Some Generalizations of a Theorem of Dold

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Abstract

We obtain generalizations of a theorem of Dold to G-cohomology theories defined on some categories of G-spaces,where G is a compact Lie group.

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Introduction

An old result of Dold says that if $h^\star$ is a generalized multiplicative cohomology theory defined on the category of CW-complexes and suppose $h^\star(X)$ is a projective module over $h^\star(point)$, then the product map

$$h^\star(X) \otimes_{\overline{h}} h^\star(Y) \to h^\star(X \times Y)$$

is an isomorphism for any Y, where $\overline{h}$ denotes $h^\star(point)$.

The generalization of this theorem is not trivial. There is a well-known example of L. Hodgkin (see [4]): although $h^\star_G(X)$ is a free $h^\star_G(point)$-module, this does not imply, in general, that the above product map is an isomorphism.

The beginning steps towards the generalization of Dold's Theorem come from R. M. Seymour who considered in [7] G-cohomology theories which are "complete" with regard to a family S of closed subgroups of G.

In this paper, we use one of the two notions of S-completeness defined by Seymour and we state a generalization of the Dold Theorem in terms of conditions concerning S and the associated H-cohomology theories for $H \in S$. Unfortunately we did not obtain satisfactory information regarding the category $\mathcal{C}$ where $h^\star_G$ is defined.

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Let us recall some elements which we shall need. First, we shall use the words *generalized cohomology theory* to mean a family $h^\star = (h^q)_{q \in Z}$ of contravariant ∂-functors in the sense of [2], defined on a category of pairs .

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We shall assume $h^\star$ has the following properties:

(i) If $f \simeq g$, then $h^\star(f) = h^\star(g)$.

(ii) If f is a relative homeomorphism, then $h^\star(f)$ is an isomorphism.

(iii)

$$h^\star(\coprod_\alpha X_\alpha) \stackrel{\cong}{\to} \prod_\alpha h^\star(X_\alpha),$$

for any family of spaces (X_α), where $\coprod$ denotes the topological sum (for this definition see also [3]).

We consider now G-cohomology theories defined on the category of G-pairs in the sense of [5], section 5 (here G is a compact Lie group). In fact, we shall recall the generalized Borel construction to obtain a large class of examples: for any family S of closed subgroups of G and any G-space X, define a topological category $\mathcal{C}(X;S)$ whose objects are $\coprod_{H\in S} X^H$ and whose morphisms are

$$\coprod_{(H,K)\in S\times S} \mathrm{Hom}_G(G/H, G/K) \times X^K.$$

Next we define the space $X(S)$ to be the geometric realization without degeneracies of $\mathcal{C}(X;S)$, in the sense of [6], appendix A.

Clearly, if (X,A) is a G-pair, then $A(S)$ is a closed subspace of $X(S)$ and also, $(.,S)$ preserves all the above properties. Thus, if $h^\star$ is a generalized cohomology theory, we can define a G-cohomology theory on G-pairs by the formula

$$h^\star_G(X,A;S) = h^\star(X(S), A(S))$$

and we shall say that $h^\star_G(.;S)$ is the *G-cohomology theory* obtained from $h^\star$ by generalized Borel construction with regard to S.

Finally in this section, for a G-cohomology theory $h^\star_G$ and for a closed subgroup H of G, we define the *associated H-cohomology theory* $h^\star_H$ by the formula

$$h^\star_H(X,A) = h^\star_G(G\times_H X, G\times_H A)$$

where (X,A) is an H-pair and $G\times_H X$ denotes the orbit space of $G\times X$ under the action of H given by

$$h\cdot(g,x) = (gh^{-1}, hx).$$

Observe that,for a G-space X,then $G\times_H X$ is naturally G-homeomorphic to $G/H\times X$ (here G acts on both factors). Thus the projection $G/H\times X \to X$ induces a natural transformation

$$c_H : h^\star_G(X) \to h^\star_H(X).$$

Because the Weyl group of H in G acts as a group of automorphisms of $h^\star_H$ restricted to G-spaces (see [7] section 3), the transformation c_H maps $h^\star_G(X)$ into the invariants of $h^\star_H(X)$ under this action. Again note that c_H is a ring homomorphism.

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The following notions were introduced by R. M. Seymour (see [7]).

Definiton 1. Let S be a family of closed subgroups of the compact group G. A G-map $f : X \to Y$ will be called a *strong* (G, S)- *equivalence* if f is an H-equivariant homotopy equivalence for each $H \in S$.

In view of Lemma 2.1 in [7], for the rest of this paper, we shall assume that the family S is closed with respect to conjugation and finite intersection.Such a family is called a *complete family* of subgroups.

Remark. If $G \in S$, then a strong (G, S)-equivalence is just a G-homotopy equivalence. On the other hand, if $S = \{1\}$, then f is a strong (G, S)-equivalence if f is a G-map which is a homotopy equivalence. Generally G-homotopy equivalence implies strong (G, S)-equivalence .

Definition 2. A G-cohomology theory $h_G^\star$ is *weakly S-complete* if the morphism $f^\star : h_G^\star(Y) \to h_G^\star(X)$, induced by a strong (G, S)-equivalence $f : X \to Y$ is an isomorphism.

Example. For a compact and connected Lie group G, we consider its maximal torus T and let $K_G^\star$ be the equivariant K(cf. [7], 5.1).

Remark. The equivariant K-theory is not weakly (1)-complete. This is proved in [1], where an example of a G-map, homotopy equivalence $f : X \to Y$ is constructed, but which does not induce the isomorphism $K_G^\star(Y) \to K_G^\star(X)$.

To conclude this section, if $h_G^\star$ is a multiplicative G-cohomology theory, we are given a natural, associative pairing

$$\mu_G : h_G^\star(X) \otimes_{\overline{h}_G} h_G^\star(Y) \to h_G^\star(X \times Y),$$

where $\overline{h}_G$ denotes the graded ring $h_G^\star(point)$. We assume $\overline{h}_G$ has an identity and μ_G is graded commutative.

Again, if H is a closed subgroup of G and X, Y are H-spaces, then there are the pairings

$$\mu_H : h_H^\star(X) \otimes_{\overline{h}_H} h_H^\star(Y) \to h_H^\star(X \times Y).$$

Here we write $\overline{h}_H$ for $h_H^\star(point)$.

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Let $\mathcal{C}$ be some full subcategory of G-spaces. We shall recall a result of R.M.Seymour (6.1 in [7]):

Proposition 1. *For $h_G^\star$ a multiplicative G-cohomology theory we suppose $h_G^\star(X)$ is a projective $\overline{h}_G$-module and that*

$$\overline{\mu}_G : h_G^\star(X) \otimes_{\overline{h}_G} \overline{h}_H \to h_H^\star(X)$$

is an isomorphism for each $H \in S$. Then the map μ_G from section 2 is an isomorphism for each Y in $\mathcal{C}$.

For weakly S-complete theories we obtain the following result:

Proposition 2. *Let S be a complete family and suppose $h_G^\star$ is a weakly S-complete multiplicative theory. Let $h_G^\star(X)$ be a projective $\overline{h}_G$-module and suppose $\overline{\mu}_G$ is an isomorphism for each $H \in S$, $X \in \mathcal{C}$. Then the map*

$$\mu_G : h_G^\star(X) \otimes_{\overline{h}_G} h_G^\star(Y) \to h_G^\star(X \times Y)$$

is also an isomorphism for each $Y \in \mathcal{C}$.

Proof. We consider the natural transformation between G- cohomology theories

$$h_G^\star(X) \otimes_{\overline{h}_G} h_G^\star(.) \to h_G^\star(X \times .).$$

Because $h_G^\star(G/H) = h_H^\star(point)$ and $h_G^\star(X \times G/H) \cong h_H^\star(X \times point) = h_H^\star(X)$, using one of the hypotheses, we obtain that the induced morphisms

$$h_G^\star(X) \otimes_{\overline{h}_G} h_H^\star(point) \to h_H^\star(X \times point)$$

are isomorphisms for each $H \in S$ and $X \in \mathcal{C}$.

By a standard induction over the skeleton of the CW-complex L we conclude that

$$h_G^\star(X) \otimes_{\overline{h}_G} h_G^\star(L_n) \to h_G^\star(X \times L_n)$$

is an isomorphism for each $n \geq 0$.

By the device of [7], Proposition 3.3, one can obtain that μ_G is an isomorphism for any CW-complex L.

Now define a topological category as follows:

$$\text{Object space is } \coprod_{H \in S} G/H \times Y^H$$
$$\text{Morphism space is } \coprod_{(H,K) \in S \times S} G/H \times \mathrm{Hom}_G(G/H, G/K) \times Y^K.$$

Next observe that G acts on the objects and morphisms of this category by the left action of G on the factors G/H and all structural maps are equivariant regarding this action.Thus,we have an induced G-action on the geometric realization $E(Y,S)$ of the above category.

Because S is a complete family,then $E(Y,S)$ is strongly G-contractible with respect to S(see [7],Proposition 2.2).It follows that the induced morphism

$$p^\star : h_G^\star(X) \to h_G^\star(E(Y,S) \times X)$$

is an isomorphism,where $p : E(Y,S) \times X \to X$ is projection onto the second factor.

Finally we take $E(Y,S)$ as the previous CW-complex L and that gives the result.

Corollary 1. *Let S be a complete family and $h_G^\star$ be a weakly S-complete multiplicative theory.We assume that $h_G^\star(X)$ is a free $\overline{h}_G$-module on basis B and that c_H maps B bijectively onto a basis for $h_H^\star(X)$ as a free $\overline{h}_H$-module for each $H \in S$.Then, if the morphism*

$$\mu_H : h_H^\star(X) \otimes_{\overline{h}_H} h_H^\star(Y) \to h_H^\star(X \times Y)$$

is an isomorphism for each $H \in S$ and $X, Y \in \mathcal{C}$,it follows that μ_G is an isomorphism for X, Y objects of the category $\mathcal{C}$.

Proof. We denote by $[B]R$ the free R-module on basis B.We have the following diagram of morphisms:

$$\begin{array}{lll}
[B]\overline{h}_G \otimes_{\overline{h}_G} \overline{h}_H & \xrightarrow{\cong} h_G^\star(X) \otimes_{\overline{h}_G} \overline{h}_H & \xrightarrow{\mu} h_H^\star(X) \\
c_H \otimes 1 \downarrow & c_H \otimes 1 \downarrow & \qquad ||
\end{array}$$

$$[c_H(B)]\overline{h}_H \otimes_{\overline{h}_H} \overline{h}_H \stackrel{\cong}{\to} h^\star_H(X) \otimes_{\overline{h}_H} \overline{h}_H \stackrel{\cong}{\to} h^\star_H(X)$$

The two vertical maps are isomorphisms by hypothesis and the first two horizontal isomorphisms are induced by the inclusions $B \subset h^\star_G(X)$ and

$c_H(B) \subset h^\star_H(X)$.Finally the second map from the lower line of diagram is a canonical isomorphism. It follows that μ is an isomorphism and applying the previous proposition we obtain the result.

In [7] R. M. Seymour obtained some conditions under which the hypotheses of above corollary hold (see proposition 6.3, [7]).

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