Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 51 (2010), No. 1, 31-44.

# Hyperelliptic Plane Curves of Type $(d, d-2)$

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**Abstract.** In [7], we classified and constructed all rational plane curves of type $(d, d-2)$. In this paper, we generalize these results to irreducible plane curves of type $(d, d-2)$ with positive genus.

MSC 2000: 14H50, 14E07

## 1. Introduction

Let $C \subset \mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$ be a plane curve of degree $d$. We call $C$ a *plane curve of type* $(d, \nu)$ if the maximal multiplicity of singular points on $C$ is equal to $\nu$. A unibranched singularity is called a *cusp*. Rational cuspidal plane curves of type $(d, d-2)$ and $(d, d-3)$ were classified by Flenner-Zaidenberg [5], [6] (see also [4], [8] for some cases). In [7], we classified rational plane curves of type $(d, d-2)$ with arbitrary singularities. In order to describe a multibranched singularity $P$, we introduced the notion of the system of the multiplicity sequences $\underline{m}_P(C)$ (see

*partially supported by Post Doctoral Fellowship for Foreign Researchers, 1705292, JSPS

Section 2). We denote by Data($C$), the collection of such systems of multiplicity sequences. The purpose of this paper is to complete the classification of irreducible plane curves of type $(d, d-2)$ with positive genus $g$. We remark that if $g \geq 2$, then $C$ is a hyperelliptic curve, for the projection from the singular point of multiplicity $d-2$ induces a double covering of $C$ over $\mathbf{P}^1$.

**Theorem 1.** *Let $C$ be a plane curve of type $(d, d-2)$ with genus $g$. Let $Q \in C$ be the singular point of multiplicity $d-2$. Then, we have*

(i) $\mathrm{Data}(C) = \left[\underline{m}_Q(C), \binom{1}{1}_{b_1}, \ldots, \binom{1}{1}_{b_n}, (2_{b_{n+1}}), \ldots, (2_{b_{n+n'}})\right]$, *where*

$$\underline{m}_Q(C) = \left\{ \begin{pmatrix} k_1 \\ k_1' \\ \vdots \\ k_s \\ k_s' \\ k_{s+1} \\ \vdots \\ k_N \end{pmatrix} \begin{matrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{a_1} \\ \vdots \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{a_s} \\ 2_{a_{s+1}} \\ \vdots \\ 2_{a_N} \end{matrix} \right\}$$

*and the following conditions are satisfied:*

(1) $\sum_{h=1}^{N} k_h + \sum_{h'=1}^{s} k'_{h'} = d-2$ *and* $\sum_{i=1}^{N} a_i + \sum_{j=1}^{n+n'} b_j = d-g-2$, *where* $a_i \geq 0$ *($a_i > 0$ for $i = 1, \ldots, s$), $b_j > 0$,*

(2) *we have $n, n', s \geq 0$ and $n' + s' \leq 2g+2$, where $s' = \#\{j | a_{s+j} > 0\}$,*

(3) *for $i = 1, 2, \ldots, s$, if $k_i' = k_i$, then $a_i \geq k_i$ and if $k_i' > k_i$, then $a_i = k_i$,*

(4) *for $i = s+1, \ldots, N$, if $a_i > 0$, then either $k_i$ is even and $a_i \geq k_i/2$ or $k_i$ is odd and $a_i = (k_i - 1)/2$.*

*Note that the $N$ is the number of the different tangent lines to $C$ at $Q$.*

(ii) Data($C$) *can be derived from Degtyarev's 2-formula $T(C)$ defined for the defining equation of $C$ (see Proposition 10 for details).*

**Corollary.** *Let $C$ be an irreducible plane curve of type $(d, d-2)$ with genus $g$.*

(i) *If $C$ has only cusps, then $C$ has the following data ($b_i > 0, k > 0, j \geq 0$):*

| Class | Data($C$) | |
|---|---|---|
| (a) | $[(k), (2_{b_1}), \ldots, (2_{b_{n'}})]$ | $(k = g + \sum_{i=1}^{n'} b_i)$<br>$(n' \leq 2g+2)$ |
| (b) | $[(2k+1, 2_k), (2_{b_1}), \ldots, (2_{b_{n'}})]$ | $(k+1 = g + \sum_{i=1}^{n'} b_i)$<br>$(n' \leq 2g+1)$ |
| (c) | $[(2k, 2_{k+j}), (2_{b_1}), \ldots, (2_{b_{n'}})]$ | $(k = g + j + \sum_{i=1}^{n'} b_i)$<br>$(n' \leq 2g+1)$ |

(ii) *If $C$ has only bibranched singularities, then $C$ has the following data ($b_i > 0, k > 0, r > 0, j \geq 0, l \geq 0$):*

| Class | Data($C$) | |
|---|---|---|
| (e) | $\left[\binom{k}{k}\binom{1}{1}_{k+j}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k = g + j + \sum_{i=1}^{n} b_i)$ |
| (f) | $\left[\binom{k}{k+r}\binom{1}{1}_{k}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k + r = g + \sum_{i=1}^{n} b_i)$ |
| (aa) | $\left[\binom{k}{r}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k + r = g + \sum_{i=1}^{n} b_i)$ |
| (ab) | $\left[\left\{\binom{2k+1}{r}^{2_k}\right\}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k + r + 1 = g + \sum_{i=1}^{n} b_i)$ |
| (ac) | $\left[\left\{\binom{2k}{r}^{2_{k+j}}\right\}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k + r = g + j + \sum_{i=1}^{n} b_i)$ |
| (bb) | $\left[\left\{\binom{2k+1}{2r+1}{}^{2_k}_{2_r}\right\}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k + r + 2 = g + \sum_{i=1}^{n} b_i)$ |
| (bc) | $\left[\left\{\binom{2k+1}{2r}{}^{2_k}_{2_{r+l}}\right\}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k + r + 1 = g + l + \sum_{i=1}^{n} b_i)$ |
| (cc) | $\left[\left\{\binom{2k}{2r}{}^{2_{k+j}}_{2_{r+l}}\right\}, \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}\right]$ | $(k + r = g + j + l + \sum_{i=1}^{n} b_i)$ |

**Theorem 2.** (cf. Coble [1], Coolidge [2]) *Let $C$ be an irreducible plane curve of type $(d, d-2)$ with genus $g$. Then, there exists a Cremona transformation which transforms $C$ into a plane curve:*

$$\Gamma : y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i),$$

*with some distinct $\lambda_i$'s.*

*Conversely, given a plane curve $\Gamma$ as above and a collection of systems of multiplicity sequences $M$ satisfying the conditions* (1)–(4) *in Theorem* 1, (i) *for $d \geq g + 2$, then we can find an irreducible plane curve $C$ of type $(d, d-2)$ such that*

(a) Data($C$) $= M$,

(b) *$C$ is Cremona birational to $\Gamma$.*

**Remark 3.** For the first half of Theorem 2, we refer to Coble [1], p. 125 and Coolidge [2], Book III, Chapter V. As for the second half of Theorem 2, the particular cases in which $M = [(g)]$, $[\binom{g}{g}\binom{1}{1}_g]$ were discussed in [2], Book III, Chapter V, Theorems 8, 10.

In Section 2, we review the system of the multiplicity sequences, the 2-formula and quadratic Cremona transformations. In Section 3 (resp. Section 4), we will prove Theorem 1 (resp. Theorem 2). In Section 5, we discuss the defining equations for those curves given in Corollary.

## 2. Preliminaries

A cusp $P$ can be described by its multiplicity sequence $\underline{m}_P = (m_0, m_1, m_2, \ldots)$. For a multibranched singular point $P$ on $C$, we introduced the system of the multiplicity sequences of $P$.

**Definition 4.** ([7]) *Let $P \in C$ be a multibranched singular point, having $r$ local branches $\gamma_1, \ldots, \gamma_r$. Let $\underline{m}(\gamma_i) = (m_{i0}, m_{i1}, m_{i2}, \ldots)$ denote the multiplicity sequences of the branches $\gamma_i$, respectively. We define the system of the multiplicity sequences, which will be denoted by the same symbol $\underline{m}_P(C)$, to be the combination of $\underline{m}(\gamma_i)$ with brackets indicating the coincidence of the centers of the infinitely near points of the branches $\gamma_i$. For instance, for the case in which $r = 3$, we write it in the following form:*

$$\left\{ \begin{pmatrix} m_{1,0} \\ m_{2,0} \\ m_{3,0} \end{pmatrix} \cdots \begin{pmatrix} m_{1,\rho} \\ m_{2,\rho} \\ m_{3,\rho} \end{pmatrix} \begin{array}{l} \begin{pmatrix} m_{1,\rho+1} \\ m_{2,\rho+1} \end{pmatrix} \cdots \begin{pmatrix} m_{1,\rho'} \\ m_{2,\rho'} \end{pmatrix} \begin{array}{l} m_{1,\rho'+1}, \ldots, m_{1,s_1} \\ m_{2,\rho'+1}, \ldots, m_{2,s_2} \end{array} \\ m_{3,\rho+1}, \ldots, m_{3,s_3} \end{array} \right\}.$$

*We also use some simplifications such as*

$$(2_a) = (\overbrace{2, \ldots, 2}^{a}, 1, 1), \quad (2_0) = (1), \quad \binom{1}{1}_a = \overbrace{\binom{1}{1} \cdots \binom{1}{1}}^{a}.$$

**Example 5.** We examine our notations for ADE singularities.

| $P$ | $A_{2n}$ | $A_{2n-1}$ | $D_{2n-1}$ | $D_{2n}$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|---|---|
| $\underline{m}_P(C)$ | $(2_n)$ | $\binom{1}{1}_n$ | $\left\{ \binom{2}{1}^{2_{n-3}} \right\}$ | $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \binom{1}{1}_{n-2} \right\}$ | $(3)$ | $\binom{2}{1}\binom{1}{1}$ | $(3, 2)$ |

**Example 6.** The hyperelliptic curve $y^2 = \prod_{i=1}^{2g+2}(x - \lambda_i)$ has one singularity $Q$ on the line at infinity with $\underline{m}_Q = \binom{g}{g}\binom{1}{1}_g$.

Let $C$ be an irreducible plane curve of type $(d, d-2)$. Let $Q \in C$ be the singular point with multiplicity $d-2$. Choosing homogeneous coordinates $(x, y, z)$ so that $Q = (0, 0, 1)$, the curve $C$ is defined by an equation:

$$F(x, y)z^2 + 2G(x, y)z + H(x, y) = 0,$$

where $F$, $G$ and $H$ are homogeneous polynomials of degree $d-2$, $d-1$ and $d$, respectively. Set $\Delta = G^2 - FH$. Let $t_1, \ldots, t_l \in \mathbf{P}^1$ be all the distinct roots of the equation $F(t)\Delta(t) = 0$. For each $i$, let $(p_i, q_i) = (\mathrm{ord}_{t_i}(F), \mathrm{ord}_{t_i}(\Delta))$, where $\mathrm{ord}_{t_i}(F)$ (resp. $\mathrm{ord}_{t_i}(\Delta)$) is the multiplicity of the root $t_i$ of the equation $F(t) = 0$ (resp. $\Delta(t_i) = 0$). Set $T(C) = \{(p_1, q_1), \ldots, (p_l, q_l)\}$. This unordered $l$-tuple $T(C)$ is called the 2-*formula* of $C$ (Degtyarev [3]). We remark that $T(C)$ does not depend on the choice of the coordinates $(x, y, z)$ with $Q = (0, 0, 1)$.

**Lemma 7.** *The 2-formula $T(C)$ satisfies the following properties:*

(i) $\sum_{i=1}^{l} q_i = 2\sum_{i=1}^{l} p_i + 2$,

(ii) $p_i = q_i$ *or* $\min\{p_i, q_i\}$ *is even for each* $i$,

(iii) *there exists a pair* $(p_i, q_i)$ *such that* $q_i$ *is an odd number.*

*Proof.* (i) By definition, $\sum_{i=1}^{l} p_i = d-2$ and $\sum_{i=1}^{l} q_i = 2d-2$.
(ii) Suppose that $p_i \neq q_i$. We may assume $t_i = (0,1)$. We can write $F$, $G$ and $\Delta$ as $F = x^{p_i}F_0$, $\Delta = x^{q_i}\Delta_0$ and $G = x^m G_0$, respectively, where $m = \mathrm{ord}_{t_i}(G)$. If $p_i > q_i$, then $x^{2m}G_0^2 = x^{q_i}(\Delta_0 + x^{p_i - q_i}F_0 H)$. Thus $q_i = 2m$. If $0 < p_i < q_i$, then we get $x^{2m}G_0^2 = x^{p_i}(x^{q_i - p_i}\Delta_0 + F_0 H)$, which implies that $2m \geq p_i > 0$. We have $x \not| H$, since $C$ is irreducible. Hence $p_i = 2m$.
(iii) Suppose that all $q_i$'s are even. Then we can write as $\Delta = \Delta_0^2$. We have $F(Fz^2 + 2Gz + H) = (Fz + G + \Delta_0)(Fz + G - \Delta_0)$. Since $\deg(Fz + G \pm \Delta_0) = d-1$, we infer that $Fz^2 + 2Gz + H$ is reducible. This is a contradiction. □

**Remark 8.** We note that $P(x, y, z) = Fz^2 + 2Gz + H$ is irreducible if (a) $\mathrm{GCD}(F, G, H) = 1$, and if (b) the property (iii) holds. Indeed, under the assumption (a), if $P$ is reducible, then $P = (Az + B)(Cz + D)$ with $A, B, C, D \in \mathbf{C}[x, y]$. But, in this case, $4\Delta = (AD - BC)^2$, which contradicts the property (iii).

**Example 9.** Let $C$ be the quartic curve $x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x + y + z) = 0$. We have $T(C) = \{(2,0), (0,3), (0,3)\}$.

The (degenerate) quadratic Cremona transformation

$$\varphi_c : (x, y, z) \longrightarrow (xy, y^2, x(z - cx)) \quad (c \in \mathbf{C})$$

played an important role in [6, 7]. We find that $\varphi_c^{-1}(x, y, z) = (x^2, xy, yz + cx^2)$. We use the notations:

$$l : x = 0,\ t : y = 0,\ O = (0,0,1), A = (1,0,c), B = (0,1,0).$$

Note that $\varphi_c(l \setminus \{O\}) = B$ and $\varphi_c(t \setminus \{O, A\}) = O$.

Let $C$ be an irreducible plane curve of type $(d, d-2)$ with $d \geq 4$. Suppose the singular point $Q \in C$ of multiplicity $d-2$ has coordinates $O$. We have seen in [7] that the strict transform $C' = \varphi_c(C)$ is an irreducible plane curve of type $(d', d'-2)$ for some $d'$. In [7, 8], by analyzing how a local branch $\gamma$ at $P \in \mathrm{Sing}(C)$ is transformed by $\varphi_c$, we described $\mathrm{Data}[C']$ from $\mathrm{Data}[C]$.

## 3. Proof of Theorem 1

(i) We easily see that $P \in \mathrm{Sing}(C) \setminus \{Q\}$ is a double point, because $LC = (d-2)Q + 2P$, where $L$ is the line passing through $P, Q$. Let $\pi : \tilde{\mathbf{P}}^2 \to \mathbf{P}^2$ be the blowing-up at $Q$. Let $E$ denote the exceptional curve. Take a line $L$ passing through $Q$. Let $C'$ (resp. $L'$) be the strict transform of $C$ (resp. $L$). We have

$C'L' = 2$. It follows that $P \in \mathrm{Sing}(C') \cap E$ is also a double point. Thus, $\mathrm{Data}(C)$ has the shape as in Theorem 1. Clearly, $\sum k_h + \sum k'_{h'} = \mathrm{mult}_Q(C) = d-2$. The second part of the condition (1) follows from the genus formula. The condition (2) follows from the Hurwitz formula applied to the double covering $\tilde{C} \to \mathbf{P}^1$, which corresponds to the projection of $C$ from $Q$, where the $\tilde{C}$ is the non-singular model of $C$. For the proof of the conditions (3), (4), we refer to [7]. We will give an alternative, direct proof in Proposition 10.

(ii) Let $F(x,y)z^2 + 2G(x,y)z + H(x,y) = 0$ be the defining equation of $C$ as in Section 2. Let $T(C)$ be the 2-formula of $C$. Setting

$$T'(C) = \{(p,q) \in T(C) \mid p > 0 \text{ or } q \geq 2\},$$

we renumber the pairs $(p_i, q_i) \in T'(C)$ in the following way:

(1) $p_i > 0$, $q_i > 0$ and $q_i$ is even for $i = 1, \dots s$,

(2) either $p_i > 0$, $q_i > 0$ and $q_i$ is odd, or $p_i > 0, q_i = 0$ for $i = s+1, \dots, N$,

(3) $p_i = 0$, $q_i > 0$ and $q_i$ is even, for $i = N+1, \dots, N+n$,

(4) $p_i = 0$, $q_i \geq 3$ and $q_i$ is odd, for $i = N+n+1, \dots, N+n+n'$.

**Proposition 10.** *Set*

(1) *for* $i = 1, \dots s$,
$$\begin{cases} k_i = k'_i = p_i/2, a_i = q_i/2 & \text{if } p_i \leq q_i, \\ k_i = q_i/2, k'_i = p_i - q_i/2, a_i = q_i/2 & \text{if } p_i > q_i, \end{cases}$$

(2) *for* $i = s+1, \dots N$,
$$\begin{cases} k_i = p_i, a_i = (q_i - 1)/2 & \text{if } q_i > 0, \\ k_i = p_i, a_i = 0 & \text{if } q_i = 0, \end{cases}$$

(3) $b_j = q_{N+j}/2$, *for* $j = 1, \dots, n$,

(4) $b_j = (q_{N+j} - 1)/2$, *for* $j = n+1, \dots, n+n'$.

*Then* $\mathrm{Data}(C)$ *is given as in Theorem* 1, (i).

*Proof.* Take $(p_i, q_i) \in T'(C)$. Write $t_i$ as $t_i = (\alpha_i, \beta_i)$. Let $L_i$ be the line $\beta_i x = \alpha_i y$. By arranging the coordinates, we may assume $(\alpha_i, \beta_i) = (0,1)$. Write $F$, $G$ and $\Delta$ as $F = x^{p_i} F_0$, $G = x^m G_0$ and $\Delta = x^{q_i} \Delta_0$, where $m = \mathrm{ord}_{t_i}(G)$.

We first consider the case in which $p_i = 0$. Since $\Delta(t_i) = 0$, we have

$$F(t_i)z^2 + 2G(t_i)z + H(t_i) = F(t_i)(z + G(t_i)/F(t_i))^2.$$

It follows that $CL_i = (d-2)Q + 2P$, where $P = (0, 1, -G(t_i)/F(t_i))$. Let $U$ be a neighbourhood of $P$ such that $y \neq 0$ and $F(x,y) \neq 0$ for all $(x,y,z) \in U$. We use the affine coordinates $(\overline{x}, \overline{z}) = (x/y, z/y)$. We have

$$\begin{aligned} &F(x,y)(F(x,y)z^2 + 2G(x,y)z + H(x,y)) \\ &\qquad = y^{2d-2}((F(\overline{x},1)\overline{z} + G(\overline{x},1))^2 - \Delta(\overline{x},1)). \end{aligned}$$

Thus $C$ is defined by the equation $(F(\overline{x},1)\overline{z}+G(\overline{x},1))^2=\Delta(\overline{x},1)$ on $U$. Letting $u=F(\overline{x},1)\overline{z}+G(\overline{x},1)$ and $v=(\sqrt[q_i]{\Delta_0(\overline{x},1)})\overline{x}$, $C$ is defined by the equation $u^2=v^{q_i}$ around $P$. Thus $P\in \mathrm{Sing}(C)\setminus\{Q\}$ if $q_i\geq 2$. In this case, we have

$$\underline{m}_P(C)=\begin{cases}\binom{1}{1}_{q_i/2} & \text{if } q_i \text{ is even,}\\ (2_{(q_i-1)/2}) & \text{if } q_i \text{ is odd,}\end{cases}$$

which gives the assertions (3), (4).

Conversely, take $P\in\mathrm{Sing}(C)\setminus\{Q\}$. Let $L$ be the line passing through $P,Q$. Write $L:\beta x=\alpha y$. Since $CL=(d-2)Q+2P$, we have $F(\alpha,\beta)\neq 0$ and $\Delta(\alpha,\beta)=0$. For $(\alpha,\beta)\in\mathbf{P}^1$, we find a pair $(0,q)\in T(C)$. We see from the above argument that $C$ is defined by the equation $u^2=v^q$ near $P$. Thus $q\geq 2$.

We now consider the case in which $p_i>0$. Let $\pi:\tilde{\mathbf{P}}^2\to\mathbf{P}^2$ be the blowing-up at $Q$ and $E$ the exceptional curve of $\pi$. We use the affine coordinates $(\overline{x},\overline{y})=(x/z,y/z)$ of $U:=\{(x,y,z)\in\mathbf{P}^2\mid z\neq 0\}$. Put $V=\pi^{-1}(U)$. There exist an open cover $V=V_1\cup V_2$ $(V_j\cong\mathbf{C}^2)$ with standard coordinates $(u_j,v_j)$ of $V_j$ such that $\pi|_{V_1}:V_1\ni(u_1,v_1)\mapsto(u_1v_1,u_1)$ and $\pi|_{V_2}:V_2\ni(u_2,v_2)\mapsto(u_2,u_2v_2)$. Note that $E$ is defined by $u_j=0$ on $V_j$. The strict transform $L_i'$ of $L_i$ is defined by $v_1=0$ on $V_1$. Let $P$ be the unique point $E\cap L_i'$. We have $P=(0,0)$ on $V_1$. The strict transform $C'$ of $C$ is defined by the equation: $F(v_1,1)+2G(v_1,1)u_1+H(v_1,1)u_1^2=0$ on $V_1$. By the definition of $p_i$ and $m$, the curve $C'$ is defined by the equation: $F_0v_1^{p_i}+2G_0v_1^mu_1+Hu_1^2=0$. In particular, we have $(C'E)_P=p_i$. If $q_i=0$, then we must have $m=0$ (see the proof of Lemma 7). Hence $C'$ is smooth at $P$. If $q_i>0$, then we have $m>0$ (cf. the proof of Lemma 7). Since $C$ is irreducible, we see that $H(t_i)\neq 0$. We have $H(F+2G_0v_1^mu_1+Hu_1^2)=(Hu_1+G_0v_1^m)^2-\Delta$. This means that $C'$ is defined by the equation:

$$(H(v_1,1)u_1+G_0(v_1,1)v_1^m)^2-\Delta(v_1,1)=0$$

in a neighborhood of $P$.

Letting $u=H(v_1,1)u_1+G_0(v_1,1)v_1^m$ and $v=(\sqrt[q_i]{\Delta_0(v_1,1)})v_1$, $C'$ is defined by the equation $u^2=v^{q_i}$ around $P$. We have

$$\underline{m}_P(C')=\begin{cases}\binom{1}{1}_{q_i/2} & \text{if } q_i \text{ is even,}\\ (2_{(q_i-1)/2}) & \text{if } q_i \text{ is odd,}\\ (1) & \text{if } q_i=0,\end{cases}$$

which gives the values of $a_i$ in (1), (2). We prove the remaining assertions in (1). If $q_i$ is even, then $C'$ has two branches $\gamma_+,\gamma_-$ at $P$ defined by

$$H(v_1,1)u_1+G_0(v_1,1)v_1^m\pm v_1^{q_i/2}\sqrt{\Delta_0(v_1,1)}=0.$$

In case $p_i>q_i$, we have $m=q_i/2$ (see the proof of Lemma 7). We infer that one of the intersection numbers $(E\gamma_+)_P$ and $(E\gamma_-)_P$ is equal to $q_i/2$. The other one must be equal to $p_i-q_i/2$, because $(EC')_P=p_i$. In case $p_i\leq q_i$, we have $m\geq p_i/2$ (cf. the proof of Lemma 7). Thus $(E\gamma_\pm)_P\geq p_i/2$, hence $(E\gamma_\pm)_P=p_i/2$. Consequently, we obtain the pair $(k_i,k_i')$.

Conversely, take $P \in C' \cap E$. We assume $P \in V_1$. Write the coordinates of $P$ as $P = (0, \beta)$. The equation $F(\beta, 1) + 2G(\beta, 1)u_1 + H(\beta, 1)u_1^2 = 0$ has the solution $u_1 = 0$ as $C'$ passes through $P$. Thus $F(\beta, 1) = 0$. For $(\beta, 1) \in \mathbf{P}^1$, we find a pair $(p, q) \in T(C)$ with $p > 0$. □

**Remark 11.** For $i = s+1, \ldots, N$, if $(k_i, a_i) = (1, 0)$, then we have either $(p_i, q_i) = (1, 1)$ or $(1, 0)$. The case $(p_i, q_i) = (1, 1)$ occurs if and only if the line $L_i$ is a flex-tangent line to the corresponding branch at $Q$.

## 4. Proof of Theorem 2

Let $C$ be given by the equation (see Section 2):

$$F(x, y)z^2 + 2G(x, y)z + H(x, y) = 0.$$

Put $\Delta = G^2 - FH$. Via linear coordinates change of $x$ and $y$, we may assume that $y \not| F\Delta$. We then define a Cremona transformation (cf. [2], Book III, Chapter V):

$$\Phi(x, y, z) = (xy^{d-2}, y^{d-1}, Fz + G).$$

We find that $\Phi^{-1}(x, y, z) = (xF, yF, y^{d-2}z - G)$. We see easily that the strict transform $C' = \Phi(C)$ is defined by the equation:

$$y^{2(d-2)}z^2 = \Delta.$$

Write $\Delta = \prod_{i=1}^{k}(x - \lambda_i y)^{q_i}$, where the $\lambda_i$'s are distinct. Renumber $q_i$'s so that $q_i$'s are odd for $i = 1, \ldots, l$ and $q_i$'s are even for $i = l+1, \ldots, k$. Letting $s_i = [q_i/2]$ for $i = 1, \ldots, k$, we put $S = \prod_{i=1}^{k}(x - \lambda_i y)^{s_i}$ and $s = \sum_{i=1}^{k} s_i$. Note that $2d - 2 = \sum_{i=1}^{k} q_i = 2s + l$. We next define a Cremona transformation:

$$\Psi(x, y, z) = (xS, yS, y^s z).$$

We find that $\Psi^{-1}(x, y, z) = (xy^s, y^{s+1}, Sz)$. We see that $\Gamma' = \Psi(C')$ is defined by the equation:

$$y^{2(d-2)}z^2 = \prod_{i=1}^{l}(x - \lambda_i y).$$

We see that $l = 2g + 2$ and $g = d - s - 2$. Take a projective transformation: $\iota : (x, y, z) \to (x, z, y)$. Finally, the image $\Gamma = \iota(\Gamma')$ has the affine equation:

$$y^2 = \prod_{i=1}^{2g+2}(x - \lambda_i).$$

We now prove the second half of Theorem 2. We start with the curve $\Gamma$ and a collection of systems of multiplicity sequences:

$$M = \left[ m, \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{b_1}, \ldots, \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{b_n}, (2_{b_{n+1}}), \ldots, (2_{b_{n+n'}}) \right],$$

where the $m$ is the system of the multiplicity sequences of the singular point with multiplicity $d-2$. Let $r(M)$, $N(M)$ denote the number of the branches and the number of the different tangent lines of $m$. We have to construct an irreducible plane curve of type $(d, d-2)$ with $\mathrm{Data}(C) = M$. In [7], we considered the case in which $g = 0$. We here assume that $g \geq 1$. We follow the arguments in [7].

First we deal with the cuspidal case given in Corollary of Theorem 1. See also Proposition 13.

Case (a): $M = [(k), (2_{b_1}), \dots, (2_{b_{n'}})]$, where $k = g + \sum b_i$. We use the induction on $n'$. (i) $M = [(g)]$. Interchanging coordinates, we start with the curve:

$$\Gamma_0 : x^{2g} z^2 = \prod_{i=1}^{2g+2} (y - \lambda_i x).$$

After a linear change of coordinates, we may assume that $c = \prod_{i=1}^{2g+2}(-\lambda_i) \neq 0$. Letting $c_1 = \sqrt{c}$, we have $\Gamma_0 t = (2g)O + A_1 + A_1'$, where $A_1 = (1, 0, c_1), A_1' = (1, 0, -c_1)$. Let $\Gamma_1$ be the strict transform of $\Gamma_0$ via $\varphi_{c_1}$. Using Lemma 1, (a) and Lemma 2, (e)* in [7], we see that $\Gamma_1 t = (2g-1)O + A_2$. Write $A_2 = (1, 0, c_2)$. Let $\Gamma_2$ be the strict transform of $\Gamma_1$ via $\varphi_{c_2}$. In this way, we successively choose $c_1, \dots, c_g$. It turns out that $\mathrm{Data}(\Gamma_g) = [(g)]$.

(ii) Suppose we have constructed $C_0$ with $\mathrm{Data}(C_0) = [(k_0), (2_{b_1}), \dots, (2_{b_{n'-1}})]$, where $k_0 = g + \sum_{i=1}^{n'-1} b_i$. After a suitable change of coordinates, we may assume $C_0 l = k_0 O + 2B_1$ and $C_0 t = (k_0+1)O + A_1$. Note that the double covering $\tilde{C} \to \mathbf{P}^1$ defined through the projection from $O$ to a line, must have $2g+2$ branch points. Since $n' - 1 < 2g + 2$, we see that a line passing through $O$ is tangent to $C_0$ at a smooth point $B_1$. Write $A_1 = (1, 0, c_1)$. Let $C_1$ be the strict transform of $C_0$ via $\varphi_{c_1}$. We have $C_1 l = (k_0+1)O + 2B$ and $C_1 t = (k_0+2)O + A_2$. Write $A_2 = (1, 0, c_2)$. Let $C_2$ be the strict transform of $C_1$ via $\varphi_{c_2}$. We have again $C_2 t = (k_0+3)O + A_3$. Repeating in this way, we successively choose $c_1, \dots, c_{b_{n'}}$ and define $C_1, \dots, C_{b_{n'}}$. Then, the curve $C = C_{b_{n'}}$ has the desired property.

Case (b): $M = [(2k+1, 2_k), (2_{b_1}), \dots, (2_{b_{n'}})]$, where $k + 1 = g + \sum b_i$. As in Case (a), we can similarly prove this case. For the first step: $M = [(2g-1, 2_{g-1})]$, it suffices to arrange coordinates so that $\Gamma_0 t = gO + 2A_1$ with $A_1 = (1, 0, 0)$. Put $c_1 = 0$ and choose $c_2, \dots, c_g$ arbitrarily. Then we obtain $\mathrm{Data}(\Gamma_g) = M$ (cf. Lemma 1, (b) and Lemma 2, (e)* in [7]).

Case (c): $M = [(2k, 2_{k+j}), (2_{b_1}), \dots, (2_{b_{n'}})]$, where $k = g + j + \sum b_i$. We also use the induction on $n'$ as in Case (a). For the first step: $M = [(2(g+j), 2_{g+2j})]$, we start with a curve $C_0$ with $\mathrm{Data}(C_0) = [(g+j), (2_j)]$ constructed in Case (a). We again arrange coordinates so that $C_0 t = (g+j)O + 2R$, where $\underline{m}_R(C_0) = (2_j)$ and $R = (1, 0, a)$. Choose $c_1 \neq a$ and $c_2, \dots, c_{g+j}$ arbitrarily. Then we have $\mathrm{Data}(C_{g+j}) = M$ (cf. Lemma 1, (a)*, (c) in [7]).

Starting with the cuspidal case, we can prove the general case in a similar manner to that in [7]. We have three subcases:

I. $N(M) = r(M) = 1$, II. $N(M) = 1, r(M) = 2$, III. $N(M) \geq 2$.

Here, we only give a proof for $M = [(k), \binom{1}{1}_{b_1}, \dots, \binom{1}{1}_{b_n}, (2_{b_{n+1}}), \dots, (2_{b_{n+n'}})]$, where $k = g + \sum_{j=1}^{n+n'} b_j$, which is one of the remaining cases in I. We use the induction on $n$.

(i) We constructed a cuspidal curve $C$ with $\mathrm{Data}(C) = [(k), (2_{b_{n+1}}), \dots, (2_{b_{n+n'}})]$.

(ii) Suppose we have already constructed $C_0$ with

$$\mathrm{Data}(C_0) = [(k_0), \begin{pmatrix}1\\1\end{pmatrix}_{b_1}, \dots, \begin{pmatrix}1\\1\end{pmatrix}_{b_{n-1}}, (2_{b_{n+1}}), \dots, (2_{b_{n+n'}})],$$

where $k_0 = g + \sum_{j=1}^{n-1} b_j + \sum_{j=n+1}^{n+n'} b_j$. By arranging coordinates, we have $C_0 l = k_0 O + B_1 + B_1'$ and $C_0 t = (k_0+1)O + A_1$. Letting $A_1 = (1, 0, c_1)$, the strict transform $C_1$ of $C_0$ via $\varphi_{c_1}$ has the property $C_0 t = (k_0 + 2)O + A_2$. Write $A_2 = (1, 0, c_2)$. We successively choose $c_2, \dots, c_{b_n}$ in this way. Then the strict transform $C$ of $C_0$ via $\varphi_{c_{b_n}} \circ \cdots \circ \varphi_{c_1}$ has the desired property (cf. Lemma 1, (d) and Lemma 2, (tn) in [7]). In particular, $C$ contains a tacnode $\binom{1}{1}_{b_n}$ at $B = (0, 1, 0)$.

## 5. Defining equations

We now describe the defining equations for those curves listed in Corollary. In [6, 7, 8], the defining equations were computed step by step by using quadratic Cremona transformations. But, for some cases, we encountered a difficulty to evaluate points in some special positions. We here employ the method used by Degtyarev in [3].

**Lemma 12.** *Consider two polynomials*

$$g(t) = \sum_{i=0}^{d} c_i t^i, \quad \delta(t) = \sum_{i=0}^{2d} d_i t^i \in \mathrm{C}[t].$$

*Suppose $\delta(0) = d_0 \neq 0$. For $k \leq d$, we have $t^k \mid (g^2 - \delta)$ if and only if*

(1) $c_0 = \pm\sqrt{d_0}$,

(2) $c_j = (d_j - \sum_{i=1}^{j-1} c_i c_{j-i})/(2c_0)$ *for* $j = 1, \dots, k-1$.

*Proof.* Write $g(t)^2 = \sum_{j=0} b_j t^j$. We see that $b_j = \sum_{i=0}^{j} c_i c_{j-i}$ for $j \leq d$. □

**Proposition 13.** *The defining equations of irreducible plane curves of type $(d, d-2)$ with genus $g$ having only cusps are the following (up to projective equivalence, the $\lambda_i$'s are distinct).*

(a) $y^k z^2 + 2Gz + \left\{G^2 - \Delta\right\}/y^k = 0$, *where*

$$\Delta(x, y) = \prod_{i=1}^{n'} (x - \lambda_i y)^{2b_i+1} \prod_{i=n'+1}^{2g+2} (x - \lambda_i y).$$

*Letting $G(x, y) = \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h$, the coefficients $c_0, \dots, c_{k-1}$ are determined by the condition $y^k \mid (G(1, y)^2 - \Delta(1, y))$ (see Lemma 12).*

(b) $(y^k z + \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h)^2 y - \prod_{i=1}^{n'} (x - \lambda_i y)^{2b_i+1} \prod_{i=n'+1}^{2g+1} (x - \lambda_i y) = 0.$

(c) $(y^k z + \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h)^2 - y^{2j+1} \prod_{i=1}^{n'} (x - \lambda_i y)^{2b_i+1} \prod_{i=n'+1}^{2g+1} (x - \lambda_i y) = 0,$
*where* $c_0 \neq 0$.

*Proof.* Class (a). In this case, in view of the argument in the proof of Theorem 1, (ii), we have $T(C) = \{(k, 0), (0, 2b_1 + 1), \dots, (0, 2b_{n'} + 1), (0, 1), \dots, (0, 1)\}$. Thus, we can write $F = y^k$ and $\Delta$ as above. We must have $y^k | (G^2 - \Delta)$. In view of Lemma 12, the coefficients $c_0, \dots, c_{k-1}$ are uniquely determined. In particular, $c_0 = \pm 1$. So by Remark 8, the defining equation is irreducible.

Class (b): We have $T(C) = \{(2k+1, 2k+1), (0, 2b_1+1), \dots, (0, 2b_{n'+1}), (0, 1), \dots, (0, 1)\}$. We can arrange coordinates as

$$F = y^{2k+1}, \quad \Delta = y^{2k+1} \prod_{i=1}^{n'} (x - \lambda_i y)^{2b_i+1} \prod_{i=n'+1}^{2g+1} (x - \lambda_i y).$$

We infer that $G = y^{k+1} G_0$ for some $G_0$.

Class (c): We have $T(C) = \{(2k, 2k+2j+1), (0, 2b_1+1), \dots, (0, 2b_{n'+1}), (0, 1), \dots, (0, 1)\}$. We can arrange coordinates as

$$F = y^{2k}, \quad \Delta = y^{2k+2j+1} \prod_{i=1}^{n'} (x - \lambda_i y)^{2b_i+1} \prod_{i=n'+1}^{2g+1} (x - \lambda_i y).$$

It follows that $G = y^k G_0$ for some $G_0$. If we write $G_0 = \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h$, then we must have $c_0 \neq 0$, for otherwise the defining equation becomes reducible (see Remark 8). $\square$

**Example 14.** We give the defining equation of a cuspidal septic curve $C$ with $\mathrm{Data}(C) = [(5), (2), (2), (2), (2)]$ which are birational to the elliptic curve $y^2 = (x^2 - 1)(x^2 - \lambda^2)$, $(\lambda \neq \pm 1, 0)$.

$$\begin{aligned} &y^5 z^2 + \{2x^4 - 3(\lambda^2+1)x^2y^2 + \frac{3}{4}(\lambda^4 + 6\lambda^2 + 1)y^4\}x^2 z \\ &\quad - \frac{1}{8}(\lambda^2+1)(\lambda^4 - 10\lambda^2 + 1)x^6 y + \frac{3}{64}\{3\lambda^8 - 28\lambda^6 - 78\lambda^4 - 28\lambda^2 + 3\}x^4 y^3 \\ &\qquad + 3\lambda^4(\lambda^2+1)x^2y^5 - \lambda^6 y^7 = 0 \end{aligned}$$

**Proposition 15.** *The defining equations of irreducible plane curves of type $(d, d-2)$ with genus $g$ having only bibranched singularities are the following (up to projective equivalence, the $\lambda_i$'s are distinct).*

(e) $(y^k z + \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h)^2 - y^{2j} \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+2} (x - \lambda_i y) = 0,$
*where* $c_0 \neq 0$.

(f) $y^{2k+r}z^2 + 2y^kG_0z + \Big\{G_0^2 - \Delta_0\Big\}/y^r = 0$,
*where*

$$\Delta_0(x,y) = \prod_{i=1}^{n}(x-\lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+2}(x-\lambda_i y)$$

*and the coefficients* $c_0, \ldots, c_{r-1}$ *of* $G_0(x,y) = \sum_{h=0}^{k+r+1} c_h x^{k+r+1-h} y^h$ *are determined by the condition* $y^r \mid (G_0(1,y)^2 - \Delta_0(1,y))$ *(cf. Lemma 12) and* $c_r$ *is chosen so that* $y^{r+1} \nmid (G_0(1,y)^2 - \Delta_0(1,y))$.

(aa) $x^r y^k z^2 + 2Gz + \Big\{G^2 - \Delta\Big\}/(x^r y^k) = 0$,
*where*

$$\Delta(x,y) = \prod_{i=1}^{n}(x-\lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+2}(x-\lambda_i y) \quad (\lambda_i \neq 0 \text{ for all } i).$$

*Write* $G(x,y) = \sum_{h=0}^{k+r+1} c_h x^{k+r+1-h} y^h$. *The coefficients* $c_0, \ldots, c_{k-1}$, $c_{k+2}, \ldots, c_{k+r+1}$ *are determined by the conditions* $y^k \mid (G(1,y)^2 - \Delta(1,y))$ *and* $x^r \mid (G(x,1)^2 - \Delta(x,1))$ *(cf. Lemma 12).*

(aa+) $x(y^k z + 2G_0)z + \Big\{xG_0^2 - \Delta_0\Big\}/y^k = 0$,
*where*

$$\Delta_0(x,y) = \prod_{i=1}^{n}(x-\lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+1}(x-\lambda_i y) \quad (\lambda_i \neq 0 \text{ for all } i).$$

*Write* $G_0(x,y) = \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h$. *The coefficients* $c_0, \ldots, c_{k-1}$ *are determined by the condition* $y^k \mid (G_0(1,y)^2 - \Delta_0(1,y))$.

(aa1) $xyz^2 - (x-\lambda y)^4 = 0 \quad (\lambda \neq 0,\ g = 0)$.

(aa2) $xyz^2 - (x-\lambda_1 y)^2(x-\lambda_2 y)^2 = 0 \quad (\lambda_1\lambda_2 \neq 0,\ g = 0)$.

(aa3) $xyz^2 - (x-\lambda_1 y)^2(x-\lambda_2 y)(x-\lambda_3 y) = 0 \quad (\lambda_1\lambda_2\lambda_3 \neq 0,\ g = 1)$.

(aa4) $xyz^2 - \prod_{i=1}^{4}(x-\lambda_i y) = 0 \quad (\lambda_i \neq 0 \text{ for all } i,\ g = 2)$.

(ab) $x^r y^{2k+1} z^2 + 2y^{k+1}G_0 z + \Big\{yG_0^2 - \Delta_0\Big\}/x^r = 0$,
*where*

$$\Delta_0(x,y) = \prod_{i=1}^{n}(x-\lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+1}(x-\lambda_i y) \quad (\lambda_i \neq 0 \text{ for all } i)$$

*and the coefficients* $c_{k+2}, \ldots, c_{k+r+1}$ *of* $G_0(x,y) = \sum_{h=0}^{k+r+1} c_h x^{k+r+1-h} y^h$ *are determined by the condition* $x^r \mid (G_0(x,1)^2 - \Delta_0(x,1))$.

(ab+) $$(y^k z + \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h)^2 xy - \prod_{i=1}^{n}(x-\lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g}(x-\lambda_i y) = 0,$$
*where* $\lambda_i \neq 0$ *for all* $i$.

(ac) $x^r y^{2k} z^2 + 2y^k G_0 z + \left\{ G_0^2 - \Delta_0 \right\} / x^r = 0,$
*where*

$$\Delta_0(x,y) = y^{2j+1} \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+1} (x - \lambda_i y) \quad (\lambda_i \neq 0 \; for \; all \; i)$$

*and the coefficients* $c_{k+2}, \ldots, c_{k+r+1}$ *of* $G_0(x,y) = \sum_{h=0}^{k+r+1} c_h x^{k+r+1-h} y^h$ *are determined by the condition* $x^r \mid (G_0(x,1)^2 - \Delta_0(x,1))$ *and* $c_0 \neq 0$, *which is required for the irreducibility of the defining equation (Remark* 8*).*

(ac+) $(y^k z + \sum_{h=0}^{k+1} c_h x^{k+1-h} y^h)^2 x - y^{2j+1} \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g} (x - \lambda_i y) = 0,$
*where* $\lambda_i \neq 0$ *for all* $i$ *and* $c_0 \neq 0$.

(bb) $(x^r y^k z + \sum_{h=0}^{k+r+1} c_h x^{k+r+1-h} y^h)^2 xy - \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g} (x - \lambda_i y) = 0,$
*where* $\lambda_i \neq 0$ *for all* $i$.

(bc) $(x^r y^k z + \sum_{h=0}^{k+r+1} c_h x^{k+r+1-h} y^h)^2 y \quad - x^{2l+1} \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g} (x - \lambda_i y) = 0,$
*where* $\lambda_i \neq 0$ *for all* $i$ *and* $c_{k+r+1} \neq 0$.

(cc) $(x^r y^k z + \sum_{h=0}^{k+r+1} c_h x^{k+r+1-h} y^h)^2$
$-x^{2l+1} y^{2j+1} \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g} (x - \lambda_i y) = 0,$
*where* $\lambda_i \neq 0$ *for all* $i$ *and* $c_0 c_{k+r+1} \neq 0$.

*Proof.* Class (e): In this case, we have $T(C) = \{(2k, 2k+2j), (0, 2b_1), \ldots, (0, 2b_n), (0,1), \ldots, (0,1)\}$. We can arrange coordinates as

$$F = y^{2k}, \quad \Delta = y^{2k+2j} \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+2} (x - \lambda_i y).$$

We infer that $G = y^k G_0$ for some $G_0$.
Class (f): We have $T(C) = \{(2k+r, 2k), (0, 2b_1), \ldots, (0, 2b_n), (0,1), \ldots, (0,1)\}$. We can arrange coordinates as

$$F = y^{2k+r}, \quad \Delta = y^{2k} \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+2} (x - \lambda_i y).$$

We infer that $G = y^k G_0$ for some $G_0$. Write $\Delta = y^{2k} \Delta_0$. Furthermore, we must have $y^r | (G_0^2 - \Delta_0)$.
Class (aa): We may assume $k \geq r$. We have the case in which $T(C) = \{(k,0), (r,0), (0, 2b_1), \ldots, (0, 2b_n), (0,1), \ldots, (0,1)\}$. We can then arrange coordinates as

$$F = x^r y^k \quad \Delta = \prod_{i=1}^{n} (x - \lambda_i y)^{2b_i} \prod_{i=n+1}^{n+2g+2} (x - \lambda_i y) \quad (\lambda_i \neq 0 \text{ for all } i).$$

We infer that $y^k | (G^2 - \Delta)$ and $x^r | (G^2 - \Delta)$.

In case $r = 1$, we also have the case in which $T(C) = \{(k, 0), (1, 1), (0, 2b_1), \ldots, (0, 2b_n), (0, 1), \ldots, (0, 1)\}$. We obtain Class (aa+). If $d = 4$, then we have four more classes:

| Class | $T(C)$ | $g$ |
|---|---|---|
| (aa1) | $\{(1,1),(1,1),(0,4)\}$ | 0 |
| (aa2) | $\{(1,1),(1,1),(0,2),(0,2)\}$ | 0 |
| (aa3) | $\{(1,1),(1,1),(0,2),(0,1),(0,1)\}$ | 1 |
| (aa4) | $\{(1,1),(1,1),(0,1),(0,1),(0,1),(0,1)\}$ | 2 |

For the remaining classes, we omit the details. □

**Acknowledgement.** The first author would like to thank Prof. I. V. Dolgachev for comments on the book [1].

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Received January 7, 2008