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Commutativity Conditions on Derivations and Lie Ideals in σ-prime Rings

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Abstract. Let R be a 2-torsion free σ-prime ring, U a nonzero square closed σ-Lie ideal of R and let d be a derivation of R. In this paper it is shown that:
1) If d is centralizing on U, then $d = 0$ or $U \subseteq Z(R)$.
2) If either $d([x, y]) = 0$ for all $x, y \in U$, or $[d(x), d(y)] = 0$ for all $x, y \in U$ and d commutes with σ on U, then $d = 0$ or $U \subseteq Z(R)$.
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1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. Recall that R is said to be 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$ for all a and b in R. If σ is an involution in R, then R is said to be σ-prime if $aRb = aR\sigma(b) = 0$ implies that $a = 0$ or $b = 0$. It is obvious that every prime ring equipped with an involution σ is also σ-prime, but the converse need not be true in general. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all x, y in R. A mapping $F : R \to R$ is said to be centralizing on a subset S of R

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if $[F(s), s] \in Z(R)$ for all $s \in S$. In particular, if $[F(s), s] = 0$ for all $s \in S$, then F is commuting on S. In all that follows $Sa_\sigma(R)$ will denote the set of symmetric and skew-symmetric elements of R; i.e., $Sa_\sigma(R) = \{x \in R/\sigma(x) = \pm x\}$. For any $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal U which satisfies $\sigma(U) \subseteq U$ is called a σ-Lie ideal. If U is a Lie (resp. σ-Lie) ideal of R, then U is called a square closed Lie (resp. σ-Lie) ideal if $u^2 \in U$ for all $u \in U$. Since $(u + v)^2 \in U$ and $[u, v] \in U$, we see that $2uv \in U$ for all $u, v \in U$. Therefore, for all $r \in R$ we get $2r[u, v] = 2[u, rv] - 2[u, r]v \in U$ and $2[u, v]r = 2[u, vr] - 2v[u, r] \in U$, so that $2R[U, U] \subseteq U$ and $2[U, U]R \subseteq U$. This remark will be freely used in the whole paper.

Many works concerning the relationship between commutativity of a ring and the behavior of derivations defined on this ring have been studied. The first important result in this subject is Posner's theorem, which states that the existence of a nonzero centralizing derivation on a prime ring forces this ring to be commutative ([9]). This result has been generalized by many authors in several ways.

In [3], I. N. Herstein proved that if R is a prime ring of characteristic not 2 which has a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Motivated by this result, H. E. Bell, in [1], studied derivations d satisfying $d([x, y]) = 0$ for all $x, y \in R$. In [4] and [7], L. Oukhtite and S. Salhi generalized these results to σ-prime rings. In particular, they proved that if R is a 2-torsion free σ-prime ring equipped with a nonzero derivation which is centralizing on R, then R is necessarily commutative.

Our purpose in this paper is to extend these results to square closed σ-Lie ideals.

2. Preliminaries and results

In order to prove our main theorems, we shall need the following lemmas.

Lemma 1. ([8], Lemma 4) *If $U \not\subset Z(R)$ is a σ-Lie ideal of a 2-torsion free σ-prime ring R and $a, b \in R$ such that $aUb = \sigma(a)Ub = 0$ or $aUb = aU\sigma(b) = 0$, then $a = 0$ or $b = 0$.*

Lemma 2. ([5], Lemma 2.3) *Let $0 \neq U$ be a σ-Lie ideal of a 2-torsion free σ-prime ring R. If $[U, U] = 0$, then $U \subseteq Z(R)$.*

Lemma 3. ([6], Lemma 2.2) *Let R be a 2-torsion free σ-prime ring and U a nonzero σ-Lie ideal of R. If d is a derivation of R which commutes with σ and satisfies $d(U) = 0$, then either $d = 0$ or $U \subseteq Z(R)$.*

Remark. One can easily verify that Lemma 3 is still valid if the condition that d commutes with σ is replaced by $d \circ \sigma = -\sigma \circ d$.

Theorem 1. *Let R be a 2-torsion free σ-prime ring and U a square closed σ-Lie ideal of R. If d is a derivation of R satisfying $[d(u), u] \in Z(R)$ for all $u \in U$, then $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose that $U \nsubseteq Z(R)$. As $[d(x), x] \in Z(R)$ for all $x \in U$, by linearization $[d(x), y] + [d(y), x] \in Z(R)$ for all $x, y \in U$. Since $\mathrm{char} R \neq 2$, the fact that $[d(x), x^2] + [d(x^2), x] \in Z(R)$ yields $x[d(x), x] \in Z(R)$ for all $x \in U$; hence

$$[r, x][d(x), x] = 0 \quad \text{for all} \quad x \in U, r \in R,$$

and therefore $[d(x), x]^2 = 0$ for all $x \in U$. Since $[d(x), x] \in Z(R)$,

$$[d(x), x]R[d(x), x]\sigma([d(x), x]) = 0 \ \text{ for all } \ x \in U$$

and the σ-primeness of R yields $[d(x), x] = 0$ or $[d(x), x]\sigma([d(x), x]) = 0$. If $[d(x), x]\sigma([d(x), x])=0$, then $[d(x), x]R\sigma([d(x), x])=0$; and the fact that $[d(x), x]^2 = 0$ gives

$$[d(x), x]R\sigma([d(x), x]) = [d(x), x]R[d(x), x] = 0.$$

Since R is σ-prime, we obtain

$$[d(x), x] = 0 \ \text{ for all } \quad x \in U.$$

Let us consider the map $\delta : R \longmapsto R$ defined by $\delta(x) = d(x) + \sigma \circ d \circ \sigma(x)$. One can easily verify that δ is a derivation of R which commutes with σ and satisfies

$$[\delta(x), x] = 0 \ \text{ for all } \quad x \in U.$$

Linearizing this equality, we obtain

$$[\delta(x), y] + [\delta(y), x] = 0 \quad \text{for all } \ x, y \in U.$$

Writing $2xz$ instead of y and using $\mathrm{char} R \neq 2$, we find that

$$\delta(x)[x, z] = 0 \quad \text{for all } \ x, z \in U.$$

Replacing z by $2zy$ in this equality, we conclude that $\delta(x)z[x, y] = 0$, so that

$$\delta(x)U[x, y] = 0 \ \text{ for all } \ x, y \in U. \tag{1}$$

By virtue of Lemma 1, it then follows that

$$\delta(x) = 0 \ \text{ or } \ [x, U] = 0, \ \text{ for all } \ x \in U \cap Sa_\sigma(R).$$

Let $u \in U$. Since $u - \sigma(u) \in U \cap Sa_\sigma(R)$, it follows that

$$\delta(u - \sigma(u)) = 0 \ \text{ or } \ [u - \sigma(u), U] = 0.$$

If $\delta(u - \sigma(u)) = 0$, then $\delta(u) \in Sa_\sigma(R)$ and (1) yields $\delta(u) = 0$; or $[u, U] = 0$. If $[u - \sigma(u), U] = 0$, then $[u, y] = [\sigma(u), y]$ for all $y \in U$ and (1) assures that

$$\delta(u)U[u, y] = 0 = \delta(u)U\sigma([u, y]), \ \text{ for all } \ y \in U.$$

Applying Lemma 1, we find that $\delta(u) = 0$ or $[u, U] = 0$. Hence, U is a union of two additive subgroups G_1 and G_2, where

$$G_1 = \{u \in U \text{ such that } \delta(u) = 0\} \ \text{ and } \ G_2 = \{u \in U \text{ such that } [u, U] = 0\}.$$

Since a group cannot be a union of two of its proper subgroups, we are forced to $U = G_1$ or $U = G_2$. Since $U \nsubseteq Z(R)$, Lemma 2 assures that $U = G_1$ and therefore $\delta(U) = 0$. Now applying Lemma 3, we get $\delta = 0$ and therefore $d \circ \sigma = -\sigma \circ d$. As $[d(x), x] = 0$ for all $x \in U$, in view of the above Remark, similar reasoning leads to $d = 0$. □

Corollary 1. ([7], Theorem 1.1) *Let R be a 2-torsion free σ-prime ring and d a nonzero derivation of R. If d is centralizing on R, then R is commutative.*

Theorem 2. *Let U be a square closed σ-Lie ideal of a 2-torsion free σ-prime ring R and d a derivation of R which commutes with σ on U.*
If $[d(x), d(y)] = d([y, x])$ for all $x, y \in U$, then $d = 0$ or $U \subseteq Z(R)$.

Proof. Suppose that $U \not\subset Z(R)$. We have

$$[d(x), d(y)] = d([y, x]) \text{ for all } x, y \in U. \tag{2}$$

Substituting $2xy$ for y in (2) and using $\mathrm{char}R \neq 2$, we get

$$d(x)[y, x] = [d(x), x]d(y) + d(x)[d(x), y] \text{ for all } x, y \in U. \tag{3}$$

Replacing y by $2[y, z]x$ and using (3), we find that

$$[d(x), x][y, z]d(x) + d(x)[y, z][d(x), x] = 0 \text{ for all } x, y, z \in U. \tag{4}$$

Replace y by $2[y, z]d(x)$ in (3) to get

$$d(x)[y, z][d(x), x] - [d(x), x][y, z]d^2(x) = 0 \text{ for all } x, y, z \in U. \tag{5}$$

From (4) and (5) we obtain

$$[d(x), x][y, z](d(x) + d^2(x)) = 0 \text{ for all } x, y, z \in U. \tag{6}$$

Writing $2[u, v](d(x) + d^2(x))y$ instead of y in (6), where $u, v \in U$, we obtain $[d(x), x][u, v]z(d(x) + d^2(x))y(d(x) + d^2(x)) = 0$, so that

$$[d(x), x][u, v]z(d(x) + d^2(x))U(d(x) + d^2(x)) = 0 \text{ for all } x, u, v, z \in U. \tag{7}$$

If $x \in U \cap Sa_\sigma(R)$, then Lemma 1 together with (7) assures that

$$d(x) + d^2(x) = 0 \quad \text{or} \quad [d(x), x][u, v]z(d(x) + d^2(x)) = 0 \quad \text{for all } u, v, z \in U.$$

Suppose that $[d(x), x][u, v]z(d(x) + d^2(x)) = 0$. Then

$$[d(x), x][u, v]U(d(x) + d^2(x)) = 0. \tag{8}$$

Since d commutes with σ and $x \in Sa_\sigma(R)$, in view of (8), Lemma 1 gives

$$d(x) + d^2(x) = 0 \text{ or } [d(x), x][u, v] = 0 \text{ for all } u, v \in U. \tag{9}$$

If $[d(x),x][u,v]=0$, then replacing u by $2uw$ in (9) where $w\in U$, we obtain

$$[d(x),x]U[u,v]=0. \tag{10}$$

As $\sigma(U)=U$ and $[U,U]\neq 0$, by (10), Lemma 2 yields that $[d(x),x]=0$. Thus, in any event,

$$\text{either } [d(x),x]=0 \text{ or } d(x)+d^2(x)=0 \text{ for all } x\in U\cap Sa_\sigma(R).$$

Let $x\in U$. Since $x+\sigma(x)\in U\cap Sa_\sigma(R)$, either $d(x+\sigma(x))+d^2(x+\sigma(x))=0$ or $[d(x+\sigma(x)),x+\sigma(x)]=0$.

If $d(x+\sigma(x))+d^2(x+\sigma(x))=0$, then $d(x)+d^2(x)\in Sa_\sigma(R)$ and (7) yields that $d(x)+d^2(x)=0$ or $[d(x),x][u,v]U(d(x)+d^2(x))=0$.

If $[d(x),x][u,v]U(d(x)+d^2(x))=0$, once again using $d(x)+d^2(x)\in Sa_\sigma(R)$, we find that $d(x)+d^2(x)=0$, or $[d(x),x][u,v]$ for all $u,v\in U$, in which case $[d(x),x]=0$.

Now suppose that $[d(x+\sigma(x)),x+\sigma(x)]=0$. As $x-\sigma(x)\in U\cap Sa_\sigma(R)$ we have to distinguish two cases:

1) If $d(x-\sigma(x))+d^2(x-\sigma(x))=0$, then $d(x)+d^2(x)\in Sa_\sigma(R)$. Reasoning as above we get $d(x)+d^2(x)=0$ or $[d(x),x]=0$.

2) If $[d(x-\sigma(x)),x-\sigma(x)]=0$, then $[d(x),x]\in Sa_\sigma(R)$. Replace u by $2yu$ in (7), with $y\in U$, to get $[d(x),x]y[u,v]z(d(x)+d^2(x))U(d(x)+d^2(x))=0$, so that

$$[d(x),x]U[u,v]z(d(x)+d^2(x))U(d(x)+d^2(x))=0 \text{ for all } x,u,v,z\in U. \tag{11}$$

Since $[d(x),x]\in Sa_\sigma(R)$, from (11) it follows that

$$[d(x),x]=0 \text{ or } [u,v]U(d(x)+d^2(x))U(d(x)+d^2(x))=0 \text{ for all } u,v\in U.$$

Suppose $[u,v]U(d(x)+d^2(x))U(d(x)+d^2(x))=0$. As $\sigma(U)=U$ and $[U,U]\neq 0$, then

$$(d(x)+d^2(x))U(d(x)+d^2(x))=0. \tag{12}$$

In (6), write $2[u,v](d(x)+d^2(x))r$ instead of z, where $u,v\in U$ and $r\in R$, to obtain

$$[d(x),x][u,v]y(d(x)+d^2(x))r(d(x)+d^2(x))=0, \text{ for all } u,v,y\in U, r\in R. \tag{13}$$

Replacing r by $r\sigma(d(x)+d^2(x))z$ in (13), where $z\in U$, we find that

$$[d(x),x][u,v]y(d(x)+d^2(x))r\sigma(d(x)+d^2(x))z(d(x)+d^2(x))=0,$$

which leads us to

$$[d(x),x][u,v]y(d(x)+d^2(x))U\sigma(d(x)+d^2(x))U(d(x)+d^2(x))=0. \tag{14}$$

Since $\sigma(d(x)+d^2(x))U(d(x)+d^2(x))$ is invariant under σ, by virtue of (14), Lemma 1 yields

$$\sigma(d(x)+d^2(x))U(d(x)+d^2(x))=0 \text{ or } [d(x),x][u,v]y(d(x)+d^2(x))=0.$$

If $\sigma(d(x)+d^2(x))U(d(x)+d^2(x))=0$, then (12) implies that $d(x)+d^2(x)=0$. Now assume that

$$[d(x),x][u,v]y(d(x)+d^2(x))=0 \text{ for all } u,v,y\in U. \tag{15}$$

Replace v by $2wv$ in (15), where $w\in U$, and use (15) to get

$$[d(x),x]w[u,v]y(d(x)+d^2(x))=0,$$

so that

$$[d(x),x]U[u,v]y(d(x)+d^2(x))=0 \text{ for all } u,v,y\in U. \tag{16}$$

As $[d(x),x]\in Sa_\sigma(R)$, (16) yields $[u,v]U(d(x)+d^2(x))=0$, in which case $d(x)+d^2(x)=0$, or $[d(x),x]=0$.
In conclusion, for all $x\in U$ we have either $[d(x),x]=0$ or $d(x)+d^2(x)=0$.
Now let $x\in U$ such that $d(x)+d^2(x)=0$. In (2), put $y=2[y,z]d(x)$ to get

$$d([y,z])[d(x),x]-[[y,z],x]d(x)+[d(x),[y,z]]d(x)=[y,z][d(x),x]. \tag{17}$$

If in (2) we put $y=2[y,z]x$, we get

$$[[y,z],x]d(x)=[d(x),[y,z]]d(x)+d([y,z])[d(x),x]=0. \tag{18}$$

From (17) and (18) it then follows that

$$[y,z][d(x),x]=0 \text{ for all } y,z\in U,$$

hence $[y,z]U[d(x),x]=0$ for all $y,z\in U$. Applying Lemma 1, this leads to

$$[d(x),x]=0, \text{ for all } x\in U.$$

By virtue of Theorem 1, this yields that $d=0$. $\square$

Note that if d is a derivation of R which acts as an anti-homomorphism on U, then d satisfies the condition $[d(x),d(y)]=d([y,x])$ for all $x,y\in U$. Thus we have the following corollary.

Corollary 2. ([6], Theorem 1.1) *Let d be a derivation of a 2-torsion free σ-prime ring R which acts as an anti-homomorphism on a nonzero square closed σ-Lie ideal U of R. If d commutes with σ, then either $d=0$ or $U\subseteq Z(R)$.*

Theorem 3. *Let U be a square closed σ-Lie ideal of a 2-torsion free σ-prime ring R and d a derivation of R. If either $d([x,y])=0$ for all $x,y\in U$, or $[d(x),d(y)]=0$ for all $x,y\in U$ and d commutes with σ on U, then $d=0$ or $U\subseteq Z(R)$.*

Proof. Suppose that $U \not\subseteq Z(R)$. Assume that $d([x, y]) = 0$; for all $x, y \in U$. Let δ be the derivation of R defined by $\delta(x) = d(x) + \sigma \circ d \circ \sigma(x)$.
Clearly, δ commutes with σ and $\delta([x, y]) = 0$ for all $x, y \in U$, so that

$$[\delta(x), y] = [\delta(y), x] \quad \text{for all } \ x, y \in U. \tag{19}$$

Writing $[x, y]$ instead of y in (19), we find that

$$[\delta(x), [x, y]] = 0 \quad \text{for all } \ x, y \in U. \tag{20}$$

Replacing x by x^2 in (19), we conclude that

$$\delta(x)[x, y] + [x, y]\delta(x) = 0 \quad \text{for all } \ x, y \in U. \tag{21}$$

As char$R \neq 2$, from (20) and (21) it follows that

$$\delta(x)[x, y] = 0 \quad \text{for all } \ x, y \in U. \tag{22}$$

Replacing y by $2zy$ in (22), we get $\delta(x)z[x, y] = 0$, so that

$$\delta(x)U[x, y] = 0 \quad \text{for all } \ x, y \in U.$$

From the proof of Theorem 1, we conclude that $\delta = 0$ and thus $d \circ \sigma = -\sigma \circ d$. Since d satisfies $d([x, y]) = 0$ for all $x, y \in U$, by similar reasoning, we are forced to $d = 0$.
Now assume that d commutes with σ and satisfies $[d(x), d(y)] = 0$ for all $x, y \in U$. The fact that $[d(x), d(2xy)] = 0$ implies that

$$d(x)[d(x), y] + [d(x), x]d(y) = 0 \quad \text{for all } \ x, y \in U. \tag{23}$$

Replace y by $2[y, z]d(u)$ in (23), where $z, u \in U$, to find that

$$[d(x), x][y, z]d^2(u) = 0 \quad \text{for all } \ x, y, u \in U. \tag{24}$$

Write $2[s, t]d^2(w)y$ instead of y in (24), where $s, t, w \in U$, thereby concluding that $[d(x), x]z[s, t]d^2(w)yd^2(u) = 0$. Accordingly,

$$[d(x), x]z[s, t]d^2(w)Ud^2(u) = 0 \ \text{ for all } \ s, t, u, w, x \in U. \tag{25}$$

Since d commutes with σ and $\sigma(U) = U$, using (25) we find that

$$d^2(U) = 0 \ \text{ or } \ [d(x), x]U[s, t]d^2(w) = 0.$$

Suppose that

$$[d(x), x]U[s, t]d^2(w) = 0 \ \text{ for all } \ s, t, w, x \in U. \tag{26}$$

Replacing t by $2tv$ in (26), where $v \in U$, we are forced to

$$[d(x), x][s, t]vd^2(w) = 0$$

and hence

$$[d(x), x][s, t]Ud^2(w) = 0 \text{ for all } s, t, w, x \in U. \tag{27}$$

Since $\sigma(U) = U$ and d commutes with σ, then (27) implies that either $d^2(U) = 0$, or $[d(x), x][s, t] = 0$ for all $s, t, x \in U$, in which case $[d(x), x] = 0$ for all $x \in U$. Thus, in any event, we find that

$$d^2(U) = 0 \text{ or } [d(x), x] = 0 \text{ for all } x \in U.$$

If $d^2(U) = 0$, then [5], Theorem 1.1 assures that $d = 0$.

If $[d(x), x] = 0$ for all $x \in U$, then Theorem 1 yields $d = 0$. $\square$

Corollary 3. ([4], Theorem 3.3) *Let d be a nonzero derivation of a 2-torsion free σ-prime ring R. If $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative.*

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