Classical polar spaces (sub–)weakly embedded in projective spaces

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1 Introduction and statement of the Main Theorem

In this paper we are concerned with classical polar spaces, i. e. with the set of points and lines of some vector space W on which a non–degenerate (σ, ϵ)-hermitian form or pseudo-quadratic form vanishes.

To state the Main Theorem we introduce some notation. Let L be a division ring and W be a (left–)vector space over L endowed with a (σ, ϵ)–hermitian form or a pseudo–quadratic form q (with associated (σ, ϵ)–hermitian form f) in the sense of [Ti, §8]. We may assume that $\epsilon = \pm 1$ and $\sigma^2 = id$. We let

$$\begin{aligned} \mathrm{Rad}(W, f) &= \{w \in W \mid f(w, x) = 0 \text{ for all } x \in W\}, \\ x^{\perp} &= \{w \in W \mid f(w, x) = 0\} \text{ for } x \in W, \\ \Lambda_{min} &= \{c - \epsilon c^{\sigma} \mid c \in L\}, \\ \Lambda_{max} &= \{c \in L \mid \epsilon c^{\sigma} = -c\}. \end{aligned}$$

If $\mathrm{Rad}(W, f) = 0$, then f is said to be non–degenerate. Further f is trace–valued, if $f(w, w) \in \{c + \epsilon c^{\sigma} \mid c \in L\}$ for all $w \in W$. A subspace U of W is called singular, if $f(u, u') = 0$ resp. $q(u) = 0$ for all $u, u' \in U$. The 1–, 2– and 3–dimensional subspaces of W are called points, lines, planes respectively. Let S be the set of singular points

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of W. For each subspace U of W, we denote by $U \cap S$ the set of singular points in U. The subspace of a vector space which is spanned by a subset M is denoted by $<M>$.

We prove the following result:

Main Theorem . *Let L and K be division rings and let W be a vector space over L (not necessarily finite–dimensional). We assume that there is either a non–degenerate (trace–valued) (σ, ϵ)–hermitian form f on W such that $\Lambda_{min} = \Lambda_{max}$ or that there is a pseudo–quadratic form q on W with corresponding (σ, ϵ)–hermitian form f such that $Rad(W, f) = 0$. We suppose that there are singular lines in W and that $dim W \geq 5$. Further let V be a vector space over K.*

We assume that the following hypotheses are satisfied:

(a) *There is an injective mapping π from the set of singular points of W into the set of points of V. (We set $\pi(U \cap S) := \{\pi(u) \mid u \in U \cap S\}$ for each subspace U of W.)*

(b) *If L_1 is a singular line of W, then the subspace $<\pi(L_1 \cap S)>$ of V generated by $\pi(L_1 \cap S)$ is a line in V.*

(c) *For each singular point x of W we have: If y is a singular point of W with $\pi(y) \subseteq <\pi(x^\perp \cap S)>$, then $y \subseteq x^\perp$.*

Then there exists an embedding $\alpha : L \to K$ and an injective semi–linear (with respect to α) mapping $\varphi : W \to V$ such that $\pi(Lx) = K\varphi(x)$ for all $x \in W$, x singular (i. e. π is induced by a semi–linear mapping).

The mapping π defined by $x \mapsto \pi(x)$, $L_1 \mapsto <\pi(L_1 \cap S)>$, where x is a singular point and L_1 is a singular line in W, yields a sub–weak embedding of the polar space $\mathcal{S}$ associated with W and f resp. q into the projective space $\mathbf{P}(V_0)$, $V_0 = <\pi(W \cap S)>$ in the sense of [TVM1].

In the paper [TVM1] it is shown that for a non–degenerate polar space the concept of sub–weak embeddings is the same as the one of weak embeddings of polar spaces in projective spaces introduced by Lefevre-Percsy [Lef1], [Lef2]. In [TVM1], [TVM2] Thas and Van Maldeghem classified all polar spaces (degenerate or not) of rank at least 3 of orthogonal, symplectic or unitary type, which are sub–weakly embedded in a finite–dimensional projective space over a commutative field (except one possibility in the symplectic case over non–perfect fields of characteristic 2). In the non–degenerate case with the radical of the bilinear form of dimension at most 1, their result is that the polar space is fully embedded over a subfield.

The Main Theorem shows that this conclusion remains valid, if f resp. q satisfy the hypotheses of the Main Theorem. The polar space $\mathcal{S}$ associated with W and f resp. q is fully embedded in the projective space $\mathbf{P}(\varphi(W))$, where $\varphi(W)$ is a vector space over the sub-division ring L^α of K. For the mapping $x \mapsto \varphi(x)$, $L_1 \mapsto \varphi(L_1)$, every point in $\varphi(L_1)$ has an inverse image under φ. The Main Theorem does not require rank at least 3, finite dimension or rank, or commutative fields.

By the classification of non–degenerate polar spaces of rank at least 3, every such polar space is associated to a non–degenerate trace–valued (σ, ϵ)–hermitian form or a non–degenerate pseudo–quadratic form (apart from two classes of exceptions in rank 3). The assumptions $\Lambda_{min} = \Lambda_{max}$ resp. $\mathrm{Rad}(W, f) = 0$ are always satisfied if char $L \neq 2$. Thus the set of polar spaces handled in the Main Theorem is sufficiently rich.

Sections 2 to 5 are devoted to the proof of the Main Theorem. First, inspired by [Ti, (8.19)] we derive some properties of the mapping π (Section 2), which enables us to extend π to arbitrary points of W (Section 3). For this we use that every point is the intersection of two hyperbolic lines (i. e. lines spanned by two singular points x, y with $x \not\subseteq y^\perp$), if $\mathrm{Rad}(W, f) = 0$. For the construction of a semi–linear mapping φ which induces π (Section 5), we need the intermediate step where we construct such a semi–linear mapping for the restriction of π to a 4^+–space in W (Section 4). By a 4^+–space we mean the orthogonal sum of two hyperbolic lines.

2 Properties of the mapping π

In this first part of the proof of the Main Theorem we derive some properties of the mapping π.

2.1. *If a, b are singular points in W with $L_1 = <a,b>$ a singular line, then $<\pi(L_1 \cap S)> = <\pi(a), \pi(b)>$.*

Proof. Since π is injective on singular points, we see that $<\pi(a), \pi(b)>$ is a line which is contained in $<\pi(L_1 \cap S)>$. By (b) the claim follows. ∎

2.2. *If L_1 is a singular line in W and x is a singular point in W with $\pi(x) \subseteq <\pi(L_1 \cap S)>$, then $x \subseteq L_1$.*

Proof. We first consider the case that $L_1 \not\subseteq x^\perp$. Then $a := L_1 \cap x^\perp$ is a point, without loss $x \neq a$. Let b be a singular point with $L_1 = <a,b>$. Then $b \not\subseteq x^\perp$. We have $<\pi(x), \pi(a), \pi(b)> \subseteq <\pi(L_1 \cap S)> = <\pi(a), \pi(b)>$ by (2.1). Hence $\pi(b) \subseteq <\pi(x), \pi(a)> \subseteq <\pi(x^\perp \cap S)>$. Now (c) yields $b \subseteq x^\perp$, a contradiction.

Thus we are left with the case $L_1 \subseteq x^\perp$. Without loss $E := <L_1, x>$ is a singular plane. Let y be a singular point in W with $y \subseteq {L_1}^\perp$, $y \not\subseteq x^\perp$. Then $\pi(x) \subseteq <\pi(L_1 \cap S)> \subseteq <\pi(y^\perp \cap S)>$. Using (c) this yields $x \subseteq y^\perp$, a contradiction. ∎

2.3. *If L_1, L_2 are singular lines in W with $<\pi(L_1 \cap S)> = <\pi(L_2 \cap S)>$, then $L_1 = L_2$.*

Proof. Using (2.2) we obtain $L_1 \cap S = L_2 \cap S$, hence $L_1 = L_2$. ∎

2.4. *Let $Q = <L_1, L_2>$ be a 4^+–space in W, where $L_1 = <x_1, x_2>$, $L_2 = <y_1, y_2>$ are singular lines with $x_1 \not\subseteq {y_1}^\perp$, $x_1 \subseteq {y_2}^\perp$, $x_2 \subseteq {y_1}^\perp$, $x_2 \not\subseteq {y_2}^\perp$. Then $<\pi(Q \cap S)> = <\pi(x_1), \pi(x_2), \pi(y_1), \pi(y_2)>$ is 4–dimensional.*

Proof. By (c) $\pi(y_1) \not\subseteq <\pi(x_1), \pi(x_2)>$ and similarly $\pi(y_2) \not\subseteq <\pi(x_1), \pi(x_2), \pi(y_1)>$. Hence $<\pi(L_1 \cap S), \pi(L_2 \cap S)> = <\pi(x_1), \pi(x_2), \pi(y_1), \pi(y_2)>$ is 4–dimensional.

By [Ti, (8.10)] $Q \cap S$ is the smallest subset X of S containing $L_1 \cap S$ and $L_2 \cap S$ such that for every singular line L' of W which has two points in X necessarily $L' \cap S$ is contained in X. Let $Y := \{y \in S \mid \pi(y) \subseteq <\pi(L_1 \cap S), \pi(L_2 \cap S)>\}$. Then Y is a subset of S having the properties mentioned above. Hence $Q \cap S \subseteq Y$ and $<\pi(Q \cap S)> \subseteq <\pi(L_1 \cap S), \pi(L_2 \cap S)>$. This yields (2.4). ■

2.5. *If H is a hyperbolic line of W and x is a singular point of W with $\pi(x) \subseteq <\pi(H \cap S)>$, then $x \subseteq H$.*

Proof. Since W contains singular lines, $H^\perp$ is generated by its singular points. If a is a singular point in $H^\perp$, then $\pi(x) \subseteq <\pi(H \cap S)> \subseteq <\pi(a^\perp \cap S)>$. Using (c) we obtain $x \subseteq a^\perp$, hence $H^\perp \subseteq x^\perp$. This yields $x \subseteq H^{\perp\perp} = H$, since $\mathrm{Rad}(W, f) = 0$. ■

2.6. *If H_1, H_2 are hyperbolic lines in W with $<\pi(H_1 \cap S)> = <\pi(H_2 \cap S)>$, then $H_1 = H_2$.*

Proof. Using (2.5) we obtain $H_1 \cap S = H_2 \cap S$, hence $H_1 = H_2$. ■

2.7. *Let U be a subspace of W with $U = <U \cap S> \neq 0$. If $u \in U$, $u \notin Rad(U, f)$, then $u = cx_1 + y_1$ for some $c \in L$ and a hyperbolic pair (x_1, y_1) of U.*

Proof. Since $u \notin \mathrm{Rad}(U, f)$, there exists a singular point a in U with $a \not\subseteq u^\perp$. Let $a = <x_1>$ with $f(x_1, u) = 1$. If there is a pseudo–quadratic form on W, let $q(u) = c + \Lambda_{min}$ and set $y_1 = -cx_1 + u$. If there is a trace–valued (σ, ϵ)–hermitian form on W, let $f(u, u) = c + \epsilon c^\sigma$ and set $y_1 = cx_1 + u$. ■

2.8. *If Q is a 4^+–space in W and x is a singular point of W with $\pi(x) \subseteq <\pi(Q \cap S)>$, then $x \subseteq Q$.*

Proof. Since Q is finite–dimensional with $\mathrm{Rad}(Q, f) = 0$, we have $W = Q \perp Q^\perp$. Let $x = L(w + s)$ where w and s are vectors in Q, $Q^\perp$ respectively. Without loss $s \neq 0$.

We first consider the case that s is singular. Then W contains singular planes and $Q^\perp$ contains hyperbolic lines. Let a be a singular point in $Q^\perp$ with $a \not\subseteq s^\perp$, hence $a \not\subseteq x^\perp$. Then $\pi(x) \subseteq <\pi(Q \cap S)> \subseteq <\pi(a^\perp \cap S)>$. Using (c) this yields $x \subseteq a^\perp$, a contradiction.

Thus we are left with the case that s is non–singular. Then w is non–singular. Let $w = cx_1 + y_1$ as in (2.7) and let $Q = <x_1, y_1> \perp <x_2, y_2>$ with (x_2, y_2) a hyperbolic pair.

By (c) $<\pi(x_1), \pi(x_2), \pi(y_1)>$ is 3–dimensional. Hence

$$<\pi(x_1), \pi(x_2), \pi(y_1)> = <\pi(Q \cap S)> \cap <\pi({x_2}^\perp \cap S)>,$$

since otherwise $<\pi(Q \cap S)> \subseteq <\pi({x_2}^\perp \cap S)>$ and $y_2 \in {x_2}^\perp$ by (c), a contradiction. Similarly $<\pi(Q \cap S)> \cap <\pi({y_2}^\perp \cap S)> = <\pi(x_1), \pi(y_1), \pi(y_2)>$. Since $x \subseteq {x_2}^\perp \cap {y_2}^\perp$, this yields

$$\pi(x) \subseteq <\pi(x_1), \pi(x_2), \pi(y_1)> \cap <\pi(x_1), \pi(y_1), \pi(y_2)> = <\pi(x_1), \pi(y_1)>.$$

The last equality is obtained using (c). By (2.5) we see that $x \subseteq <x_1, y_1> \subseteq Q$. ■

2.9. *If Q_1, Q_2 are 4^+–spaces in W with $\langle\pi(Q_1 \cap S)\rangle = \langle\pi(Q_2 \cap S)\rangle$, then $Q_1 = Q_2$.*

Proof. Using (2.8) we obtain $Q_1 \cap S = Q_2 \cap S$, hence $Q_1 = Q_2$. ■

2.10. *Let H_1 be a hyperbolic line in W and let z be an arbitrary point in ${H_1}^\perp$. Then z is the intersection of two hyperbolic lines in ${H_1}^\perp$ or we have the following exceptional situation:*
$|L| \leq 4$, $dim W \leq 6$, W is associated to a quadratic form. If H_2 is a hyperbolic line in ${H_1}^\perp$ which contains z, then there exists a singular point b in ${H_1}^\perp$ with $b \not\subseteq H_2$, $b \subseteq z^\perp$.

Proof. Let H_2 be a hyperbolic line which contains z. We write $z = \langle cx_2 + y_2\rangle$ where (x_2, y_2) is a hyperbolic pair in H_2, using (2.7). Since H_2 and $z^\perp$ are proper subspaces of ${H_1}^\perp$, there exists a singular point a in ${H_1}^\perp$ with $a \not\subseteq H_2$, $a \not\subseteq z^\perp$ by [Ti, (8.17)], provided that $|L| \geq 4$. Then $\langle z, a\rangle$ is a second hyperbolic line, which contains z.

Further, if there are singular lines in ${H_1}^\perp$, then we let $\langle x_2, y_2\rangle \perp \langle x_3, y_3\rangle \subseteq {H_1}^\perp$ with (x_3, y_3) a hyperbolic pair and choose $\langle z, x_2 + x_3\rangle$ as second hyperbolic line containing z.

So we are left with the case $|L| \leq 3$ and $\dim {H_1}^\perp \leq 4$. Under the assumptions of the Main Theorem we may assume that W is equipped with a quadratic form, since in the case $|L| = 3$, f a symplectic form ($\sigma = id, \epsilon = -1$) we can proceed as in the previous paragraph. Let $H_2 \perp \langle a\rangle \subseteq {H_1}^\perp$ with $q(a) \neq 0$. Then $b = -q(a)x_2 + y_2 + a$ is singular. If $c \neq q(a)$, then $\langle z, b\rangle$ is a second hyperbolic line containing z. If $c = q(a)$, then $\langle b\rangle$ is a singular point in ${H_1}^\perp$, $b \notin H_2$, $b \in z^\perp$. This yields the claim. ■

2.11. *If E is a plane of W with $E = \langle E \cap S\rangle$, then $\langle\pi(E \cap S)\rangle$ is a plane of V.*

Proof. We first consider the case that E is a singular plane. Then

$$E = \langle x_1, x_2, x_3\rangle \subseteq \langle x_1, y_1\rangle \perp \langle x_2, y_2\rangle \perp \langle x_3, y_3\rangle$$

where (x_i, y_i) is a hyperbolic pair ($i = 1, 2, 3$). If a is a singular point in E then $a = \langle\alpha x_1 + \beta x_2 + \gamma x_3\rangle$ with $\alpha, \beta, \gamma \in L$. Hence by (2.1)

$$\pi(a) \subseteq \langle\pi(x_1), \pi(\beta x_2 + \gamma x_3)\rangle \subseteq \langle\pi(x_1), \pi(x_2), \pi(x_3)\rangle.$$

This shows that $\langle\pi(E \cap S)\rangle = \langle\pi(x_1), \pi(x_2), \pi(x_3)\rangle$, which is a plane by (c).

Thus we are left with the case that E is not singular. Then $E = \langle x_1, y_1\rangle \perp z$, where x_1, y_1 are singular points in E with $x_1 \not\subseteq {y_1}^\perp$. Since $\langle\pi(E \cap S)\rangle$ cannot equal $\langle\pi(x_1), \pi(y_1)\rangle$ by (c), we see that $\dim\langle\pi(E \cap S)\rangle \geq 3$. Let $H_1 := \langle x_1, y_1\rangle$ and choose a singular point z' in ${H_1}^\perp$ with $z' \not\subseteq z^\perp$. Let $H_2 := \langle z, z'\rangle$. Then $Q := H_1 \perp H_2$ is a 4^+-space which contains E.

We first neglect the special situation occurring in (2.10) and choose a hyperbolic line H_3 in ${H_1}^\perp$ which contains z and is different from H_2. Then $Q_1 := H_1 \perp H_3$ is a second 4^+–space containing E. Since $<\pi(E \cap S)> \subseteq <\pi(Q \cap S)> \cap <\pi(Q_1 \cap S)>$, we obtain $\dim <\pi(E \cap S)> \leq 3$ by (2.9).

In the special situation we choose a singular point $b \subseteq E^\perp$ as constructed in the proof of (2.10). We show directly that $\dim <\pi(E \cap S)> \leq 3$. For this we assume that $<\pi(E \cap S)> = <\pi(Q \cap S)>$. Then $\pi(x) \subseteq <\pi(E \cap S)> \subseteq <\pi(b^\perp \cap S)>$ for all $x \in Q \cap S$. Hence $Q \subseteq b^\perp$ by (c), a contradiction. ∎

2.12. *Let E be a plane of W with $E = <E \cap S>$ and E not singular. If x is a singular point in W with $\pi(x) \subseteq <\pi(E \cap S)>$, then $x \subseteq E$.*

Proof. If we do not have the special situation occurring in (2.10), then we write $E = Q \cap Q_1$, where Q and Q_1 are 4^+–spaces in W, as in the proof of (2.11). Then (2.8) yields $x \subseteq Q \cap Q_1 = E$.

In the special situation we let b be a singular point in $E^\perp$ as constructed in the proof of (2.10). Then $\pi(x) \subseteq <\pi(E \cap S)> \subseteq <\pi(b^\perp \cap S)>$. Now (c) yields that $x \subseteq b^\perp$. Further, $x \subseteq Q$ by (2.8) as above. Thus $x \subseteq b^\perp \cap Q = E$. ∎

2.13. *If a, b are singular points in W with $H := <a, b>$ a hyperbolic line, then $<\pi(H \cap S)> = <\pi(a), \pi(b)>$.*

Proof. Since the line $<\pi(a), \pi(b)>$ is contained in $<\pi(H \cap S)>$, we have to show that $<\pi(H \cap S)>$ is a line. Let $H = <x_1, y_1> \subseteq <x_1, y_1> \perp <x_2, y_2> =: Q$ with (x_i, y_i) a hyperbolic pair $(i = 1, 2)$. With $E := <x_1, y_1, x_2>$ and $E_1 := <x_1, y_1, y_2>$ we obtain that $<\pi(H \cap S)> \subseteq <\pi(E \cap S)> \cap <\pi(E_1 \cap S)>$. By (2.11) and (c) $<\pi(E \cap S)>$ and $<\pi(E_1 \cap S)>$ are different planes of V, hence (2.13). ∎

2.14. *Let x, y, z be singular points in W with $E := <x, y, z>$ a plane. Then $<\pi(x), \pi(y), \pi(z)>$ is 3–dimensional.*

Proof. If E is singular compare the proof of (2.11). If without loss $H := <x, y>$ is a hyperbolic line, then the assumption $\pi(z) \subseteq <\pi(x), \pi(y)>$ leads to $\pi(z) \subseteq <\pi(H \cap S)>$. Hence $z \subseteq H$ by (2.5), a contradiction. ∎

3 The extension of π to arbitrary points of W

In this section we extend the mapping π to arbitrary points, using that every point is the intersection of two hyperbolic lines (compare the approach in the proof of [HO, (8.1.5)]).

3.1. *Let a be a non–singular point in W. Then $\bigcap <\pi(H \cap S)>$, where H is a hyperbolic line which contains a, is a point in V.*

Proof. For every hyperbolic line H which contains a we set $H' := <\pi(H \cap S)>$. By (2.10) $a = H_0 \cap H_1$ with hyperbolic lines H_0, H_1. By (2.6), (2.13) H_0' and H_1' are different lines in V. Since $E := H_0 + H_1$ is a plane with $E = <E \cap S>$, we obtain $\dim H_0' + H_1' = 3$ by (2.11). Hence $P' := H_0' \cap H_1'$ is a point.

If H_2 is an arbitrary hyperbolic line containing a, then $P' \subseteq H_2{}'$. Otherwise $H_0{}' \cap H_1{}' \cap H_2{}' = 0$. Let $H_0{}' \cap H_1{}' =: K_1$, $H_0{}' \cap H_2{}' =: K_2$, $H_1{}' \cap H_2{}' =: K_3$. As above K_1, K_2, K_3 are points in V. Since $H_0{}' \cap H_1{}' \cap H_2{}' = 0$, we see that K_1, K_2, K_3 are pairwise distinct and $K_1 + K_2 + K_3$ is 3–dimensional. Hence $H_0{}' = K_1 + K_2$, $H_1{}' = K_1 + K_3$, $H_2{}' = K_2 + K_3$.

Let H be a hyperbolic line with $a \subseteq H$. As above $H' \cap H_i{}' \neq 0$ for $i = 0, 1, 2$. Without loss $H' \cap H_0{}' \neq H' \cap H_1{}'$, since $H_0{}' \cap H_1{}' \cap H_2{}' = 0$. Hence

$$H' = (H' \cap H_0{}') + (H' \cap H_1{}') \subseteq H_0{}' + H_1{}' \subseteq K_1 + K_2 + K_3.$$

This yields that $\pi(x) \subseteq H' \subseteq K_1 + K_2 + K_3$ for all singular points x in W with $H := <a, x>$ a hyperbolic line.

Let next x be a singular point with $x \subseteq a^\perp$. We choose a singular point y with $y \subseteq x^\perp$, $y \nsubseteq a^\perp$. Then $L_1 = <x, y>$ is a singular line. Let z be a point in L_1 with $z \neq x, y$, then $z \nsubseteq a^\perp$. Hence $\pi(x) \subseteq <\pi(L_1 \cap S)> = <\pi(z), \pi(y)> \subseteq K_1 + K_2 + K_3$ by the preceding paragraph.

Hence $\pi(x) \subseteq K_1 + K_2 + K_3$ for all singular points in W, a contradiction by (2.4). Thus $P' \subseteq H_2{}'$ and $P' = \bigcap <\pi(H \cap S)>$, where H is a hyperbolic line which contains a, is a point in V. ∎

3.2. Definition:
For each non–singular point a of W we set

$$\pi(a) := \bigcap <\pi(H \cap S)>, \text{ where } H \text{ is a hyperbolic line which contains } a.$$

By (3.1) $\pi(a)$ is a point in V.

Using (2.6), (2.13) we see that the definition given above is also valid for singular points a of W. Thus we have extended π to arbitrary points of W.

3.3. *If x is a singular point and a is a non–singular point of W with $a \subseteq x^\perp$, then $\pi(a) \subseteq <\pi(x^\perp \cap S)>$.*

Proof. Let $x = Lx_1$ and $H_1 = <x_1, y_1>$ with (x_1, y_1) a hyperbolic pair. Since $W = H_1 \perp H_1{}^\perp$, there are $\alpha \in L$ and $s \in H_1{}^\perp$ with $a = L(\alpha x_1 + s)$. As in (2.7) we write $s = \beta x_2 + y_2$, where $0 \neq \beta \in L$ and (x_2, y_2) is a hyperbolic pair in $H_1{}^\perp$. With $H = <\alpha x_1 + \beta x_2, y_2>$ we obtain

$$\begin{aligned} \pi(a) &\subseteq <\pi(H \cap S)> && \text{by (3.2)} \\ &= <\pi(L(\alpha x_1 + \beta x_2)), \pi(Ly_2)> && \text{by (2.13).} \end{aligned}$$

Since $L(\alpha x_1 + \beta x_2)$ and Ly_2 are singular points in $x^\perp$, this yields the claim. ∎

3.4. *π is injective on the set of all points of W.*

Proof. Let a, b be points in W with $\pi(a) = \pi(b)$. If a and b are singular, then $a = b$, since π is injective on singular points.

We next assume that a is singular and b is not singular and lead this to a contradiction. Let H be a hyperbolic line which contains b. Then $\pi(a) = \pi(b) \subseteq$

$<\pi(H\cap S)>$ by (3.2). Using (2.5) we obtain $a \subseteq H$ for every hyperbolic line H containing b. Hence $a = b$, a contradiction.

Thus we are left with the case that a and b are non–singular. We assume that $a \neq b$. Then there exists a singular point x with $x \subseteq a^\perp$ and $x \not\subseteq b^\perp$. With $H := <x, b>$ we obtain $\pi(x), \pi(b) \subseteq <\pi(H\cap S)>$ using (3.2). Since $\pi(x) \neq \pi(b)$ by the paragraph above, (2.13) yields that $<\pi(H\cap S)> = <\pi(x), \pi(b)> = <\pi(x), \pi(a)> \subseteq <\pi(x^\perp \cap S)>$. By (c) this shows $H \subseteq x^\perp$, a contradiction. ∎

3.5. *Let a be a singular point in W and z be an arbitrary point in W with $L_1 := <a, z>$ a line. Then $\pi(y) \subseteq <\pi(a), \pi(z)>$ for every point y in L_1.*

Proof. If $z \not\subseteq a^\perp$, then (3.2), (2.13) and (3.4) yield

$$\pi(y) \subseteq <\pi(L_1 \cap S)> = <\pi(a), \pi(z)>.$$

If $z \subseteq a^\perp$ and z singular, then the claim follows by (2.1).

So let $z \subseteq a^\perp$ and z not singular. Let $a = Lx_1$, $H_1 = <x_1, y_1>$ with (x_1, y_1) a hyperbolic pair and $z = L(cx_1 + s)$, where $c \in L$, $s \in {H_1}^\perp$. We choose different hyperbolic lines H and H_0 in ${H_1}^\perp$ containing s, using (2.10) and excluding the special case mentioned there. Then $E := H \perp a$ and $E_0 := H_0 \perp a$ are different planes of W which are generated by their singular points. We write $H = <s, t>$, $H_0 = <s, t_0>$ where t and t_0 are singular points in ${H_1}^\perp$.

For $a \neq y \subseteq <a, z>$, y is contained in the two different hyperbolic lines $M := <y, t>$ and $M_0 := <y, t_0>$. Because of $\pi(y) = <\pi(M\cap S)> \cap <\pi(M_0 \cap S)>$ by (3.2), $\pi(y)$ is contained in $E' := <\pi(E\cap S)>$ and in ${E_0}' := <\pi(E_0 \cap S)>$. By (2.11) and (2.12) E' and ${E_0}'$ are different planes in V. Since $X := <z, t> \subseteq E$ is a hyperbolic line, (3.2) yields that $\pi(z) \subseteq <\pi(X\cap S)> \subseteq <\pi(E\cap S)> = E'$. Similarly we have $\pi(z) \subseteq {E_0}'$. Hence $E' \cap {E_0}'$ contains $<\pi(a), \pi(z)>$. We obtain $\pi(y) \subseteq E' \cap {E_0}' = <\pi(a), \pi(z)>$ by (3.4).

In the remaining special situation we let b be a singular point in $<x_1, y_1, z>^\perp$, as constructed in the proof of (2.10). As above $<\pi(y), \pi(a), \pi(z)> \subseteq E'$. We assume that $\pi(y) \not\subseteq <\pi(a), \pi(z)>$. Then $E' = <\pi(y), \pi(a), \pi(z)> \subseteq <\pi(b^\perp \cap S)>$ by (3.3). Now (c) yields $E \subseteq b^\perp$, a contradiction. ∎

4 Construction of a semi–linear mapping on 4^+–spaces

In this section we show that the mapping π restricted to the set of points of a 4^+–space in W is induced by a semi–linear mapping. We use the idea of the proof of [JW, (1.2.4)]. The main ingredient in Section 4 is (3.5). For $w \in W$, we often write $\pi(w)$ instead of $\pi(Lw)$.

4.1. *Let Q be a 4^+-space in W. Then there is an embedding $\alpha : L \to K$ and an injective semi–linear (with respect to α) mapping $\varphi : Q \to V$ with $\pi(Lx) = K\varphi(x)$ for $x \in Q$.*

Proof. Let $Q = <x_1, y_1> \perp <x_2, y_2>$ be a 4^+–space in W. Let x_1, x_2, y_1, y_2 be generated by e_1, e_2, e_3, e_4 respectively and let $\pi(e_i)$ be generated by f_i $(i = 1, \ldots, 4)$. By (2.4) f_1, f_2, f_3, f_4 are linearly independent. For $j \geq 2$ we have $L(e_1 + e_j) \subseteq <e_1, e_j>$, hence $\pi(e_1 + e_j) \subseteq <\pi(e_1), \pi(e_j)>$ by (3.5). Thus there is a $b \in K$ with $\pi(e_1 + e_j) = K(f_1 + bf_j)$. Further, $b \neq 0$, since π is injective. Replacing f_j by bf_j we may assume

$$\pi(e_i) = Kf_i, \quad \pi(e_1 + e_j) = K(f_1 + f_j)$$

for $i, j = 1, \ldots, 4$, $j \geq 2$.

For $j \geq 2$ and $c \in L$ we have $\pi(e_1 + ce_j) \neq \pi(e_j)$. Hence there is a scalar denoted by $\alpha_j(c)$ in K with $\pi(e_1 + ce_j) = K(f_1 + \alpha_j(c)f_j)$ as in the preceding paragraph. This defines a mapping $\alpha_j : L \to K$ with $\alpha_j(0) = 0$ and $\alpha_j(1) = 1$.

We show:

$(*)$ $\alpha_j = \alpha_2 =: \alpha$ is an embedding from L in K for $j \geq 2$.

Let $i, j \geq 2$, $i \neq j$, and $c_i, c_j \in L$. Then $L(e_1 + c_je_j + c_ie_i)$ is contained in $<e_1 + c_je_j, e_i>$ and in $<e_1 + c_ie_i, e_j>$. Hence (3.5) and the fact that f_1, f_i, f_j are linearly independent yield

$$\pi(e_1 + c_je_j + c_ie_i) = K(f_1 + \alpha_j(c_j)f_j + \alpha_i(c_i)f_i)$$

for $i, j \geq 2$, $i \neq j$.

Similarly, $L(c_je_j + e_i)$ is contained in $<e_j, e_i>$ and in $<e_1, e_1 + c_je_j + e_i>$. Using the above paragraph and $\alpha_i(1) = 1$, we obtain

$$\pi(c_je_j + e_i) = K(\alpha_j(c_j)f_j + f_i)$$

for $i, j \geq 2$, $i \neq j$.

Let $i \geq 3$ and $c, d \in L$. Then $L(e_1 + (c + d)e_2 + e_i) \subseteq <e_1 + ce_2, de_2 + e_i>$. Since $e_1 + ce_2$ is singular, applying π and (3.5) yield $K(f_1 + \alpha(c + d)f_2 + f_i) \subseteq <f_1 + \alpha(c)f_2, \alpha(d)f_2 + f_i>$. Hence

$$\alpha(c + d) = \alpha(c) + \alpha(d)$$

for $c, d \in L$ by comparing coefficients.

Similarly, for $c, d \in L$ we have $L(e_1 + cde_2 + ce_i) \subseteq <e_1, de_2 + e_i>$. Thus $K(f_1 + \alpha(cd)f_2 + \alpha_i(c)f_i) \subseteq <f_1, \alpha(d)f_2 + f_i>$ as above. This leads to

$$\alpha(cd) = \alpha_i(c)\alpha(d)$$

for $i \geq 3$, $c, d \in L$ by comparing coefficients.

The special case $d = 1$ yields $\alpha_i = \alpha$ for $i \geq 3$ and α is a homomorphism.

Further, if $c \in L$ with $\alpha(c) = 0$ then $\pi(e_1 + ce_2) = Kf_1 = \pi(e_1)$. Since π is injective on singular points we obtain $c = 0$, and α is injective. Thus $\alpha : L \to K$ is an embedding and $(*)$ holds.

Next we show:

$(**)$ If $x = \sum_{i=1}^{4} c_ie_i \in W$, then $\pi(Lx) = K(\sum_{i=1}^{4} \alpha(c_i)f_i)$.

If only one of the coefficients c_i is different from 0 the claim holds.

We first assume that exactly two of the coefficients c_i are different from 0. If $c_1 \neq 0$, we use the fact that $\pi(e_1 + ce_i) = K(f_1 + \alpha(c)f_i)$ for $c \in L$. If $c_1 = 0$, then there are $i, j \geq 2$, $i \neq j$ with $Lx = L(c_ie_i + c_je_j) \subseteq < e_1, e_1 + c_ie_i + c_je_j >$. Further, $Lx \subseteq < e_i, e_j >$. We apply π and use the intermediate step of the proof that α is an embedding. This yields the claim.

We now assume that exactly three of the coefficients c_i are different from 0. If $c_1 \neq 0$, we use the intermediate step as in the preceding paragraph. If $c_1 = 0$, then $Lx \subseteq < e_1 - c_2e_2 - c_3e_3, e_1 + c_4e_4 >$ and $Lx \subseteq < e_1 - c_3e_3 - c_4e_4, e_1 + c_2e_2 >$. Since $e_1 + c_4e_4$ and $e_1 + c_2e_2$ are singular, we can use (3.5) and the claim follows.

Finally, assume that all coefficients c_i are different from 0. Then we have $Lx \subseteq < c_1e_1 + c_3e_3 + c_4e_4, e_2 >$ and $Lx \subseteq < c_1e_1 + c_2e_2, c_3e_3 + c_4e_4 >$ with e_2 and $c_1e_1 + c_2e_2$ singular. Hence we can finish the proof of $(**)$ as above.

The mapping $\varphi : Q \to V$ defined by $\varphi(\sum_{i=1}^4 c_ie_i) := \sum_{i=1}^4 \alpha(c_i)f_i$ is semi–linear (with respect to the embedding $\alpha : L \to K$) and satisfies $\pi(Lx) = K\varphi(x)$ for $x \in Q$. Further φ is injective, since α is. ∎

5 The construction of a semi–linear mapping inducing π

In this last section of the proof of the Main Theorem we show that the mapping π from the set of all points of W into the set of points of V constructed in (3.2) is induced by a semi–linear mapping. We proceed similarly as in the proof of the Fundamental Theorem of Projective Geometry in [Ba, p. 44].

5.1. *Let Lx and Ly be different points of W. At least one of Lx and Ly is assumed to be singular. Let $\pi(Lx) = Kx'$. Then there exists a unique $y' \in V$ such that $\pi(Ly) = Ky'$ and $\pi(L(x - y)) = K(x' - y')$. We write $h(x, x', y) := y'$ and set $h(x, x', 0) := 0$.*

Proof. Since $L(x - y) \subseteq Lx + Ly$, (3.5) yields that $\pi(L(x - y)) \subseteq \pi(Lx) + \pi(Ly)$. Hence $\pi(L(x - y)) = Kt$, where $t = cx' - z$ with $c \in K$ and $z \in \pi(Ly)$. Necessarily $c \neq 0$ and $z \neq 0$, since π is injective by (3.4). Hence $y' := c^{-1}z$ satisfies the above conditions. The uniqueness of y' is straight forward. ∎

5.2. *If Lx and Ly are different points with at least one singular, then we have $h(x, x', y) = y'$ if and only if $h(y, y', x) = x'$.*

Proof. This is obvious from the definition of h in (5.1). ∎

5.3. *Let $u, v, t \in W$ be linearly independent with u, v singular and $u \notin v^{\perp}$. Let $\pi(Lu) = Ku'$, $\pi(Lv) = Kv'$, $\pi(Lt) = Kt'$. If $h(u, u', v) = v'$ and $h(u, u', t) = t'$, then $h(v, v', t) = t'$.*

Proof. We have to show that $\pi(L(v - t)) = K(v' - t')$. Since $< u, v >$ is a hyperbolic line, the plane $E := < u, v, t >$ is contained in some 4^+–space Q. Let $a \in E$ be singular with $E = <u, v, a>$. Then $\pi(Lu) + \pi(Lv) + \pi(La)$ is 3–dimensional by (2.14). Using (3.5) we obtain that $\pi(La) \subseteq \pi(Lu) + \pi(Lv) + \pi(Lt)$. Hence u', v', t' are linearly independent. By (4.1) there is an embedding $\alpha : L \to K$ and an injective semi–linear mapping $\varphi : Q \to V$ with $\pi(Lx) = K\varphi(x)$ for $x \in Q$. Hence $Ku' = K\varphi(u)$, $Kz' = K\varphi(z)$, $K(u' - z') = K\varphi(u - z)$ for $z \in \{v, t\}$. Comparing coefficients yields $\varphi(v - t) = \varphi(v) - \varphi(t) = \lambda(v' - t')$ for some $\lambda \in K^*$. Hence $\pi(L(v - t)) = K(v' - t')$. ∎

5.4. *Let $x, a, b \in W$ be linearly independent and singular and let $\pi(Lx) = Kx'$. Then $h(x, x', a + b) = h(x, x', a) + h(x, x', b)$.*

Proof. We set $a' := h(x, x', a)$, $b' := h(x, x', b)$. By (2.14) x', a', b' are linearly independent. We have to prove that $h(x, x', a + b) = a' + b'$. By definition of h we have to show that $\pi(L(a + b)) = K(a' + b')$ and that $\pi(L(x - a - b)) = K(x' - a' - b')$.

We first consider the second equation. Since $L(x - a - b) \subseteq L(x - a) + Lb$ and $L(x - a - b) \subseteq L(x - b) + La$ with b and a singular, we can apply (3.5). Thus $\pi(L(x - a - b))$ is contained in $K(x' - a') + Kb'$ and in $K(x' - b') + Ka'$. Comparing coefficients yields $\pi(L(x - a - b)) = K(x' - a' - b')$.

Further, $L(a + b) \subseteq La + Lb$ and $L(a + b) \subseteq Lx + L(x - a - b)$. Since a and x are singular, (3.5) and the preceding paragraph show that $\pi(L(a + b))$ is contained in $Ka' + Kb'$ and in $Kx' + K(x' - a' - b')$. Hence $\pi(L(a + b)) = K(a' + b')$. ∎

5.5. *Let $x, a, b \in W$ be singular with $Lx \not\subseteq La + Lb$ and let $\pi(Lx) = Kx'$. Then $h(x, x', a + b) = h(x, x', a) + h(x, x', b)$.*

Proof. If x, a, b are linearly independent this is (5.4). Thus, we may assume that x, a, b are linearly dependent and hence $La = Lb \neq Lx$. Since W contains singular lines, there exists a $w \in W$ with w, $w + a$ singular and x, a, w linearly independent. We obtain

$$
\begin{aligned}
&h(x, x', w) + h(x, x', a + b) \\
&= h(x, x', w + a + b) && \text{by (5.4)} \\
&= h(x, x', w + a) + h(x, x', b) && \text{by (5.4)} \\
&= h(x, x', w) + h(x, x', a) + h(x, x', b) && \text{by (5.4).}
\end{aligned}
$$

In the first application of (5.4) $x, w, a + b$ are singular and linearly independent (or $a + b = 0$). Subtraction of $h(x, x', w)$ yields the claim. ∎

5.6. Since W contains singular lines, there are $u, v, w \in W$ singular and linearly independent with $u \notin v^\perp$, $u \notin w^\perp$, $v \notin w^\perp$. We let $\pi(Lu) = Ku'$ and set $v' := h(u, u', v)$, $w' := h(u, u', w)$.

5.7. *If $x, y \in \{u, v, w\}$ with $x \neq y$, then $h(x, x', y) = y'$.*

Proof. By (5.2) we have to consider the cases $(x, y) = (u, v), (u, w), (v, w)$. In the first two cases the claim holds by definition and in the third case we use (5.3). ∎

5.8. *Let $0 \neq t \in W$. Then two of the three expressions $h(u, u', t)$, $h(v, v', t)$ and $h(w, w', t)$ are defined and equal. We denote this value by $\varphi(t)$.*

Proof. We have $Lt \nsubseteq (Lu + Lv) \cap (Lu + Lw) \cap (Lv + Lw) = 0$. Hence without loss $Lt \nsubseteq Lu + Lv$ and u, v, t are linearly independent. Now $h(u, u', v) = v'$ and $h(u, u', t) := t'$ yields $h(v, v', t) := t'$ by (5.3). ∎

5.9. *The mapping $\varphi : W \to V$ defined in* (5.8) *(we set $\varphi(0) = 0$) satisfies $\pi(Lt) = K\varphi(t)$ for $0 \neq t \in W$.*

Proof. We have $\varphi(t) = h(x, x', t)$ for a suitable $x \in \{u, v, t\}$, hence $\pi(Lt) = K\varphi(t)$. ∎

5.10. *We have $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in W$ with a, b singular.*

Proof. Without loss $a, b \neq 0$. We first consider the case that $a+b \neq 0$. Then by (5.8) $\varphi(a + b) = h(x, x', a + b) = h(y, y', a + b)$ for suitable x, y with $\{u, v, w\} = \{x, y, z\}$ and $L(a + b) \nsubseteq Lx + Ly$. Hence $Lx \nsubseteq La + Lb$ or $Ly \nsubseteq La + Lb$. Without loss $Lx \nsubseteq La + Lb$. Since x, a, b are singular, (5.5) yields $h(x, x', a+b) = h(x, x', a)+h(x, x', b)$. Since $La \nsubseteq Lx + Ly$ or $La \nsubseteq Lx + Lz$, we have $h(x, x', a) = \varphi(a)$ by (5.8). Similarly $h(x, x', b) = \varphi(b)$, and the claim follows.

If $a + b = 0$, we choose $c \neq 0$ singular with $c \neq a$ and $b + c$ singular. Then the first case yields $\varphi(c) = \varphi(a + b + c) = \varphi(a) + \varphi(b + c) = \varphi(a) + \varphi(b) + \varphi(c)$, i. e. (5.10). ∎

5.11. *We have $\varphi(a + b) = \varphi(a) + \varphi(b)$ for $a, b \in W$ with a singular.*

Proof. Let $a = x_1$, $H_1 = <x_1, y_1>$ with (x_1, y_1) a hyperbolic pair and $b = \alpha x_1 + \beta y_1 + s$ with $s \in {H_1}^{\perp}$. We write $s = \gamma x_2 + \delta y_2$, where (x_2, y_2) is a hyperbolic pair in ${H_1}^{\perp}$. Hence

$$
\begin{aligned}
\varphi(a + b) &= \varphi((x_1 + \alpha x_1 + \gamma x_2) + (\beta y_1 + \delta y_2)) \\
&= \varphi(x_1 + (\alpha x_1 + \gamma x_2)) + \varphi(\beta y_1 + \delta y_2) && \text{by (5.10)} \\
&= \varphi(x_1) + \varphi(\alpha x_1 + \gamma x_2) + \varphi(\beta y_1 + \delta y_2) && \text{by (5.10)} \\
&= \varphi(x_1) + \varphi(\alpha x_1 + \gamma x_2 + \beta y_1 + \delta y_2) && \text{by (5.10)} \\
&= \varphi(a) + \varphi(b).
\end{aligned}
$$

∎

5.12. *If $n \in \mathbf{N}$ and $a_1, \ldots, a_n$ are singular, then $\varphi(\sum_{i=1}^{n} a_i) = \sum_{i=1}^{n} \varphi(a_i)$.*

Proof. We may assume $a_i \neq 0$ for $i = 1, \ldots, n$. With (5.11) the claim follows by induction on n. ∎

5.13. *The mapping φ is additive, i.e. $\varphi(a + b) = \varphi(a) + \varphi(b)$ for $a, b \in W$.*

Proof. Since W is generated by its singular points, we can choose a basis $\{e_i \mid i \in I\}$ where e_i is singular ($i \in I$). We write a and b as linear combination of the basis vectors and apply (5.12). ∎

5.14. *The mapping φ is injective.*

Proof. Since $\varphi : (W, +) \to (V, +)$ is a homomorphism, we have to show that $\ker \varphi = 0$. If $0 \neq t \in W$ with $\varphi(t) = 0$, then $\pi(Lt) = K\varphi(t) = 0$, a contradiction. ■

5.15. For $0 \neq t \in W$ and $0 \neq \lambda \in L$ we have $K\varphi(t) = \pi(Lt) = \pi(L(\lambda t)) = K\varphi(\lambda t)$. Hence $\varphi(\lambda t) = \alpha(\lambda, t)\varphi(t)$, where $\alpha(\lambda, t) \in K$. We set $\alpha(0, t) := 0$.

5.16. *If $0 \neq t_1, t_2 \in W$, then $\alpha(\lambda, t_1) = \alpha(\lambda, t_2)$ for $\lambda \in L$.*

Proof. If t_1 and t_2 are linearly independent, then the claim follows by a straight forward calculation using the definition of α. If t_1 and t_2 are linearly dependent, we choose $0 \neq t \in W$ with $Lt \neq Lt_1 = Lt_2$ and restrict to the first case. ■

5.17. For $\lambda \in L$ we set $\alpha(\lambda) := \alpha(\lambda, t_0)$ where $0 \neq t_0 \in W$. By (5.16) this definition is independent of the choice of t_0 and we have $\varphi(\lambda t) = \alpha(\lambda)\varphi(t)$ for $\lambda \in L$, $t \in W$.

5.18. *The mapping $\alpha : L \to K$ is an embedding.*

Proof. This is straight forward. ■

5.19. The steps (5.1) to (5.18) show that there is an embedding $\alpha : L \to K$ and an injective semi–linear (with respect to α) mapping such that $\pi(Lx) = K\varphi(x)$ for $0 \neq x \in W$. This completes the proof of the Main Theorem.

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