CR Structures on Real Hypersurfaces of a Complex Projective Space

Jong Taek Cho and U-Hang Ki

Abstract

We define the generalized Tanaka connection for real hypersurfaces in Kählerian manifolds, and further classify a real hypersurface of a complex projective space whose shape operator or Ricci tensor is parallel with respect to the generalized Tanaka connection, respectively.

Introduction

Let $P_n(C)$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let M be an orientable real hypersurface of $P_n(C)$. Then M has an almost contact metric structure (η, ϕ, g) induced from the Kählerian structure of $P_n(C)$ (see section 2). We denote by ξ the structure vector field dual to η. One of the typical examples of M is a geodesic hypersphere. We denote by ∇, A, R and S, the Levi-Civita connection with respect to g, the shape operator, the curvature tensor and the Ricci tensor on M, respectively. It is well-known that there does not exist a real hypersurface M in $P_n(C)$ with the parallel second fundamental tensor($\nabla A = 0$). Also the second author ([4]) proved that there does not exist a real hypersurface M with the parallel Ricci tensor($\nabla S = 0$) in $P_n(C)$, $n \geq 3$. As an immediate consequence of this result, $P_n(C)(n \geq 3)$ does not admit a locally symmetric($\nabla R = 0$) real hypersurface M. For real hypersurfaces in Kählerian manifolds, the CR structure associated with the almost contact metric structure is integrable, but is not in general non-degenerate. On the other hand, N.Tanaka ([13]) defined tha canonical affine connection on a non-degenerate integrable CR manifold. And S.Tanno ([14]) defined the generalized Tanaka connection for contact metric manifolds by relaxing the integrability condition of their associated CR structures. In the present paper, for a non-zero real number k we define the generalized Tanaka connection $\check{\nabla}^{(k)}$ for real hypersurfaces in Kählerian manifolds by the naturally extended one of S.Tanno's generalized Tanaka connection ($k = 1$) for contact metric manifolds.

The generalized Tanaka connection $\check{\nabla}^{(k)}$ coincides with the Tanaka connection if real hypersurfaces satisfy $\phi A + A\phi = 2k\phi$ (see section 2). In section 3 we show that there are real hypersurfaces M of $P_n(C)$ such that its almost contact metric structures is not contact metric structure but, $\check{\nabla}^{(k)}$ defined on M for some $k \neq 0$ coincides with the Tanaka connection (see Remark 1).

R. Takagi ([11]) classified homogeneous real hypersurfaces of $P_n(C)$ by means of six model spaces of type A_1, A_2, B, C, D, and E, further he explicitly write down their principal curvatures and multiplicities in a table in [12]. T.E.Cecil and P.J.Ryan([3]) extensively investigated a real hypersurface which is realized as a tube of constant radius r over a complex submanifold of $P_n(C)$ on which ξ is a principal curvature vector field with principal curvature $\alpha = 2 \cot 2r$ and the corresponding focal map φ_r has constant rank. By making use of these two works, M. Kimura ([5]) proved the following

Theorem 0. *Let M be a connected real hypersurface of $P_n(C)$. Then M has constant principal curvatures and the structure vector field ξ is principal if and only if M is locally congruent to one of the following spaces:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $P_{n-1}(C)$), where $0 < r < \frac{\pi}{2}$;*

(A_2) *a tube of radius r over a totally geodesic $P_k(C)(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$;*

(B) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$;*

(C) *a tube of radius r over $P_1(C) \times P_{\frac{n-1}{2}}(C)$, where $0 < r < \frac{\pi}{4}$;*

(D) *a tube of radius r over a complex Grassmann $G_{2,5}(C)$, where $0 < r < \frac{\pi}{4}$ and $n = 9$;*

(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \dfrac{\pi}{4}$ and $n = 15$.*

In these circumstaces, we will investigate real hypersurfaces of $P_n(C)$ whose shape operator is parallel with respect to the generalized Tanaka connection, and further the Ricci tensor is parallel with respect to the generalized Tanaka connection. More specifically, in section 4, we prove:

Theorem 1. *Let M be a real hypersurface of $P_n(C)$. If the shape operator A is $\check{\nabla}^{(k)}$-parallel ($\check{\nabla}^{(k)}A = 0$), then ξ is a principal curvature vector field and further M is locally congruent to one of the homogeneous real hypersurfaces of type A_1, A_2 or B, and vice versa.*

Proposition 1. *Let M be a real hypersurface of $P_n(C)$ whose structure vector ξ is a principal curvature vector field. If the Ricci tensor S of M is $\check{\nabla}^{(k)}$-parallel ($\check{\nabla}^{(k)}S = 0$), then M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2, or B, and vice versa.*

Recently, a ruled real hypersurface is defined by a foliated one by complex hyperplanes $P_{n-1}(C)$ and investigated in [6]. We see that the ruled real hypersurface satisfies that $\phi A + A\phi = 0$ resticted on the distribution D which is determined by the kernel of η (or that D is integrable). In section 5, we prove that there does not exist the ruled real hypersurface with the $\check{\nabla}^{(k)}$-parallel Ricci tensor.

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

1 Almost contact metric structures and the associated CR structures

First, we give a brief review of several fundamental notions and formulas which we will need later on. An odd–dimensional Riemannian manifold M with metric tensor g is said to have an *almost contact metric structure* if it admits a (1,1)-tensor field ϕ, a vector field ξ and a 1-form η satisfying

(1.1) $\qquad \phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$

From (1.1) we get

(1.2) $\qquad\qquad\qquad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X,\xi).$

We call (η, ϕ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, g)$ an *almost contact metric manifold.* The tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM | \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. For an almost contact metric manifold $M = (M; \eta, \phi, g)$, one may define naturally an almost complex structure on the product manifold $M \times R$, where R denotes real line. If the almost complex structure is integrable, M is said to be *normal*. The integrablity condition for the almost complex structure is the vanishing of the tensor $[\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the *Nijenhuis torsion* of ϕ. Also, for an almost contact metric manifold M we define its *fundamental 2-form* Φ by $\Phi(X,Y) = g(\phi X, Y)$. If

(1.3) $\qquad\qquad\qquad\qquad\qquad \Phi = d\eta,$

M is called a *contact metric manifold.* A normal contact metric manifold is called a *Sasakian manifold.* For more details about the general theory of almost contact metric manifolds, we refer to [1], [9].

On the other hand, for an almost contact metric manifold $M = (M; \eta, \phi, g)$, the restriction $\bar{\phi} = \phi|D$ of ϕ to D defines an almost complex structure to D. If

$$[\bar{\phi}X, \bar{\phi}Y] - [X,Y] \in D$$

and

$$[\bar{\phi}, \bar{\phi}](X,Y) = 0$$

for all $X, Y \in D$, where $[\bar{\phi}, \bar{\phi}]$ is the Nijenhuis torsion of $\bar{\phi}$, then the pair $(\eta, \bar{\phi})$ is called the *integrable CR structure* associated with the almost contact metric structure (η, ϕ, g), and the associated Levi form defined by $L(X,Y) = d\eta(X, \bar{\phi}Y), X, Y \in D$. If the associated Levi form is nondegenerate (positive or negative definite, resp.) and hermitian, then $(\eta, \bar{\phi})$ is called a *non-degenerate (strongly pseudo-convex, resp.) pseudo-hermitian CR structure.* For further details about CR structures, we refer to [2], [14].

2 The generalized Tanaka connection on real hypersurfaces

Let M be a real hypersurface of a Kählerian manifold $\tilde{M} = (\tilde{M}; J, \tilde{g})$ and N (or $N' = -N$) a unit normal vector on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection

in $\tilde{M}$. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator A is called a *principal curvature vector* (resp. *principal curvature*). We denote by V_λ the eigenspace associated with an eigenvalue λ. For any vector field X tangent to M, we put

(2.1) $$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then we easily see that the structure (η, ϕ, g) is an almost contact metric structure on M, and further it is known that the associated CR structure is integrable (cf. [2]). From $\tilde{\nabla} J = 0$ and (2.1), making use of the Gauss and Weingarten formulas, we have

(2.2) $$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

(2.3) $$\nabla_X \xi = \phi AX.$$

In general, the notion of contact metric structure is equivalent to the notion of strongly pseudo-convex CR structure, not necessarily integrable (see Proposition 2.1 in [14]). From (1.3) and (2.3) we have

Proposition 2. *Let $M = (M; \eta, \phi, g)$ be a real hypersurface of a Kählerian manifold. The almost contact metric structure of M is contact metric if and only if $\phi A + A\phi = \pm 2\phi$, where $\pm$ is determined by the orientation.*

The Tanaka connection ([13]) is the canonical affine connection defined on non-degenerate integrable CR manifold. S.Tanno ([14]) defined the generalized Tanaka connection for contact metric manifolds by the unique linear connection which co-incides with the Tanaka connection if the associated CR structure is integrable. We define the generalized Tanaka connection for real hypersurfaces of Kählerian mani-folds by the naturally extended one of S.Tanno's generalized Tanaka connection for contact metric manifolds.

Now we recall the generalized Tanaka connection $\check{\nabla}$ for contact metric manifolds,

$$\check{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y,$$

for all vector fields X and Y.

Thus, taking account of (2.3) the generalized Tanaka connection $\check{\nabla}^{(k)}$ for real hypersurfaces of Kählerian manifolds is naturally defined by

(2.4) $$\check{\nabla}_X^{(k)} Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y,$$

where k is a non-zero real number. We put $F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$. Then the torsion tensor is $\check{T}^{(k)}(X, Y) = F_X Y - F_Y X$. Also, by using (1.2), (1.3), (2.2), (2.3) and (2.4) we can see that

(2.5) $$\check{\nabla}^{(k)}\eta = 0, \ \check{\nabla}^{(k)}\xi = 0, \ \check{\nabla}^{(k)}g = 0, \ \check{\nabla}^{(k)}\phi = 0.$$

and

$$\check{T}^{(k)}(X, Y) = 2d\eta(X, Y)\xi, \ \forall X, \ Y \in D.$$

We note that the associated Levi form $L(X, Y) = \frac{1}{2}g((\bar{\phi}\bar{A} + \bar{A}\bar{\phi})X, \bar{\phi}Y)$, where we denote by $\bar{A}$ the restriction A to D. We denote the extended one of L to TM by the same letter L. If M satisfies $\phi A + A\phi = 2k\phi$, then we see that the associated CR structure is strongly pseudo-convex and further satisfy $\check{T}^{(k)}(\xi, \phi Y) = -\phi\check{T}^{(k)}(\xi, Y)$, and hence the generalized Tanaka connection $\check{\nabla}$ coincides with the Tanaka connection (see [14]). That is, we have

Proposition 3. *Let $M = (M; \eta, \phi, g)$ be a real hypersurface of a Kählerian manifold. If M satisfies $\phi A + A\phi = 2k\phi$, then the generalized Tanaka connection $\check{\nabla}^{(k)}$ coincides with the Tanaka connection.*

3 Real hypersurfaces of $P_n(C)$

Let $\tilde{M}$ be a complex projective space $P_n(C)$ of constant holomorphic sectional curvature 4. Then we have the following Gauss and Codazzi equations

$$(3.1) \qquad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y -$$

$$-2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY,$$

$$(3.2) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi,$$

for any tangent vector fields X, Y, Z on M. Using (1.2), (1,3), (2.2) and (2.3) we get

$$(3.3) \qquad SX = (2n + 1)X - 3\eta(X)\xi + hAX - A^2X,$$

$$(3.4) \qquad (\nabla_X S)Y = -3\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + (Xh)AY + h(\nabla_X A)Y$$

$$-A(\nabla_X A)Y - (\nabla_X A)AY,$$

for any tangent vector fields X, Y on M, where $h = traceA$.

We recall the the following :

Proposition 4 ([7]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is constant.*

Proposition 5 ([7]). *Assume that ξ is a principal curvature vector field with corresponding principal curvature α. If $AX = \lambda X$ for X orthogonal to ξ, then we have $A\phi X = \dfrac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X$.*

Under the same hypothesis as in Proposition 5, by Proposition 4 and the table in [12] we see that a real hypersurface M is locally congruent to one of type B if and only if M satisfies $\alpha\lambda^2 + 4\lambda - \alpha = 0$, where $AX = \lambda X$ for X orthogonal to ξ. If M satisfies $\phi A + A\phi = c\phi$ for some non-zero constant c, then by using (1.1) we easily see that ξ is a principal curvature vector field, and thus taking account of Proposition 5 we have (see also [15])

Theorem 2. *Let M be a real hypersurface of $P_n(C)$. Then M satisfies $\phi A + A\phi = c\phi$ for some non-zero constant c if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1 with $\alpha = \dfrac{c^2 - 4}{2c}$ or B with $\alpha = -\dfrac{4}{c}$.*

Here we prove

Proposition 6. *Let M be a real hypersurface of $P_n(C)$. The almost contact metric structure of M is contact metric if and only if M is locally congruent to a geodesic hypersphere of radius $\dfrac{\pi}{4}$ or a tube of radius $\dfrac{\pi}{8}$ over a complex quadric Q^{n-1}.*

Proof. If the almost contact metric structure is contact metric, then from Proposition 2 we get

$$(3.5) \qquad\qquad \phi AX + A\phi X = \pm 2\phi X,$$

for all tangent vector field X on M, where $\pm$ is determined by the orientation in such a way that the almost contact metric structure is a contact metric structure. We put $X = \xi$ in (3.5), then together with (1.2) we get $\phi A\xi = 0$, from which we see that ξ is a principal curvature vector field. Assume that $X \in V_\lambda$. Then from (3.5) and Proposition 5, we have $\lambda^2 - 2\lambda + 1 \pm \alpha = 0$. Taking account of the table given in [12], we can see that M is locally congruent to a geodesic hypersphere of radius $\dfrac{\pi}{4}$ or a tube of radius $\dfrac{\pi}{8}$ over a complex quadric Q_{n-1}. (Q.E.D.)

Remark 1. From Theorem 2, Propositions 3 and 6 we see that the almost contact metric structure on geodesic hyperspheres of radius $r \neq \dfrac{\pi}{4}$ and tubes over complex quadric Q_{n-1} of radius $r \neq \dfrac{\pi}{8}$ are not contact metric, but for a constant $c \neq \pm 2$ appeared in theorem 2 the generalized Tanaka connection $\check{\nabla}^{(c/2)}$ defined on them coincides with the Tanaka connection.

Theorem 3 ([8]). *Let M be a real hypersurface of $P_n(C)$. Then M is locally congruent to one of the homogeneous real hypersurfaces of type A_1, A_2 if and only if M satisfies $\phi A = A\phi$.*

The following Theorems 4 and 5 are very useful for the proof of Theorem 1 and Proposition 1 in section 4.

Theorem 4 ([6]). *Let M be a real hypersurface of $P_n(C)$. Then the shape operator satisfies $g((\nabla_X A)Y, Z) = 0$ for any X, Y and Z which are orthogonal to ξ and ξ is a principal curvature vector field if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 or B.*

Theorem 5 ([10]). *Let M be a real hypersurface of $P_n(C)$. Then the Ricci tensor of M satisfies $g((\nabla_X S)Y, Z) = 0$ for any X, Y and Z which are orthogonal to ξ and ξ is a principal curvature vector field if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 or B.*

4   Proof of Theorem 1 and Proposition 1

We define a vector field U on M by $U = \nabla_\xi \xi$. Then from (1.2) and (2.3) we easily observe that

$$(4.1) \qquad\qquad g(U, \xi) = 0, \; g(U, A\xi) = 0, \; \| U \|^2 = g(U, U) = \beta - \alpha^2,$$

where $\beta = g(A^2\xi, \xi)$. From (1.2),(2.3) and (4.1) we have at once:

Lemma 1. *$A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.*

Taking account of (2.4), we have

$$(4.2) \qquad (\check{\nabla}_X^{(k)} A)Y = \check{\nabla}_X^{(k)} AY - A\check{\nabla}_X^{(k)} Y = (\nabla_X A)Y + F_X AY - AF_X Y$$

$$= (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY$$

$$-g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y,$$

for any vector fields X and Y on M. First, we prove that ξ is a principal curvature vector, i.e., $A\xi = \alpha\xi$. From (4.2) we see that $\check{\nabla}_\xi^{(k)} A = 0$ implies

$$(4.3) \quad (\nabla_\xi A)X = k(\phi AX - A\phi X) + \eta(AX)U - g(AX, U)\xi + g(X, U)A\xi - \eta(X)AU,$$

for any vector field X on M. From (4.1) and (4.3) we easily see that $d\alpha(\xi) = 0$, where d denotes the exterior derivative. The above equation (4.3), together with (3.2), yields

$$(4.4) \qquad (\nabla_X A)\xi = k(\phi AX - A\phi X) - \phi X - g(U, AX)\xi + \eta(AX)U$$

$$-\eta(X)AU + g(U, X)A\xi.$$

With $U = \phi A\xi$ and from (2.2),(2.3) and (4.4) we have

$$(4.5) \quad \nabla_X U = X + (\alpha - k)AX - k\phi A\phi X + \phi A\phi AX - \eta(A^2 X)\xi + (\alpha + k)\eta(AX)\xi$$

$$-\eta(X)\xi - \eta(AX)A\xi + g(U, X)U - \eta(X)\phi AU.$$

Also, it follows from (4.2) and $(\check{\nabla}_X^{(k)} A)\xi = 0$ that

$$(4.6) \qquad (\nabla_X A)\xi = g(AX, U)\xi + \alpha\phi AX + k\eta(X)U - A\phi AX$$

for any vector field X on M. From (4.4) and (4.6) we get

$$(4.7) \quad kg(\phi AX, Y) - kg(A\phi X, Y) - g(\phi X, Y) - g(U, AX)\eta(Y) + \eta(AX)g(U, Y)$$

$$-\eta(X)g(AU, Y) + g(U, X)\eta(AY)$$

$$= g(AX, U)\eta(Y) + \alpha g(\phi AX, Y) + k\eta(X)g(U, Y) - g(A\phi AX, Y)$$

for any vector fields X and Y on M. We put $Y = \xi$ in (4.7) and taking account of (1.2) and (2.1), we have

$$(4.8) \qquad\qquad (\alpha + k)g(X, U) = 3g(AX, U)$$

for any vector field X on M. The equation (4.8) yields that

$$(4.9) \qquad\qquad 3AU = (\alpha + k)U.$$

Further, from (3.2) and (4.6) we get

$$(4.10) \qquad (\nabla_\xi A)X = \phi X + g(AX, U)\xi + \alpha\phi AX + k\eta(X)U - A\phi AX$$

for any vector field X on M. From (4.10) it follows that

$$(4.11) \quad 2g(\phi X, Y) + g(AX, U)\eta(Y) - g(AY, U)\eta(X) + \alpha g(\phi AX, Y) - \alpha g(\phi AY, X)$$

$$+k\eta(X)g(U, Y) - k\eta(Y)g(U, X) - g(A\phi AX, Y) + g(A\phi AY, X) = 0$$

for all vector fields X and Y on M. Putting $X = A\xi$ and $Y = U$ in (4.11) and taking account of (2.1) and (4.9), we obtain

$$(4.12) \qquad (\alpha - 2k)g(A^2\xi, \phi U) = 3(2 + k\alpha)(\beta - \alpha^2).$$

Now, let W be the subset of M such that $\beta - \alpha^2 \neq 0$. Then W is an open subset of M. Suppose that W is non-empty, and from now on we discuss our arguments on W. If there exists a point $p \in W$ such that $\alpha(p) = 2k$, then from (4.12) we see that $\beta(p) - \alpha(p)^2 = 0$, which is impossible. Thus we see that $\alpha \neq 2k$ on W, and we get from (4.12)

$$(4.13) \qquad g(A^2\xi, \phi U) = 3\frac{2 + k\alpha}{\alpha - 2k}(\beta - \alpha^2).$$

From (4.13), taking account of $\phi U = -A\xi + \alpha\xi$, we get

$$(4.14) \qquad g(A\phi U, \phi U) = -\frac{\alpha^2 + k\alpha + 6}{\alpha - 2k}(\beta - \alpha^2).$$

Further from (4.9) we have

$$(4.15) \qquad 3\phi AU = (\alpha + k)\phi U.$$

Differentiating covariantly (4.15) with respect to ξ and taking the component of U, then yields with (2.2), (4.5), (4.10), (4.13) and (4.14) :

$$(\alpha + k)g(A\phi U, \phi U) + 3g(A^2\xi, \phi U) = (3\beta - 2\alpha^2 - 3\alpha + 2)g(U, U).$$

Thus from (4.14) and (4.15), we obtain

$$(\alpha - 2k)(3\beta - \alpha^2 - k\alpha + 9 - k^2) = 0.$$

Since $\alpha \neq 2k$ on W, we have

$$(4.16) \qquad 3\beta = \alpha^2 + k\alpha - 9 + k^2.$$

Then $3\|U\|^2 = 3(\beta - \alpha^2) = k^2 + \alpha k - 2\alpha^2 - 9 \geq 0$ independent with k. Thus for any (fixed) point we have $\alpha^2 \leq -4$, which is impossible. After all, we conclude that W is empty, and hence $A\xi = \alpha\xi$ on M.

Now the equation (4.2), together with $A\xi = \alpha\xi$, shows that

$$(4.17) \qquad g((\nabla_X A)Y, Z) = 0$$

for any vector fields X, Y and Z orthogonal to ξ. Thus by Theorem 4, we see that M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 or B. By using (2.3) and Proposition 4 we can see that a homogeneous real hypersurface M of type A_1, A_2 or B satisfies $(\check{\nabla}^{(k)} A)\xi = 0$. Further M satisfies $(\check{\nabla}_\xi^{(k)} A)X = 0$ for any vector field X orthogonal to ξ. In fact, from (4.2) taking account of (3.2) and $A\xi = \alpha\xi$, we get

$$(\check{\nabla}_\xi^{(k)} A)X = \alpha\phi AX - A\phi AX + \phi X - k(\phi AX - A\phi X),$$

for any vector field X orthogonal to ξ. Assume $X \in V_\lambda$. Then from Proposition 5 we have

$$(4.18) \qquad (\check{\nabla}_\xi^{(k)} A)X = (\alpha - 2k)(\lambda^2 - \alpha\lambda - 1)\phi X.$$

From Theorem 3, we see that a real hypersurface of type A_1 or A_2 satisfies $\lambda^2 - \alpha\lambda - 1 = 0$. Also, from Theorem 0 we see that for a real hypersurface of type B $\alpha(= 2\cot 2r)$ is non-zero constant. So, from (4.18) we see that M of type A_1, A_2 or B satisfies $(\check{\nabla}_\xi^{(k)} A)X = 0$ for any vector field X orthogonal to ξ. Therefore we have proved Theorem 1. (Q.E.D.)

Next, we prove Proposition 1. Taking account of (2.4), we have

$$(4.19) \qquad (\check{\nabla}_X^{(k)} S)Y = \check{\nabla}_X^{(k)} SY - S\check{\nabla}_X^{(k)} Y = (\nabla_X S)Y + F_X SY - SF_X Y$$

$$= (\nabla_X S)Y + g(\phi AX, SY)\xi - g(\phi AX, Y)S\xi - \eta(SY)\phi AX + \eta(Y)S\phi AX$$

$$-k\eta(X)\phi SY + k\eta(X)S\phi Y$$

for any vector fields X and Y on M. From (4.19), the hypotheses $\check{\nabla}^{(k)} S = 0$ and $A\xi = \alpha\xi$ yield

$$(4.20) \qquad g((\nabla_X S)Y, Z) = 0$$

for any vector fields X, Y and Z orthogonal to ξ. From (4.20) and Theorem 5 we see that M is locally congruent to one of the real hypersurfaces of type A_1, A_2 or B.

For a real hypersurface M of type A_1, A_2 or B, since $A\xi = \alpha\xi$, by making use of (2.3), (3.3) and Proposition 4 we easily see that M satisfies $(\check{\nabla}^{(k)} S)\xi = 0$. Now from (5.1), (3,3) and (3.4) we get

$$(4.21) \qquad (\check{\nabla}_\xi^{(k)} S)X = h(\alpha\phi AX - A\phi AX + \phi X) - (\alpha A\phi AX - A^2\phi AX + A\phi X)$$

$$-(\alpha\phi A^2 X - A\phi A^2 X + \phi AX) + kh(\phi A - A\phi)X + k(\phi A^2 - A^2\phi)X$$

for any vector field X orthogonal to ξ. Assume that $X \in V_\lambda$. Then from (4.21) and Proposition 5 we have

$$\begin{aligned} (\check{\nabla}_\xi^{(k)} S)X &= (\alpha - 2k)\{2\lambda^4 - 2(\alpha + h)\lambda^3 + 3\alpha h\lambda^2 - (2\alpha - 2h + \alpha^2 h)\lambda - \\ &\quad - 2 - \alpha h\} = (\alpha - 2k)(\lambda^2 - \alpha\lambda - 1)\{2\lambda^2 - 2h\lambda + (2 + \alpha h)\}. \end{aligned}$$

Thus, by the similar arguments in the last part in the proof of Theorem 1 we see that a real hypersurface M of type A_1, A_2 or B also satisfies $(\check{\nabla}_\xi^{(k)} S)X = 0$ for any vector field X orthogonal to ξ. Therefore we have proved Proposition 1. (Q.E.D.)

5 Ruled real hypersurfaces

A ruled real hypersurface M of $P_n(C)$ is defined by a foliated one by complex hyperplanes $P_{n-1}C$ and the shape operator A of M is given by (cf.[6])

$$(5.1) \qquad A\xi = \alpha\xi + \nu V \ (\nu \neq 0), \ AV = \nu\xi, \ AX = 0 \text{ for any } X \perp \xi, V,$$

where V is a unit vector orthogonal to ξ, and where α, ν are differentiable functions on M. From (5.1) we immediately see that $\bar{\phi}\bar{A} + \bar{A}\bar{\phi} = 0$, which is equivalent to the condition that the distribution D is integrable (cf. Proposition 5 in [6]).

From (3.3) and (5.1), we have

$$(5.2) \qquad\qquad S\xi = a\xi + bV, \; SV = cV + b\xi,$$

$$SX = (2n + 1)X \;\text{ for any } X \perp \xi, V,$$

where $a = 2(n-1) + \alpha h - \alpha^2 - \nu^2$, $b = \nu(h - \alpha)$ and $c = 2n + 1 - \nu^2$. We have

Proposition 7. *There does not exist a ruled real hypersurface with $\check{\nabla}^{(k)}$-parallel Ricci tensor in $P_n(C)$.*

Proof. Suppose that a ruled real hypersurface M satisfy $\check{\nabla}^{(k)} S = 0$. Then (5.2), $\check{\nabla}^{(k)} S = 0$ and straightforward calculations yield the following:

$$(5.3) \qquad\qquad g((\check{\nabla}_X^{(k)} S)Y, V) = \nu^2 g(\check{\nabla}_X^{(k)} Y, V) = 0,$$

$$(5.4) \qquad\qquad g((\check{\nabla}_\xi^{(k)} S)Y, V) = \nu^2 g(\check{\nabla}_\xi^{(k)} Y, V) = 0,$$

$$(5.5) \qquad\qquad g((\check{\nabla}_V^{(k)} S)Y, V) = \nu^2 g(\check{\nabla}_V^{(k)} Y, V) = 0,$$

$$(5.6) \qquad\qquad g((\check{\nabla}_X^{(k)} S)V, V) = X\nu^2 = 0,$$

$$(5.7) \qquad\qquad g((\check{\nabla}_\xi^{(k)} S)V, V) = \xi\nu^2 = 0,$$

$$(5.8) \qquad\qquad g((\check{\nabla}_V^{(k)} S)V, V) = V\nu^2 = 0.$$

From (5.6)-(5.8) we see that ν is a non-vanishing constant on M. From (2.5) and $g(V, V) = 1$, we see that $\check{\nabla}_X^{(k)} V$, $\check{\nabla}_V^{(k)} V$ and $\check{\nabla}_\xi^{(k)} V$ are all orthogonal to V and ξ, and from (5.3)-(5.5) we see that $\check{\nabla}^{(k)} V = 0$.

Thus from (2.4) we have

$$(5.9) \qquad\qquad \nabla_\xi V = k\phi V, \; \nabla_V V = 0, \; \nabla_X V = 0$$

for any $X \perp \xi, V$. Then from (2.2), (5.1) and (5.9), we get $R(V, \phi V)\phi V = \nabla_V(\nabla_{\phi V}\phi V) - \nabla_{\phi V}(\nabla_V \phi V) - \nabla_{[V,\phi V]}\phi V = 0$. But, from (3.1) and (5.1) we see that $R(V, \phi V)\phi V = 4V$. Thus we obtain $V = 0$, which is impossible. Therefore we have our conclusion. (Q.E.D.)

From the above Proposition 7, we see also that there does not exist a ruled real hypersurface with $\check{\nabla}^{(k)}$-parallel curvature tensor.

Remark 2. We will discuss on real hypersurfaces of a complex hyperbolic space $H_n(C)$ of constant holomorphic sectional curvature -4 by using the extended generalized Tanaka connection in a forthcoming paper.

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Chonnam University
Department of Mathematics
Kwangju 500-757, Korea
e-mail:jtcho@chonnam.chonnam.ac.kr

Topology and Geometry Research Center

Kyungpook University
Taegu 702-701, Korea