, if we don't refer something else, we suppose that $\bar M$ is a contact metric manifold with $\bar\xi$ belonging to the $k$-nullity distribution, i.e.,

$$\bar R(\bar X, \bar Y)\bar\xi = k(\bar\eta(\bar Y)\bar X - \bar\eta(\bar X)\bar Y), \quad \bar X, \bar Y \in \mathcal{X}(\bar M). \tag{2.2}$$

In particular, if $\bar M$ is Sasakian, then $k = 1$.

Then the following lemma is needed later ([11], [13] and [8]).

**Lemma 2.1** *Let $\bar M$ be a contact metric manifold with $\bar\xi$ belonging to the $k$-nullity distribution. Then we have*

$$\bar Q\bar\xi = (2nk)\bar\xi, \quad \bar h^2 = (k-1)\bar\phi^2 \quad (\textit{and hence } k \le 1); \tag{2.3}$$

$$(\bar\nabla_{\bar X}\bar\phi)\bar Y = -<\bar X + \bar h\bar X, \bar Y>\bar\xi + \bar\eta(\bar Y)(\bar X + \bar h\bar X) \quad (\textit{and thus } \bar\nabla_{\bar\xi}\bar\phi = 0); \tag{2.4}$$

$$\begin{aligned} -(\bar\nabla_{\bar X}\bar h)\bar Y + (\bar\nabla_{\bar Y}\bar h)\bar X &= (1-k)(2<\bar X, \bar\phi\bar Y>\bar\xi + \bar\eta(\bar X)\bar\phi\bar Y - \bar\eta(\bar Y)\bar\phi\bar X) \\ &+ \bar\eta(\bar X)\bar\phi\bar h\bar Y - \bar\eta(\bar Y)\bar\phi\bar h\bar X, \end{aligned} \tag{2.5}$$

*for any vector fields $\bar X, \bar Y \in \mathcal{X}(\bar M)$, where $\bar Q$ is the Ricci operator on $\bar M$.*

If $\bar{X}$ is a unit vector which is orthogonal to $\bar{\xi}$, we say that $\bar{X}$ and $\bar{\phi}\bar{X}$ span a $\bar{\phi}$-section. If the sectional curvature $\bar{H}(\bar{X})$ of all $\bar{\phi}$-sections is independent of $\bar{X}$, we say that $\bar{M}$ is of *pointwise constant $\bar{\phi}$-sectional curvature.*

Then we get the following theorem [8].

**Theorem 2.1** *Let $\bar{M}$ be a contact metric manifold with $\bar{\xi}$ belonging to the k-nullity distribution. If $\bar{M}$ is of pointwise constant $\bar{\phi}$-sectional curvature $\bar{H}$, then the curvature tensor has the following form:*

(2.6)

$$\begin{aligned}
4\bar{R}(\bar{X},\bar{Y})\bar{Z} &= (\bar{H}+3)(<\bar{Y},\bar{Z}>\bar{X}- <\bar{X},\bar{Z}>\bar{Y}) + (\bar{H}-1)(\bar{\eta}(\bar{X})\bar{\eta}(\bar{Z})\bar{Y} \\
&- \bar{\eta}(\bar{Y})\bar{\eta}(\bar{Z})\bar{X} + \bar{\eta}(\bar{Y}) <\bar{X},\bar{Z}> \bar{\xi} - \bar{\eta}(\bar{X}) <\bar{Y},\bar{Z}> \bar{\xi} + \\
&+ <\bar{\phi}\bar{Y},\bar{Z}> \bar{\phi}\bar{X} + \\
&- <\bar{\phi}\bar{X},\bar{Z}> \bar{\phi}\bar{Y} - 2 <\bar{\phi}\bar{X},\bar{Y}> \bar{\phi}\bar{Z} + 4(K-1)(\bar{\eta}(\bar{Y})\bar{\eta}(\bar{Z})\bar{X} \\
&- \bar{\eta}(\bar{X})\bar{\eta}(\bar{Z})\bar{Y} + \bar{\eta}(\bar{X}) <\bar{Y},\bar{Z})\bar{\xi} - \bar{\eta}(\bar{Y}) <\bar{X},\bar{Z})\bar{\xi}) \\
&+ 4(<\bar{h}\bar{Y},\bar{Z}>\bar{X}- <\bar{h}\bar{X},\bar{Z}>\bar{Y}+ <\bar{Y},\bar{Z}>\bar{h}\bar{X}- <\bar{X},\bar{Z}>\bar{h}\bar{Y} \\
&+ \bar{\eta}(\bar{X})\bar{\eta}(\bar{Z})\bar{h}\bar{Y} - \bar{\eta}(\bar{Y})\bar{\eta}(\bar{Z})\bar{h}\bar{X} + \bar{\eta}(\bar{Y}) <\bar{h}\bar{X},\bar{Z}> \bar{\xi} \\
&- \bar{\eta}(\bar{X}) <\bar{h}\bar{Y},\bar{Z}> \bar{\xi}) + 2(<\bar{h}\bar{Y},\bar{Z}>\bar{h}\bar{X}- <\bar{h}\bar{X},\bar{Z}>\bar{h}\bar{Y} \\
&+ <\bar{\phi}\bar{h}\bar{X},\bar{Z}>\bar{\phi}\bar{h}\bar{Y}- <\bar{\phi}\bar{h}\bar{Y},\bar{Z}>\bar{\phi}\bar{h}\bar{X}),
\end{aligned}$$

*where $\bar{H}$ is constant on $\bar{M}$ if $n \neq 1$.*

# 3 Invariant submanifolds

Let $\bar{M} = \bar{M}^{2r+1}(\bar{\phi},\bar{\xi},\bar{\eta},<,>)$ be a contact metric manifold. A submanifold $M = M^{2n+1}$ of $\bar{M}$ is said to be invariant if

(i) $\bar{\xi}$ is tangent to $M$ everywhere on $M$;

(ii) $\bar{\phi}X$ is tangent to $M$ for any tangent vector $X$ to $M$.

It is well-known that any invariant submanifold $M$ with induced structure tensor $(\phi,\xi,\eta,<,>)$ of $\bar{M}$ is also contact metric manifold and is minimal in $\bar{M}$ (e.g., [3], [4]). If we define an operator $h = -\frac{1}{2}\mathcal{L}_\xi\phi$ in an invariant submanifold $M(\phi,\xi,\eta,<,>)$ of a contact metric manifold $\bar{M}(\bar{\phi},\bar{\xi},\bar{\eta},<,>)$, then we have the results that $h$ is symmetric, $h$ anti-comutes with $\phi$ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$, $\eta \circ h = 0$ and Tr $h = 0$. Moreover, by the definition of $\bar{h}$, we can see that $\bar{h}X$ is tangent to $M$ and $\bar{h}X = hX$ for $X \in \mathcal{X}(M)$ (see [5]).

Let $\mathcal{X}(M)^\perp$ be the set of all vector fields normal to $M$. We denote by $\bar{\nabla}$ the covariant differentiation in $\bar{M}$ and $\nabla$ the one in $M$ determined by the induced metric on $M$. If we denote by $A$ the second fundamental form of $M$, then the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y), \quad \bar{\nabla}_X N = -A^N(X) + D_X N,$$

$X, Y \in \mathcal{X}(M)$, $N \in \mathcal{X}(M)$, where $< A^N(X), Y > = < B(X,Y), N >$ and $D$ is the linear connection in the normal bundle $T(M)^\perp$. The covariant derivative of $B$ is given by

$$(\nabla_X B)(Y,Z) = D_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

For $X, Y \in \mathcal{X}(M)$, the Gauss-Weingarten formulas implies

$$(\bar{R}(X,Y)Z)^T = R(X,Y)Z - A^{B(Y,Z)}(X) + A^{B(X,Z)}(Y), \tag{3.1}$$

where $T$ is the tangential projection on $M$. For any tangent vector field $W$ on $M$ we obtain the Gauss equation

$$\begin{aligned} < (\bar{R}(X,Y)Z, W > &= < R(X,Y)Z, W > - < B(Y,Z), B(X,W) > + \\ &+ < B(X,Z), B(Y,W) > . \end{aligned} \tag{3.2}$$

On the other hand, using the Gauss-Weingarten formulas, we get

$$(R(X,Y)N)^{\perp} = R^{\perp}(X,Y) - B(A^N(Y), X) + B(A^N(X), Y), \tag{3.3}$$

where $X, Y \in \mathcal{X}(M)$, $N \in \mathcal{X}(M)^{\perp}$ and $R^{\perp}(X,Y) = [D_X, D_Y] - D_{[X,Y]}$.

We need the following lemma later.

**Lemma 3.1** [([3], [5])]. *Let $M$ be an invariant submanifold of a contact metric manifold $\bar{M}$. Then the second fundamental form $A$ of $M$ satisfies $A^N\phi = -\phi A^N$, $A^N\xi = 0$, $N \in \mathcal{X}(M)^{\perp}$.*

Let $\bar{M}$ be a contact metric manifold with $\bar{\xi} = \xi$ belonging to the $k$-nullity distribution. Then, by (2.2), we have

$$\bar{R}(X,Y)\xi = k(\bar{\eta}(Y)X - \bar{\eta}(X)Y) = k(\eta(Y) - \eta(X)Y).$$

Here, from the Gauss equation and Lemma 3.1 we find $\bar{R}(X,Y)\xi = R(X,Y)\xi$. Therefore we get $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y)$, so that, $M$ is also a contact metric manifold with $\xi$ belonging to the $k$-nullity distribution. So, we have the following identities:

$$Q\xi = 2nk\xi, \quad h^2 = (k-1)\phi^2 \quad \text{(and hence } k \leq 1); \tag{3.4}$$

$$(\nabla_X \phi)Y = - < X + hX, Y > \xi + \eta(Y)(X + hX); \tag{3.5}$$

$$\begin{aligned} -(\nabla_X h)Y + (\nabla_Y h)X &= (1-k)(2 < X, \phi Y > \xi + \eta(X)\phi Y - \eta(Y)\phi X \\ &+ \eta(X)\phi hY - \eta(Y)\phi hX, \end{aligned} \tag{3.6}$$

for any vector fields $X, Y \in \mathcal{X}(M)$, where $Q$ is the Ricci operator on $M$.

Here we have the following.

**Lemma 3.2** *Let $\bar{M}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution. If $M$ is an invariant submanifold of $\bar{M}$, then the following are satisfied*

$$\phi A^N = A^{\phi N}; \tag{3.7}$$

$$(\bar{\nabla}_X h)N = 0; \tag{3.8}$$

$$hA^N = A^{\bar{h}N}; \tag{3.9}$$

$$A^N h = hA^N, \qquad X, Y \in \mathcal{X}(M), \quad N \in \mathcal{X}(M)^{\perp}. \tag{3.10}$$

**Proof**. At first we prove (3.7). From the Gauss formula we see that

$$\bar{\phi}(\bar{\nabla}_X Y) = \phi(\nabla_X Y) + \bar{\phi}B(X,Y). \tag{3.11}$$

On the orher hand, by (2.4) and the Gauss formula, we find

$$\begin{aligned} \bar{\phi}(\bar{\nabla}_X Y) &= \bar{\nabla}_X(\bar{\phi}Y) - (\bar{\nabla}_X\bar{\phi})Y \\ &= \bar{\nabla}_X(\phi Y) + < X + hX, Y > \xi - \eta(Y)X - \eta(Y)hX \\ &= (\nabla_X\phi)Y + \phi(\nabla_X Y) + B(X,\phi Y) \\ &+ < X + hX, Y > \xi - \eta(Y)X - \eta(Y)hX. \end{aligned} \tag{3.12}$$

Taking the normal parts of (3.11) and (3.12), we get $\bar{\phi}B(X,Y) = B(X,\phi Y)$. Thus we find $\phi A^N = A^{\phi N}$.

Next we prove (3.8) and (3.9). By Weingarten formula, we get the following two equations:

$$\bar{\nabla}_X(\bar{h}N) = -A^{\bar{h}N}(X) + D_X(\bar{h}N); \tag{3.13}$$

$$\begin{aligned} \bar{\nabla}_X(\bar{h}N) &= (\bar{\nabla}_X\bar{h})N + \bar{h}(\bar{\nabla}_X N) \\ &= (\bar{\nabla}_X\bar{h})N + \bar{h}(-A^N(X) + D_X N) \\ &= (\bar{\nabla}_X\bar{h})N - \bar{h}A^N(X) + \bar{h}D_X N). \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\mathcal{L}_\xi(\bar{\nabla}_X\bar{\phi})N - \bar{\nabla}_X(\mathcal{L}_\xi\bar{\phi})N - (\bar{\nabla}_{[\xi,X]}\bar{\phi})N = 0,$$

from which, we obtain

$$(\bar{\nabla}_X\bar{h})N = -\frac{1}{2}\mathcal{L}_\xi(\bar{\nabla}_X\bar{\phi})N + \frac{1}{2}(\bar{\nabla}_{[\xi,X]}\bar{\phi})N.$$

Here, from (2.4), we see that $(\bar{\nabla}_X\bar{\phi})N = (\bar{\nabla}_{[\xi,X]}\bar{\phi})N = 0$. Thus we get $(\bar{\nabla}_X\bar{h})N = 0$. Taking the tangential parts of (3.13) and (3.14), we find our result.

Last we prove (3.10). Using the Gauss formula, we get

$$\begin{aligned} \bar{\nabla}_X(\bar{h}Y) &= (\bar{\nabla}_X\bar{h})Y + \bar{h}(\bar{\nabla}_X Y) \\ &= (\bar{\nabla}_X\bar{h})Y + \bar{h}(\nabla_X Y + B(X,Y)) \\ &= (\bar{\nabla}_X\bar{h})Y + \bar{h}(\nabla_X Y) + \bar{h}(B(X,Y)). \end{aligned} \tag{3.15}$$

On the other hand, we find

$$\begin{aligned} \bar{\nabla}_X(\bar{h}Y) &= \bar{\nabla}_X(hY) \\ &= \nabla_X(hY) + B(X,hY) \\ &= (\nabla_X h)Y + h(\nabla_X Y) + B(X,hY). \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we have

$$(3.17) \qquad (\bar{\nabla}_X \bar{h})Y + \bar{h}(B(X,Y)) = (\nabla_X h)Y + B(X, hY).$$

Interchanging $X$ and $Y$ in the above equation, we find

$$(3.18) \qquad (\bar{\nabla}_Y \bar{h})X + \bar{h}(B(Y,X)) = (\nabla_Y h)X + B(Y, hX).$$

Subtracting (3.17) from (3.18) and using (2.5), we obtain $B(X, hY) = B(Y, hX)$, from which $A^N h = hA^N$.

Using Lemma 3.2, we get the following lemma.

**Lemma 3.3** *Let $\bar{M}^{2r+1}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution. If $M^{2n+1}$ is invariant submanifold of $\bar{M}^{2r+1}$. Then we have $Tr\, hA^2 = 0$ and $(Tr\, ha)^2 \leq 2n(1-k)\, Tr\, A^2$.*

**Proof**. At first, $h$ is represented by the following matrix form (see [13], pp. 446):

$$h = \begin{pmatrix} \mu_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \mu_n & & & & \\ & & & -\mu_1 & & & \\ & & & & \ddots & & \\ & & & & & -\mu_n & \\ 0 & & & & & & 0 \end{pmatrix}$$

where $\mu_i = \sqrt{1-k}$, $(1 \leq i \leq n)$. From (3.10) we can take the same orthogonal matrix as $h$'s one to orthogonalize $A$. Therefore, from Lemma 3.1, $A$ is expressed as in the following:

$$A = \begin{pmatrix} \nu_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \nu_n & & & & \\ & & & -\nu_1 & & & \\ & & & & \ddots & & \\ & & & & & -\nu_n & \\ 0 & & & & & & 0 \end{pmatrix}$$

Thus, we get Tr $hA = 2\sqrt{1-k}(\nu_1 + \cdots + \nu_n)$ and Tr $hA^2 = 0$, from which

$$\begin{aligned} (\mathrm{Tr}\ hA)^2 &= 4(1-k)(\nu_1 + \cdots + \nu_n)^2 \\ &\leq 4n(1-k)(\nu_1^2 + \cdots + \nu_n^2) \\ &= 2n(1-k)\mathrm{Tr}\ A^2. \end{aligned}$$

The following two theorems are known ([5], [2]).

**Theorem 3.1** *Let $M$ be an invariant submanifold of a contact manifold $\bar{M}$. Then $A^N = A^N h$ if and only if $(\nabla_X A^N)\xi = 0$.*

**Theorem 3.2** *Let $M^{2n+1}$ be a contact metric manifold with $R(X,Y)\xi = 0$ for all vectors $X$ and $Y$. Then $M^{2n+1}$ is locally the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of positive curvature 4.*

By Theorem 3.2, we get the following lemma.

**Lemma 3.4** *Let $\bar{M}^{2r+1}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution, and let $M^{2n+1}$ be an invariant submanifold of $\bar{M}$. If the second fundamental form $A$ of $M^{2n+1}$ is covariant constant, them either $M^{2n+1}$ is totally geodesic, or $M^{2n+1}$ is locally the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of constant curvature 4.*

**Proof**. By Theorem 3.1 and out assumption, we get $A^N = A^N h$. Therefore, from Lemma 2.1, i.e. follows that $A^n = A^N h = A^N h^2 = A^N(k-1)\phi^2$. By Lemma 3.1 we find

$$A^N X = (k-1)A^N(-X + \eta(X)\xi) = (1-k)A^N X.$$

This implies $kA^N X = 0$, from which $k = 0$ or $A^N X = 0$. In the case of $A^N X = 0$, $X \in \mathcal{X}(M)$, $M^{2n+1}$ is totally geodesic. When we have $k = 0$, by Theorem 3.2, $M^{2n+1}$ is locally the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of constant curvature 4.

We now suppose that the ambient manifold $\bar{M}$ is a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature $\bar{H}$. Then we have

(3.19)

$$\begin{aligned} R(X,Y)Z \;=\; & \frac{\bar{H}+3}{4}(< Y,Z > X - < X,Z > Y) + \frac{\bar{H}-1}{4}(\eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X + \eta(Y) < X,Z > \xi - \eta(X) < Y,Z > \xi \\ & + < \phi Y,Z > \phi X - < \phi X,Z > \phi Y - 2 < \phi X,Y > \phi Z) \\ & +(k-1)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \eta(X) < Y,Z > \xi \\ & -\eta(Y) < X,Z > \xi) + < hY,Z > X - < hX,Z > Y \\ & + < Y,Z > hX - < X,Z > hY + \eta(X)\eta(Z)hY \\ & -\eta(Y)\eta(Z)hX + \eta(Y) < hX,Z > \xi - \eta(X) < hY,Z > \xi) \\ & +\tfrac{1}{2}(< hY,Z > hX - < hX,Z > hY + < \phi hX,Z > \phi hY \\ & - < \phi hY,Z > \phi hX) + A^{B(Y,Z)}X - A^{B(X,Z)}Y, \end{aligned}$$

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0, \tag{3.20}$$

$$\begin{aligned} \mathrm{Ric}(Y,Z) \;=\; & \left(\frac{n(\bar{H}+3)+(\bar{H}-1)}{2} + (k-1)\right) < Y,Z > + \\ + \; & \left((2n-1)(k-1) - \frac{(n+1)(\bar{H}-1)}{2}\right)\eta(Y)\eta(Z) + \\ + \; & 2(n-1) < hY,Z > - \sum_i < B(X,e_i), B(Y,e_i) >, \end{aligned} \tag{3.21}$$

$$S_c = n^2(\bar{H}+3) + n(\bar{H}+1) + 4n(k-1) - \sum_{i,j} < B(e_i,e_j), B(e_i,e_j) >, \tag{3.22}$$

where $\{e_i\}$ is an orthonormal basis of $M$, Ric is the Ricci curvature on $M$ and $S_c$ is the scalar curvature on $M$ (see [8]).

# 4 Simon's type formula of invariant submanifolds

In this section, we assume that $\bar{M} = \bar{M}^{2r+1}(\bar{\phi}, \bar{\xi}, \bar{\eta}, <,>)$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with pointwise constant sectional curvature $\bar{H}$, and let $M = M^{2n+1}(\phi, \xi, \eta, <,>)$ be an invariant submanifold of $\bar{M}$. We put $p = r - n$. First we have (2.6). Since any invariant submanifold $M$ of a contact metric manifold $\bar{M}$ is minimal, its second fundamental form $A$ satisfies

$$\nabla^2 A = -A \circ \tilde{A} - A_* \circ A + \bar{R}(A) + \bar{R}', \tag{4.1}$$

where $\bar{R}(A)$ and $\bar{R}'$ are defined by setting
(4.2)

$$\begin{aligned} < \bar{R}(A)^W(X), Y > &= \sum_{i=1}^{2n+1} \{2 < \bar{R}(e_i,Y)B(X,e_i), W > + \\ &+ \ 2 < \bar{R}(e_i,X)B(Y,e_i), W > - \\ &- \ < A^W(X), \bar{R}(e_i,Y)e_i > - < A^W(Y), \bar{R}(e_i,X)e_i > \\ &< \ \bar{R}(e_i, B(X,Y))e_i, W > -2 < A^W(e_i), \bar{R}(e_i,X)Y >\}, \end{aligned}$$

$$< \bar{R}'(X), Y >= \sum_{i=1}^{2n+1} \left( < \bar{\nabla}_X(\bar{R})(e_i,Y)e_i, W > + < \bar{\nabla}_{e_i}(\bar{R})(e_i,X)Y, W > \right),$$

where $X, Y \in T_m(M)$, $W \in T_m(M)^\perp$ and $e_1, \ldots, e_{2n+1}$ is a frame in $T_m(M)$ (see [12], pp. 81). The operator $\tilde{A}$ is defined by

$$\tilde{A} = A^t \circ A, \tag{4.3}$$

where $A^t$ is the transpose of $A$. Let $v_1, v_2, \ldots, v_{2p}$ be a frame for $T_m(M)^\perp$. Then the operator $A_*$ is defined by setting $A_* = \sum_{i=1}^{2p} \mathrm{ad}\ A^i \mathrm{ad}\ A^i$. Here we denote $a^{v_i}$ by $A^i$ to simplify. $\tilde{A}$ and $A_*$ are symmetric and positive semi-definite operators. Since $\bar{M}$ is of constant $\bar{\phi}$-sectional curvature $\bar{H}$, from (2.1), (2.6), (3.3), (3.4), Lemma 3.1 and Lemma 3.2 we get, after lengthly computation,

$$\begin{aligned} <\bar{R}(A^W)(X), Y> + <\bar{R}'^W(X), Y> &= \tfrac{(n+2)\bar{H}+3n-2(k-1)}{2} < A^W X, Y > \\ &+ \ 2n < hA^W X, Y > -2\mathrm{Tr}hA^W < X, Y > \\ &+ \ 2\eta(X)\eta(Y)\mathrm{Tr}hA^W - \mathrm{Tr}hA^W <hX, Y> . \end{aligned}$$

Hence we get by (4.1) and Lemma 3.1

(4.4)
$$\begin{aligned} - < \nabla^2 A, A > &= < A \circ \tilde{A}, A > + < A_* \circ A, A > - \tfrac{(n+2)\bar{H}+3n-2(k-1)}{2} \|A\|^2 \\ ; &\quad -2n \sum_W \mathrm{Tr}\ h(A^W)^2 + \sum_W (\mathrm{Tr}\ hA^W)^2, \end{aligned}$$

where $\|A\|$ denotes the length of the second fundamental from $A$. From (4.3) Lemma 3.2 we can see easily that $\bar{\phi}\tilde{A} = \tilde{A}\bar{\phi}$. And $\tilde{A}$ is symmetric, positive semi-definite at each point $m \in M$. Hence we can choose an orthogonal basis $v_1, \ldots, v_p, \bar{\phi}v_1, \ldots, \bar{\phi}v_p$ of $T_m(M)^\perp$ with respect to which the matrix form of $\tilde{A}$ is of the form

$$A = \begin{pmatrix} a_1^2 & & & & & 0 \\ & \ddots & & & & \\ & & a_p^2 & & & \\ & & & a_1^2 & & \\ & & & & \ddots & \\ 0 & & & & & a_p^2 \end{pmatrix}$$

We can see that

$$\frac{1}{2p}(2\sum_{t=1}^{p} a_t^2)^2 \leq < A \circ \tilde{A}, A >= 2\sum_{t=1}^{p} a_t^4 = 2((\sum_{t=1}^{p} a_t^2)^2 - \sum_{t \neq s}^{p} a_t^2 a_s^2)$$

(also see [12], pp. 93, pp. 94). From this we have

(4.5)
$$\frac{1}{2p}\|A\|^4 \leq < A \circ \tilde{A}, A > \leq \frac{1}{2}\|A\|^4.$$

Here we define the operator $A^* = \sum_{i=1}^{2p} (A_i)^2$ which is also a symmetric, positive semi-definite operator. Taking a basis $e_1, \ldots, e_n, \phi e_1, \ldots, \phi e_n$, $\xi \in T_m(M)$, by Lemma 3.1 and (3.7) we have $\phi A^a A^b \phi A^a = -A^a A^b A^a$. Therefore we obtain

$$< A_* \circ A, A >= \sum_{a,b=1}^{2p} \|[A^a, A^b]\|^2 = 2\mathrm{Tr}\ (A^*)^2.$$

Since $A^*$ is symmetric positive semi-definite and $\phi A^* = A^*\phi$, by using a suitable frame $A^*$ is represented by the matrix form

$$A^* = \begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_{2n} & \\ 0 & & & 0 \\ , & & & \end{pmatrix}$$

where $\lambda_{n+t} = \lambda_t$, $\lambda_t \geq 0$. Then we have

$$< A_* \circ A, A >= 2\sum_{i=1}^{2n} \lambda_i^2 \geq \frac{1}{n}(\sum_{i=1}^{2n} \lambda_i)^2 = \frac{1}{n}\|A\|^4,$$

$$\begin{aligned} < A_* \circ A, A > &= 2((\sum_{i=1}^{2n} \lambda_i)^2 - \sum_{i \neq j}^{2n} \lambda_i \lambda_j) \\ &= 2(\sum_{i=1}^{2n} \lambda_i)^2 - 8 \sum_{t \neq s}^{n} \lambda_t \lambda_s - < A_* \circ A, A > . \end{aligned}$$

Consequently we get the following inequality

$$\frac{1}{n}\|A\|^4 \leq < A_* \circ A, A > \leq \|A\|^4. \tag{4.6}$$

Therefore (4.4), (4.5), and (4.6) and Lemma 3.3 imply

$$- < \nabla^2 A, A > \leq [\frac{3}{2}\|A\|^2 - \frac{1}{2}((n + 2\bar{H} + 3n) + (2n-1)(1-k)]\|A\|^2. \tag{4.7}$$

If $M$ is compact, then we get

$$- \int_M < \nabla^2 A, A > = \int_M < \nabla A, \nabla A > .$$

**Theorem 4.1** *Let $\bar{M}^{2r+1}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature $\bar{H}$, and let $M^{2n+1}$ be a compact invariant submanifold of $\bar{M}^{2r+1}$. Then either $M^{2n+1}$ is totally geodesic in $\bar{M}^{2r+1}$, or $M^{2n+1}$ is locally the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of constant curvature $4$, or at some point $m \in M^{2n+1}$ the following inequality holds*

$$\|A\|^2(m) > \frac{1}{3}((n+2)\bar{H} + 3n - 2(2n-1)(1-k)).$$

**Proof**. From (4.7), we get

$$\int_M [\frac{3}{2}\|A\|^2 - \frac{1}{2}((n+2)\bar{H} + 3n) + (2n-1)(1-k)]\|A\|^2 \geq \int_M < \nabla A, \nabla A > \geq 0.$$

Suppose $\|A\|^2 \leq \frac{1}{3}((n+2)\bar{H} + 3n - 2(2n-1)(1-k))$ everywhere. Then the second fundamental form of $M^{2n+1}$ is covariant constant, from which $M^{2n+1}$ is totally geodesic, or $M^{2n+1}$ is locally the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dinemsional manifold of constant curvature 4. Therefore, except for this possibility, at some point $m \in M^{2n+1}$,

$$\|A\|^2(m) > \frac{1}{3}((n+2)\bar{H} + 3n - 2(2n-1)(1-k)).$$

If $\bar{M}^{2r+1}$ is of constant $\bar{\phi}$-sectional curvature $\bar{H}$, then the scalar curvature $S_c$ of $M^{2n+1}$ is given by (3.22). Consequently we get the following corollary.

**Corollary 4.1** *Under the same assumption as in Theorem4.1, $M^{2n+1}$ is totally geodesic in $\bar{M}^{2r+1}$, or $M^{2n+1}$ is locally the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of constant curvature $4$, or at some point $m \in M$,*

$$S_c(m) < n^2(\bar{H}+3) + n(\bar{H}+1) + 4n(k-1) - \frac{1}{3}((n+2)\bar{H} + 3n - 2(2n-1)(1-k)).$$

**Remark 4.1** *In the case of which $\bar{M}^{2r+1}$ is Sasakian, we have $k = 1$. Then Theorem 4.1 and Corollary 4.1 become the results of Kon* ([9], pp.136).

# 5 Invariant submanifolds with trivial normal connection

In this section, we study, in the same way as Kon leads, invariant submanifolds with trivial normal connection. Let $\bar{M}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature $\bar{H}$, and let $M$ be an invariant submanifold of $\bar{M}$. Throughout in this section, we assume that the normal connection of $M$ trivial, i. e., $R^{\perp} = 0$. By the assumption and (3.3) we obtain

$$< (\bar{R}(X,\phi Y)V)^{\perp}, \bar{\phi}V >= - < B(A^V(\phi Y), X), \phi V > + < B(A^V(X), \phi Y), \bar{\phi}V >,$$

where $X, Y \in T_m(M)$ and $V$ is a unit vector in $T_m(M)^{\perp}$. From (2.6) the left hand side of this becomes

$$< (\bar{R}(X,\phi Y)V)^{\perp}, \bar{\phi}V >= \frac{1}{2}(1-\bar{H}) < \phi X, \phi Y >,$$

and by Lemma 3.1 and Lemma 3.2 we see that

$$- < B(A^V(\phi Y), X)\bar{\phi}V > + < B(A^V(X), \phi Y), \bar{\phi}V >= 2 < A^V(X), A^V(Y) > .$$

Hence we get

$$(1-\bar{H}) < \phi X, \phi Y >= 4 < A^V(X), A^V(Y) > . \tag{5.1}$$

From this we obtain the following result.

**Proposition 5.1** *Let $\bar{M}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature $\bar{H}$. If the normal connection of an invariant submanifold $M$ of $\bar{M}$ is trivial, then $\bar{H} \leq 1$ and equality holding if and only if $M$ is totally geodesic in $\bar{M}$.*

**Lemma 5.1** *Let $\bar{M}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature $\bar{H}$, and let $M$ be an invariant submanifold of $\bar{M}$ with trivial normal connection. Let $V$ and $W$ be unit vectors in $T_m(M)^{\perp}$. If $V$ is orthogonal to $W$, then*

$$A^V A^W + A^W A^V = 0. \tag{5.2}$$

**Proof**. We may construct a new unit vector $(V+W)/\sqrt{2}$ in $T_m(M)^{\perp}$. In view of (5.1) we see that for any vectors $X, Y \in T_m(M)$

$$< A^V A^W(X), Y > + < A^W A^V(X), Y >= 0,$$

which proves (5.2).

**Theorem 5.1** *Let $\bar{M}^{2r+1}$ be a contact metric manifold with $\bar{\xi}$ belonging to the $k$-nullity distribution and with constant $\bar{\phi}$-sectional curvature $\bar{H}$. If $M^{2n+1}$ is invariant submanifold of $\bar{M}^{2r+1}$ and the codimension of $M$ is greater than 2, then the following conditions are equivalent:*

*(I) the normal connection of $M$ is trivial, i.e., $R^{\perp} = 0$;*

*(II) $\bar{H} = 1$ and $M$ is totally geodesic in $\bar{M}$.*

**Proof.** From (2.6) and (3.3) it is clear that the condition (II) implies the condition (I). Let us assume the condition (I) and that $M$ is not totally geodesic in $\bar{M}$. Let $e_1, \ldots, e_{2n}, \xi$ be a $\phi$-basis in $T_m(M)$ such that $e_{n+i} = \phi e_i$. If $A^V(e_i) = 0$ for some unit $V \in T_m(M)$, then (5.1) implies that $M$ is totally geodesic. Therefore $A^V(e_i) \neq 0$ for any $V$ and $e_i$. From (5.1) we can see that $A(e_1), \ldots, A(e_{2n}$ are linearly independent. On the other hand, by (3.3) and (5.2) we get

$$< (\bar{R}(X,Y)V)^{\perp}, W >= 2 < A^V(X), A^W(Y) >,$$

where $X, Y \in T_m(M)$ and $W$ is a unit vector in $T_m(M)^{\perp}$ which is orthogonal to $V$. Using (2.6), we obtain

$$< (\bar{R}(X,Y)V)^{\perp}, W >= \frac{1}{2} << X, \phi Y > \bar{\phi} V, W > .$$

Hence we have

$$(1 - \bar{H}) << X, \phi Y > \bar{\phi} V, W) = 4 < A^V(X), A^W(Y) > . \tag{5.3}$$

If $2p > 2$, we can take $W$ which is orthogonal to $V$ and $\bar{\phi} V$. Then regarding to (5.3), it follows that $< A^V(X), A^V(Y) >= 0$ for any $X, Y \in T_m(M)$. And by the assumption, $A^W(e_i) \neq 0$ for any $i$ $(i = 1, \ldots, 2n)$. Therefore $A^V(e_i)$ is orthogonal to $A^W(e_j)$ for any $i, j$. Consequently $A^V(e_1), \ldots, A^W(e_{2n}), A^W(e_1), \ldots, A^W(e_{2n})$ are linearly independent. But each $A^V(e_i)$ and $A^W(e_i)$ are in $T_m(M)$ and we have $\dim T_m(M) = 2n + 1$. This is the contradiction. Therefore $M$ is totally geodesic in $\bar{M}$ and hence $\bar{H} = 1$.

**Remark 5.1** *In the case of which $\bar{M}$ is Sasakian, Theorem 5.1 turns out a result of Kon* ([*9*], *pp.*138).

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