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The Mean Curvature of the Second Fundamental Form of a Hypersurface

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Abstract. An expression for the first variation of the area functional of the second fundamental form is given for a hypersurface in a semi-Riemannian space. The concept of the "mean curvature of the second fundamental form" is then introduced for hypersurfaces in semi-Riemannian spaces. Some characterisations of extrinsic hyperspheres in terms of this curvature are given.

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1. Introduction and outline of the article

We shall be concerned with hypersurfaces of a semi-Riemannian manifold, for which the real-valued second fundamental form $\mathbb{I}$ is a semi-Riemannian metrical tensor. The geometry of such hypersurfaces can be explored with respect to either the first or the second fundamental form.

In analogy with the classical study of the geometry of hypersurfaces as determined by their first fundamental form, a distinction can be made between the intrinsic geometry of the second fundamental form, which is determined by measurements of $\mathbb{I}$-lengths on the hypersurface only, and the extrinsic geometry of the second fundamental form, which concerns those measurements for which the geometry of the second fundamental form of the hypersurface is compared with the corresponding geometry of nearby hypersurfaces.

It is a natural question to investigate the relation between the intrinsic geometry of the second fundamental form and the shape of the original hypersurface, and for this purpose the intrinsic curvatures of the second fundamental form have already been studied. Numerous results in this direction have been established for ovaloids, i.e., for compact hypersurfaces in a Euclidean space with a positive definite second fundamental form. For example, R. Schneider's theorem characterises the hyperspheres as the only ovaloids for which the second fundamental form has constant sectional curvature [19]. Some generalisations of this theorem for surfaces in certain Lorentzian manifolds have been found by J. A. Aledo, A. Romero, et al. [2], [3].

However, in the present article we are not concerned with the geometry of the second fundamental form from the intrinsic point of view, but we will study an aspect of the "extrinsic" geometry of the second fundamental form. As is known, the mean curvature H of a hypersurface of a semi-Riemannian manifold describes the instantaneous response of the area functional with respect to deformations of the hypersurface. Since we are studying hypersurfaces for which the second fundamental form is a semi-Riemannian metrical tensor, areas can be measured with respect to the second fundamental form as well, so we can associate to any such hypersurface M its area as measured in the geometry of the second fundamental form. This area, which will be denoted by $\mathrm{Area}_{\mathbb{I}}(M)$, is related to the classical area element $\mathrm{d}\Omega$ by

$$\mathrm{Area}_{\mathbb{I}}(M) = \int_M \sqrt{|\det A|}\,\mathrm{d}\Omega\,,$$

where A denotes the shape operator of the hypersurface.

In this article, the notion of mean curvature will be tailored to the geometry of the second fundamental form: the function which measures the rate of change of $\mathrm{Area}_{\mathbb{I}}(M)$ under a deformation of M, will be called the *mean curvature of the second fundamental form* and denoted by $H_{\mathbb{I}}$. In this way, a concept which belongs to the extrinsic geometry of the second fundamental form is introduced in analogy with a well-known concept in the classical theory of hypersurfaces. The mean curvature of the second fundamental form was defined originally by

E. Glässner [9], [10] for surfaces in $\mathbb{E}^3$. The corresponding variational problem has been studied by F. Dillen and W. Sodsiri [7] for surfaces in $\mathbb{E}^3_1$, and for Riemannian surfaces in a three-dimensional semi-Riemannian manifold in [14].

Some characterisations of the spheres in which this curvature H_{II} is involved have been found. For example, it has been shown that the spheres are the only ovaloids in $\mathbb{E}^3$ which satisfy $H_{\mathrm{II}} = C\sqrt{K}$; furthermore, the spheres are the only ovaloids on which $H_{\mathrm{II}} - K_{\mathrm{II}}$ does not change sign (see [22] and G. Stamou's [21]).

In the initiating Section 2 of this article, the notation will be explained and several useful formulae from the theory of hypersurfaces will be briefly recalled.

In the following Section 3, the first variation of the area functional of the second fundamental form is calculated and the mean curvature of the second fundamental form is defined.

In the subsequent Sections 4–7 the mean curvature of the second fundamental form will be employed to give several characterisations of extrinsic hyperspheres as the only hypersurfaces in space forms, an Einstein space, and a three-dimensional manifold, respectively, which can satisfy certain inequalities in which the mean curvature of the second fundamental form is involved.

In Section 8 the expression for H_{II} will be investigated for curves. This is of particular interest, since the length of the second fundamental form of a curve γ,

$$\mathrm{Length}_{\mathrm{II}}(\gamma) = \int \sqrt{|\kappa|}\,\mathrm{d}s\,,$$

(where κ is the geodesic curvature and s an arc-length parameter) is a modification of the classical bending energy

$$\int \kappa^2\,\mathrm{d}s$$

which has already been studied by D. Bernoulli and L. Euler. Moreover, the results we present agree with W. Blaschke's description of J. Radon's variational problem [5] and with a more recent article of J. Arroyo, O. J. Garay and J. J. Mencía [4].

In the final Section 9, the function H_{II} will be investigated for (sufficiently small) geodesic hyperspheres in a Riemannian manifold by means of the method of power series expansions, which was applied extensively by A. Gray [11], and also by B.-Y. Chen and L. Vanhecke [6], [12]. Furthermore, we address the question of whether the locally flat spaces are characterised by the property that every geodesic hypersphere has the same II-area as a Euclidean hypersphere with the same radius.

2. Definitions, notation, and useful formulae

2.1. Assumption

All hypersurfaces are understood to be embedded and connected.

2.2. Nomenclature

A hypersurface in a semi-Riemannian manifold is said to be (semi-)Riemannian if the restriction of the metric to the hypersurface is a (semi-)Riemannian metrical tensor.

2.3. Notation

Since a hypersurface M in a manifold $\overline{M}$ will be studied, geometric objects in $\overline{M}$ are distinguished from their analogues in M with a $\overline{\text{bar}}$. Geometric entities derived from the second fundamental form are distinguished from those derived from the first fundamental form by means of a sub- or superscript $\mathrm{I\!I}$. For example, the area element obtained from the second fundamental form will be written as $\mathrm{d}\Omega_{\mathrm{I\!I}}$.

2.4. Notation

The set of all vector fields on a manifold M will be denoted by $\mathfrak{X}(M)$. Furthermore, $\mathfrak{F}(M)$ stands for the set of all real-valued functions on M. If (M, g) is a semi-Riemannian submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, the set of all vector fields on M which take values in the tangent bundle $\mathrm{T}\overline{M}$ is denoted by $\overline{\mathfrak{X}}(M)$. The orthogonal projection $\mathrm{T}_p\overline{M} \to \mathrm{T}_pM$ will be denoted by $[\cdot]^T$.

2.5. The Laplacian

The sign of the Laplacian will be chosen so that $\Delta f = f''$ for a real-valued function on $\mathbb{R}$.

2.6. The fundamental forms

Let M be a semi-Riemannian hypersurface of dimension m in a semi-Riemannian manifold $(\overline{M}, \overline{g})$. We shall suppose that a unit normal vector field $U \in \overline{\mathfrak{X}}(M)$ has been chosen on M. The shape operator A, the second fundamental form $\mathrm{I\!I}$ and the third fundamental form $\mathrm{I\!I\!I}$ of the hypersurface M are defined by the formulae

$$\left\{\begin{array}{lcccccccc} A & : & \mathfrak{X}(M) & \to & \mathfrak{X}(M) & : & V & \mapsto & -\overline{\nabla}_V U\,; \\ \mathrm{I\!I} & : & \mathfrak{X}(M)\times\mathfrak{X}(M) & \to & \mathfrak{F}(M) & : & (V,W) & \mapsto & \alpha\, g(A(V),W)\,; \\ \mathrm{I\!I\!I} & : & \mathfrak{X}(M)\times\mathfrak{X}(M) & \to & \mathfrak{F}(M) & : & (V,W) & \mapsto & g(A(V),A(W))\,, \end{array}\right. \tag{1}$$

where $\alpha = \overline{g}(U, U) = \pm 1$. It will be assumed that the second fundamental form is a semi-Riemannian metric on M.

2.7. Frame fields

Let $\{E_1, \dots, E_m\}$ denote a frame field on M which is orthonormal with respect to the first fundamental form g. Define ε_i $(i = 1, \dots, m)$ by $\varepsilon_i = g(E_i, E_i) = \pm 1$. Furthermore, let $\{V_1, \dots, V_m\}$ be a frame field on M which is orthonormal with respect to the second fundamental form $\mathrm{I\!I}$. Define κ_i $(i = 1, \dots, m)$ by $\kappa_i = \mathrm{I\!I}(V_i, V_i) = \pm 1$.

2.8. Curvature

The following convention concerning the Riemann-Christoffel curvature tensor R will be made: for $X, Y, Z \in \mathfrak{X}(M)$, we define $\mathrm{R}(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$. The Ricci tensor and the scalar curvature will be denoted by Ric and S. The mean curvature H of the hypersurface M is defined by

$$H = \frac{\alpha}{m}\mathrm{tr}(A) = \frac{1}{m}\sum_{i=1}^{m} \mathrm{I\!I}(E_i, E_i)\varepsilon_i \,.$$

The (M,g)-sectional curvature of the plane spanned by two vectors v_p and w_p in T_pM, will be denoted by $K(v_p, w_p)$. The symbols $K^{\mathrm{I\!I}}(v_p, w_p)$ and $\overline{K}(v_p, w_p)$ will be used in accordance with the remark of Subsection 2.3. Similarly, the scalar curvature of the second fundamental form will be denoted by $S_{\mathrm{I\!I}}$.

2.9. The difference tensor L

The difference tensor L between the two Levi-Civita connections $\nabla^{\mathrm{I\!I}}$ and ∇ is defined by

$$L(X,Y) = \nabla^{\mathrm{I\!I}}_X Y - \nabla_X Y \,,$$

where $X, Y \in \mathfrak{X}(M)$. The trace of L with respect to $\mathrm{I\!I}$ is defined to be the vector field

$$\mathrm{tr}_{\mathrm{I\!I}} L = \sum_{i=1}^{m} L(V_i, V_i)\kappa_i \,,$$

where V_i and κ_i have been defined in Subsection 2.7.

2.10. The equations of Gauss and Codazzi

The Riemann-Christoffel curvature tensor R of the hypersurface M is related to the second fundamental form by means of the Gauss equation

$$g(\mathrm{R}(X,Y)Z,W) = \overline{g}(\overline{\mathrm{R}}(X,Y)Z,W) + \alpha\,\Big(\mathrm{I\!I}(X,Z)\,\mathrm{I\!I}(Y,W) - \mathrm{I\!I}(X,W)\,\mathrm{I\!I}(Y,Z)\Big) \,,$$

which is valid for all tangent vector fields $X, Y, Z, W \in \mathfrak{X}(M)$. As a consequence, we have

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) - \alpha\,\overline{g}(\overline{\mathrm{R}}(X,U)Y,U) + \alpha\, m\, H\,\mathrm{I\!I}(X,Y) - \alpha\,\mathrm{I\!I\!I}(X,Y)\,. \quad (2)$$

The Codazzi equation of the hypersurface is

$$(\nabla_X A)Y - (\nabla_Y A)X = \overline{\mathrm{R}}(X,Y)U \,,$$

for all $X, Y \in \mathfrak{X}(M)$.

3. The variation of the area of the second fundamental form

3.1. The area functional of the second fundamental form

Let $\mathcal{E}$ denote the set of all hypersurfaces in a semi-Riemannian manifold $(\overline{M},\overline{g})$ for which the first as well as the second fundamental form is a semi-Riemannian metrical tensor. Our first objective is to determine the critical points of the area functional of the second fundamental form

$$\mathrm{Area}_{\mathrm{I\!I}} : \mathcal{E} \to \mathbb{R} : M \mapsto \mathrm{Area}_{\mathrm{I\!I}}(M) = \int_M \mathrm{d}\Omega_{\mathrm{I\!I}}\,.$$

3.2. The mean curvature of the second fundamental form

Definition 3.1. *Let M be a hypersurface in a semi-Riemannian manifold $(\overline{M},\overline{g})$, and suppose that the first as well as the second fundamental form of M is a semi-Riemannian metrical tensor. Let*

$$\mu : \,]-\varepsilon,\varepsilon[\, \times M \to \overline{M} : (s,p) \mapsto \mu_s(p)$$

be a mapping such that

$$\begin{cases} \mu_s(M) \in \mathcal{E} \text{ for all } s; \\ \mu_s(p) = p \text{ for all } p \text{ outside of a compact set of } M \text{ and all } s; \\ \mu_0(p) = p \text{ for all } p \in M. \end{cases}$$

Then μ will be called a variation of M in $\mathcal{E}$.

Definition 3.2. *Let M be a semi-Riemannian hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$ which belongs to the class $\mathcal{E}$. The vector field $\mathcal{Z}$ in $\mathfrak{X}(M)$ is defined by*

$$\mathcal{Z} = \sum_{i=1}^{m} \kappa_i \, A^{\leftarrow}\left(\left[\overline{\mathrm{R}}(V_i,U)V_i\right]^T\right).$$

Here $A^{\leftarrow}$ denotes the inverse of the shape operator A, and V_i and κ_i were defined in Subsection 2.7.

It can easily be seen that the vector field $\mathcal{Z}$ vanishes if $(\overline{M},\overline{g})$ has constant sectional curvature. If $\overline{M}$ has dimension three, the vector field $\mathcal{Z}$ is equal to $\frac{A(Z)}{\det A}$, where the vector field Z has been defined in [3, 14] by the condition

$$\forall\, X \in \mathfrak{X}(M), \qquad \overline{\mathrm{Ric}}(U,X) = \mathrm{I\!I}(Z,X)\,.$$

Theorem 3.3. *Let M be a hypersurface in a semi-Riemannian manifold $(\overline{M},\overline{g})$ for which the first as well as the second fundamental form is a semi-Riemannian metrical tensor. Let μ be a variation of M in $\mathcal{E}$, for which the variational vector* *field has normal component* $f\,U$. *The variation of the area functional* $\mathrm{Area}_{\mathrm{I\!I}}$ *is given by*

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Area}_{\mathrm{I\!I}}(\mu_s M) = -\alpha \int_M f \cdot \frac{1}{2} \Big(m\,H - \sum_{i=1}^{m} \overline{g}(\overline{\mathrm{R}}(V_i, U)V_i, U)\,\kappa_i + \frac{\alpha}{2}\Delta_{\mathrm{I\!I}}\log|\det A| - \alpha\,\mathrm{div}_{\mathrm{I\!I}}\mathcal{Z} \Big)\, \mathrm{d}\Omega_{\mathrm{I\!I}}\,.$$

This theorem can be proved by similar methods to those used in [14] (see also [23]). The formula for the variation of the second fundamental form which was given there, can be generalised to hypersurfaces in the following way:

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{I\!I}(\mu_s)(X,Y) = \alpha\, f\, \Big(\overline{g}(\overline{\mathrm{R}}(U,X)U,Y) - \mathrm{I\!I\!I}(X,Y)\Big) + \mathrm{Hess}_f(X,Y)\,.$$

The left-hand side of this expression, which is valid if the variational vector field is equal to fU, is defined as in [14].

Definition 3.4. *Let* M *be an* m*-dimensional hypersurface in a semi-Riemannian manifold* $(\overline{M}, \overline{g})$ *for which both the first and the second fundamental forms are semi-Riemannian metrical tensors. The mean curvature of the second fundamental form* $H_{\mathrm{I\!I}}$ *is defined by*

$$H_{\mathrm{I\!I}} = \frac{1}{2}\left(m\,H - \sum_{i=1}^{m} \overline{g}(\overline{\mathrm{R}}(V_i, U)V_i, U)\,\kappa_i + \frac{\alpha}{2}\Delta_{\mathrm{I\!I}}\log|\det A| - \alpha\,\mathrm{div}_{\mathrm{I\!I}}\mathcal{Z} \right). \tag{3}$$

If $H_{\mathrm{I\!I}} = 0$, *the hypersurface will be called* $\mathrm{I\!I}$*-minimal.*

Remark 3.5. This definition extends those of [9, 10]; in [14], the sign of $H_{\mathrm{I\!I}}$ was chosen differently.

Example 3.6. The standard embedding of $\mathrm{S}^m(\frac{1}{\sqrt{2}})$ in $\mathrm{S}^{m+1}(1)$ is $\mathrm{I\!I}$-minimal. Furthermore, the standard embedding of $\mathrm{S}^k(\frac{1}{\sqrt{2}}) \times \mathrm{S}^{m-k}(\frac{1}{\sqrt{2}})$ in $\mathrm{S}^{m+1}(1)$ (see, e.g., [16]) is a $\mathrm{I\!I}$-minimal hypersurface ($k = 1, \ldots, m-1$). These assertions can be proved with ease when one takes into account the fact that these hypersurfaces are parallel (in the sense that $\nabla \mathrm{I\!I} = 0$).

Remark 3.7. As a consequence of Theorem 3.3 and Definition 3.4, we obtain the following formulae for the variation of the classical area (Area) and of the area of the second fundamental form ($\mathrm{Area}_{\mathrm{I\!I}}$):

$$\begin{cases} \left.\dfrac{\partial}{\partial s}\right|_{s=0} \mathrm{Area}(\mu_s(M)) = -m\,\alpha \displaystyle\int f H\, \mathrm{d}\Omega\,; \\[2ex] \left.\dfrac{\partial}{\partial s}\right|_{s=0} \mathrm{Area}_{\mathrm{I\!I}}(\mu_s(M)) = -\alpha \displaystyle\int f H_{\mathrm{I\!I}}\, \mathrm{d}\Omega_{\mathrm{I\!I}}\,. \end{cases}$$

Remark 3.8. The expression for $H_{\mathrm{I\!I}}$ can be rewritten in an alternative way at a point $p \in M$ where the frame fields can be chosen such that

- the g-orthonormal basis $\{E_1(p), \ldots, E_m(p)\}$ of T_pM is composed of eigenvectors of the shape operator (principal directions) at p:
$$A(E_i(p)) = \lambda_i(p)\ E_i(p)\,, \qquad (i = 1, \ldots m)\,;$$
- the $\mathrm{I\!I}$-orthonormal basis $\{V_1(p), \ldots, V_m(p)\}$ of T_pM consists of the rescaled principal directions at p:
$$V_i(p) = \frac{1}{\sqrt{|\lambda_i(p)|}} E_i(p)\,, \qquad (i = 1, \ldots m)\,.$$

In this case, the following expression for the mean curvature of the second fundamental form holds at the point p:

$$(H_{\mathrm{I\!I}})_{(p)} = \left(\frac{1}{2} \left[mH - \sum_{i=1}^{m} \frac{1}{\lambda_i} \overline{K}(E_i, U) \right] + \frac{\alpha}{4} \Delta_{\mathrm{I\!I}} \log |\det A| - \frac{\alpha}{2} \mathrm{div}_{\mathrm{I\!I}} \mathcal{Z} \right)_{(p)} . \tag{4}$$

Remark 3.9. By using the contracted Gauss equation (2), yet another expression for the mean curvature of the second fundamental form can be derived:

$$H_{\mathrm{I\!I}} = -\frac{\alpha}{2} \left(\mathrm{tr}_{\mathrm{I\!I}} \overline{\mathrm{Ric}} - \mathrm{tr}_{\mathrm{I\!I}} \mathrm{Ric} + \alpha(m^2 - 2m)H - \frac{1}{2} \Delta_{\mathrm{I\!I}} \log |\det A| + \mathrm{div}_{\mathrm{I\!I}} \mathcal{Z} \right) . \tag{5}$$

4. A comparison result for the connections

In the sequel of this article we will make use of the following lemma, which slightly extends well-known results ([15] Theorem 7, [20], and [8], Corollary 13). First we recall a useful definition.

Definition 4.1. *A totally umbilical, compact hypersurface M of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ which satisfies $A = \rho\,\mathrm{id}$ for a constant $\rho \in \mathbb{R}$, is called an extrinsic hypersphere.*

Lemma 4.2. *Let M be a compact hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Suppose that both the first and the second fundamental forms are positive definite and that these metrical tensors induce the same Levi-Civita connection. Furthermore, assume that (M, g) has either strictly positive or strictly negative sectional curvature. Then M is an extrinsic hypersphere.*

The lemma can be proved either by methods similar to those used in [20], or by means of the local de Rahm theorem (see [23, p. 96]).

5. Hypersurfaces in a space form

We shall use $\overline{M}_0^{m+1}(\overline{C})$ to denote the following Riemannian manifolds of dimension $m+1$:

$$\begin{cases} \text{the Euclidean hypersphere} & \mathrm{S}^{m+1}(\frac{1}{\sqrt{\overline{C}}}) & (\text{for } \overline{C} > 0)\,; \\ \text{the Euclidean space} & \mathbb{E}^{m+1} & (\text{for } \overline{C} = 0)\,; \\ \text{the hyperbolic space} & \mathrm{H}^{m+1}(\frac{1}{\sqrt{-\overline{C}}}) & (\text{for } \overline{C} < 0)\,. \end{cases}$$

We shall use $\overline{M}_1^{m+1}(\overline{C})$ to denote the following Lorentzian manifolds of dimension $m+1$:

$$\begin{cases} \text{the de Sitter space} & \mathrm{S}_1^{m+1}(\frac{1}{\sqrt{\overline{C}}}) & (\text{for } \overline{C} > 0)\,; \\ \text{the Minkowski space} & \mathbb{E}_1^{m+1} & (\text{for } \overline{C} = 0)\,; \\ \text{the anti-de Sitter space} & \mathrm{H}_1^{m+1}(\frac{1}{\sqrt{-\overline{C}}}) & (\text{for } \overline{C} < 0)\,. \end{cases}$$

Each of the above semi-Riemannian manifolds has constant sectional curvature $\overline{C}$.

Lemma 5.1. *Let M be a compact semi-Riemannian hypersurface in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant sectional curvature $\overline{C}$ and dimension $m+1$ (with $m \geqslant 2$). Assume that the second fundamental form of M is positive definite. The inequality*

$$S_{\mathrm{I\!I}} \leqslant 2\alpha(m-1)\left(H_{\mathrm{I\!I}} + \overline{C}\mathrm{tr}A^{\leftarrow}\right) \tag{6}$$

is satisfied if and only if the Levi-Civita connections of the first and the second fundamental forms coincide.

Proof. The following expressions are valid for the curvatures which are involved in the above inequality:

$$\begin{cases} H_{\mathrm{I\!I}} &=& \dfrac{1}{2}\left(\alpha\,\mathrm{tr}A - \overline{C}\,\mathrm{tr}A^{\leftarrow}\right) + \dfrac{\alpha}{4}\dfrac{\Delta_{\mathrm{I\!I}}\mathrm{det}A}{\mathrm{det}A} - \dfrac{\alpha}{4}\dfrac{\mathrm{I\!I}(\nabla^{\mathrm{I\!I}}\mathrm{det}A, \nabla^{\mathrm{I\!I}}\mathrm{det}A)}{(\mathrm{det}A)^2}\,; \\ \\ S_{\mathrm{I\!I}} &=& \alpha(m-1)\left(\alpha\,\mathrm{tr}A + \overline{C}\,\mathrm{tr}A^{\leftarrow}\right) + \mathrm{I\!I}(L,L) - \dfrac{1}{4}\dfrac{\mathrm{I\!I}(\nabla^{\mathrm{I\!I}}\mathrm{det}A, \nabla^{\mathrm{I\!I}}\mathrm{det}A)}{(\mathrm{det}A)^2}\,, \end{cases}$$

where the quantity $\mathrm{I\!I}(L,L)$ is defined by

$$\mathrm{I\!I}(L,L) = \sum_{i,j,k=1}^{m} (\mathrm{I\!I}(L(V_i,V_j),V_k))^2 \kappa_i\kappa_j\kappa_k = \sum_{i,j,k=1}^{m} (\mathrm{I\!I}(L(V_i,V_j),V_k))^2\,.$$

The first expression is an immediate consequence of equation (4). The second expression can be found in, e.g., [19] (if $(\overline{M}, \overline{g})$ is the Euclidean space of dimension $m+1$), [2] (if $(\overline{M}, \overline{g})$ is the de Sitter space of dimension $m+1$), or [1] (if $(\overline{M}, \overline{g})$

is a Riemannian space form of dimension $m+1$). The inequality (6) is equivalent to

$$0 \leqslant \frac{(m-1)}{2}\frac{\Delta_{\mathrm{II}}\det A}{\det A} - \frac{(2m-3)}{4}\frac{\mathrm{II}(\nabla^{\mathrm{II}}\det A, \nabla^{\mathrm{II}}\det A)}{(\det A)^2} - \mathrm{II}(L,L)\,,$$

and this implies

$$\det A = \text{constant} \qquad \text{and} \qquad \nabla = \nabla^{\mathrm{II}}\,.$$

Conversely, if $\nabla = \nabla^{\mathrm{II}}$, it follows that $\nabla\mathrm{II}$ vanishes. Consequently, $\det A$ is a constant and the inequality is satisfied. $\square$

Theorem 5.2. *Let M be a compact Riemannian hypersurface in the space form $\overline{M}_e^{m+1}(\overline{C})$ (for $m \geqslant 2$). Assume that the second fundamental form of M is positive definite. The inequality*

$$S_{\mathrm{II}} \leqslant 2\alpha(m-1)\left(H_{\mathrm{II}} + \overline{C}\,\mathrm{tr}A^{\leftarrow}\right) \tag{7}$$

is satisfied if and only if M is an extrinsic hypersphere.

Proof. Three cases will be treated separately.

1. $\overline{M}_e^{m+1}(\overline{C})$ is a Riemannian space form.
   It has already been shown that inequality (7) implies that M is parallel, in the sense that $\nabla\mathrm{II}$ vanishes. Such hypersurfaces were classified in Theorem 4 of [16]. If $\overline{C} \geqslant 0$, the only hypersurfaces with a positive definite second fundamental form which appear in this classification are the extrinsic hyperspheres. If $\overline{C} < 0$, the extrinsic hyperspheres are the only compact hypersurfaces in the classification.
2. $\overline{M}_e^{m+1}(\overline{C})$ is a Lorentzian space form with $\overline{C} \leqslant 0$.
   It follows from the Gauss equation that (M,g) has strictly negative sectional curvature. The result follows from Lemmas 4.2 and 5.1.
3. $\overline{M}_e^{m+1}(\overline{C})$ is the de Sitter space.
   It follows from (7) that ∇A vanishes. Consequently, M has constant mean curvature and an application of Theorem 4 of [18] concludes the proof.

$\square$

6. Hypersurfaces in an Einstein space

Theorem 6.1. *Let $(\overline{M},\overline{g})$ be a Riemannian Einstein manifold of dimension $m+1$ (with $m \geqslant 3$) with strictly positive scalar curvature $\overline{S}$. Any compact hypersurface $M \subseteq \overline{M}$ with positive definite second fundamental form satisfies*

$$H_{\mathrm{II}} + m\sqrt{\left(\frac{m-2}{m+1}\right)\overline{S}} \geqslant \frac{1}{2}\mathrm{tr}_{\mathrm{II}}\mathrm{Ric} \tag{8}$$

if and only if it is an extrinsic hypersphere with $A = \sqrt{\frac{\overline{S}}{(m-2)(m+1)}}\,\mathrm{id}$. Moreover, in this case there holds $H_{\mathrm{II}} = \sqrt{\frac{\overline{S}}{(m-2)(m+1)}}$.

Proof. Since $\overline{\mathrm{Ric}} = \frac{\overline{S}}{m+1}\,\overline{g}$, we deduce that $\mathrm{tr}_{\mathrm{I\!I}}\overline{\mathrm{Ric}} = \frac{\overline{S}}{m+1}\mathrm{tr}A^{\leftarrow}$. Define β and ρ by

$$\beta = \sqrt{\left(\frac{m-2}{m+1}\right)\overline{S}} \qquad \text{and} \qquad \rho = \sqrt{\frac{\overline{S}}{(m-2)(m+1)}}\,.$$

Furthermore, the principal curvatures will be denoted by λ_i ($i = 1, \ldots, m$). It follows now from (5) and the assumption (8) that

$$\begin{aligned} \int \mathrm{tr}_{\mathrm{I\!I}}\overline{\mathrm{Ric}}\,\mathrm{d}\Omega_{\mathrm{I\!I}} &= \int \left[2\,H_{\mathrm{I\!I}} + \beta\sum_{i=1}^{m}\left(\frac{\rho}{\lambda_i} + \frac{\lambda_i}{\rho}\right)\right]\,\mathrm{d}\Omega_{\mathrm{I\!I}} \\ &\geqslant \int 2\left(H_{\mathrm{I\!I}} + m\,\beta\right)\,\mathrm{d}\Omega_{\mathrm{I\!I}} \geqslant \int \mathrm{tr}_{\mathrm{I\!I}}\overline{\mathrm{Ric}}\,\mathrm{d}\Omega_{\mathrm{I\!I}}\,. \end{aligned}$$

This is only possible if all principal curvatures are equal to ρ. □

7. Surfaces in a three-dimensional semi-Riemannian manifold

All previous results agree with [14] if the surrounding space is three-dimensional (except for the sign convention of $H_{\mathrm{I\!I}}$). Moreover, some results can be sharpened. Assume $M \in \mathcal{E}$ and $m = 2$. Let $K_{\mathrm{I\!I}}$ denote the Gaussian curvature of $(M, \mathrm{I\!I})$. Consequently, the relation $2K_{\mathrm{I\!I}} = S_{\mathrm{I\!I}}$ is valid.

Theorem 7.1. *Let M be a compact surface in a three-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and suppose that the first as well as the second fundamental form of M is positive definite. Suppose that the Gaussian curvature K of M is strictly positive. Then M is an extrinsic hypersphere if and only if*

$$K_{\mathrm{I\!I}} \geqslant \alpha\,H_{\mathrm{I\!I}} + \frac{1}{2}\mathrm{tr}_{\mathrm{I\!I}}\overline{\mathrm{Ric}}\,. \tag{9}$$

Proof. Assume first that (9) is satisfied. A minor adaptation of the proof of Proposition 5 of [14] shows that M is totally umbilical, and that equality is attained in (9). An application of Theorem 6 of [14] shows that we have

$$K_{\mathrm{I\!I}} = \alpha\,H_{\mathrm{I\!I}} + \frac{1}{2}\mathrm{tr}_{\mathrm{I\!I}}\overline{\mathrm{Ric}} - \frac{1}{4}\Delta_{\mathrm{I\!I}}\log(\det A)\,,$$

and consequently $\det A$ is a constant. The converse follows since, if M is an extrinsic hypersphere, Theorem 6 of [14] shows that equality holds in (9). □

The following corollary, which follows immediately from the above theorem and Theorem 5.2, generalises a result of [17], [21].

Corollary 7.2. *Let M be a compact Riemannian surface in the space form $\overline{M}^3_0(\overline{C})$ (with $\overline{C} \in \mathbb{R}$) or the de Sitter space. Assume that the second fundamental form of M is positive definite and that the Gaussian curvature of (M, g) is strictly positive. Then either*

$$H_{\mathrm{I\!I}} - \alpha\,K_{\mathrm{I\!I}} + 2\frac{\overline{C}H}{K - \overline{C}}$$

changes sign or M is an extrinsic sphere.

8. Curves in a semi-Riemannian surface

Let $\gamma :]a,b[\to (\overline{M},\overline{g}) : s \mapsto \gamma(s)$ be an arcwise parametrised time-like or space-like curve in a semi-Riemannian surface. Let T denote the unit tangent vector γ' along γ. It will be supposed that $\overline{g}(\overline{\nabla}_T T, \overline{\nabla}_T T)$ vanishes nowhere. By virtue of this property, γ is sometimes called a Frenet curve. On the other hand, this requirement precisely means that II is a semi-Riemannian metrical tensor on γ. Let $\{T, U\}$ be the Frenet frame field along γ:

$$T = \gamma', \qquad U = \frac{1}{\sqrt{\left|\overline{g}(\overline{\nabla}_T T, \overline{\nabla}_T T)\right|}} \overline{\nabla}_T T\,.$$

Further, we set $\beta = \overline{g}(T,T) = \pm 1$ and $\alpha = \overline{g}(U,U) = \pm 1$. The geodesic curvature κ of γ in $(\overline{M},\overline{g})$ is determined by the Frenet-Serret formulae:

$$\begin{pmatrix} \overline{\nabla}_T T \\ \overline{\nabla}_T U \end{pmatrix} = \begin{pmatrix} 0 & \beta\kappa \\ -\alpha\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ U \end{pmatrix}. \tag{10}$$

The geodesic curvature κ is equal to the mean curvature of $\gamma \subseteq (\overline{M},\overline{g})$. The functional which measures the length of a curve with respect to the second fundamental form, which will be denoted by $\mathrm{Length}_{\mathrm{II}}$ instead of $\mathrm{Area}_{\mathrm{II}}$, can be computed as the integral

$$\mathrm{Length}_{\mathrm{II}}(\gamma) = \int_\gamma \sqrt{|\kappa|}\, \mathrm{d}s\,.$$

Let $\overline{K}$ denote the Gaussian curvature of $(\overline{M},\overline{g})$. A calculation shows

$$H_{\mathrm{II}} = \frac{1}{2}\left(\frac{-\alpha\overline{K}}{\kappa} + \kappa + \frac{\alpha\beta}{4}\left(2\frac{\kappa''}{\kappa^2} - 3\frac{(\kappa')^2}{\kappa^3} \right) \right). \tag{11}$$

Example 8.1. A curve γ (with $\kappa > 0$) in $\mathbb{E}^2$ is II-minimal if and only if the curvature κ, when regarded as a function of the arc-length, satisfies

$$4\kappa^4 + 2\kappa\kappa'' - 3(\kappa')^2 = 0\,.$$

Since the formula

$$\kappa(s) = \frac{A}{A^2(s+Q)^2 + 1} \qquad \text{(where } A \in]0,+\infty[\text{ and } Q \in \mathbb{R}\text{)}$$

describes the general solution of this differential equation, the catenaries are exactly the II-minimal planar curves. (Compare [5, § 27] for the corresponding variational problem for space curves.)

Example 8.2. For curves on the unit sphere, the equation $H_{\mathrm{II}} = 0$ can be rewritten as

$$4\kappa^2 - 4\kappa^4 - 2\kappa''\kappa + 3(\kappa')^2 = 0\,.$$

This is equation (4) of [4], if the length functional of the second fundamental form $\mathrm{Length}_{\mathrm{II}}$ is interpreted as the so-called curvature energy functional. As is proved and beautifully illustrated in [4], there exists a discrete family of closed, immersed, II-minimal curves on the unit sphere. Then $S^1(\frac{1}{\sqrt{2}}) \subseteq S^2(1)$ is an embedded "II-minimal" curve which belongs to this family. This curve is, as is remarked in [4], actually a local maximum of $\mathrm{Area}_{\mathrm{II}}$.

9. Geodesic hyperspheres in a Riemannian manifold

As a final example we shall investigate the (sufficiently small) geodesic hyperspheres in a Riemannian manifold, since these provide us with a naturally defined class of hypersurfaces with a positive definite second fundamental form. We will use the method of power series expansions. The interested reader is referred to [13] or [23] for the technical calculations which are too extensive to be presented here.

It is a straightforward consequence of the calculations of [6], [11], [12] that the locally flat spaces are the only Riemannian manifolds for which all geodesic hyperspheres have either constant mean curvature which is equal to the inverse of their radius, or constant Gauss-Kronecker curvature which is equal to the inverse of the m-th power of their radius. An analysis of the coefficients appearing in the power series expansion of the mean curvature of the second fundamental form for small geodesic spheres (as a function of their radius) establishes the following corresponding property.

Theorem 9.1. *A Riemannian manifold (of dimension $m+1$) is locally flat if and only if the mean curvature of the second fundamental form of every geodesic hypersphere is equal to the constant $\frac{m}{2r}$ (where r is the radius of the geodesic hypersphere).*

Let us denote the geodesic hypersphere of centre n and radius r by $\mathcal{G}_n(r)$. It was asked in [12] whether the Riemannian geometry of the ambient manifold $(\overline{M}, \overline{g})$ is fully determined by the area functions

$$\overline{M} \times]0, +\infty[\,\to \mathbb{R} : (n, r) \mapsto \mathrm{Area}(\mathcal{G}_n(r)) \qquad (r \text{ sufficiently small})$$

of the geodesic hyperspheres. It appears that a decisive answer has not yet been given. Similarly, it may be asked whether a Riemannian manifold for which every geodesic hypersphere has the same II-area as a Euclidean hypersphere of the same radius, is locally flat. We were only able to find the following partial answer, which should be compared with Theorem 4.1 of [12] and makes use of similar methods.

Theorem 9.2. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold of dimension $m+1$, and suppose that the area of every geodesic hypersphere of $\overline{M}$, as seen in the geometry of the second fundamental form, is equal to $r^{\frac{m}{2}}\alpha_m$ (where r is the radius of the geodesic hypersphere, and α_m is the area of the unit hypersphere of $\mathbb{E}^{m+1}$). Then there holds*

$$\begin{cases} \overline{S} &= 0\,; \\ \|\overline{\mathrm{R}}\|^2 &= \|\overline{\mathrm{Ric}}\|^2\,. \end{cases} \tag{12}$$

Further, $\overline{M}$ is locally flat if any of the following additional hypotheses is made:

(i) $\dim \overline{M} \leqslant 5$;

(ii) *the Ricci tensor of $\overline{M}$ is positive or negative semi-definite (in particular if M is Einstein);*

(iii) *$\overline{M}$ is conformally flat and* $\dim \overline{M} \neq 6$;

(iv) *$\overline{M}$ is a Kähler manifold of complex dimension* $\leqslant 5$;

(v) *$\overline{M}$ is a Bochner flat Kähler manifold of complex dimension* $\neq 6$;

(vi) *$\overline{M}$ is a product of surfaces (with an arbitrary number of factors).*

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