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On Involution Rings with Unique Minimal *-subring

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Abstract. The structure of certain involution rings having a unique minimal *-subring, is described.

1. Introduction

Kruse and Price [5] determined the structure of nilpotent p-rings with unique minimal subring. Using their result, Hirano [4] completely described the structure of right (or left) artinian rings with a unique minimal subring and considered the problem under the general situation. In [10], Wiegandt determined the structure of rings having a unique minimal subring, under the two different interpretations: (i) rings in which the intersection of nonzero subrings is nonzero and (ii) rings having exactly one atom in their lattice of subrings. In this paper, we consider which involution rings have a unique minimal *-subring, considering the two interpretations and thus establishing the involutive versions of the main results in [4] and [10].

All rings considered are associative. Let us recall that an involution ring A is a ring with an additional unary operation *, called involution, subjected to the identities: $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in A$. A biideal B of a ring A is a subring of A satisfying the inclusion $BAB \subseteq B$. An ideal (respectively: biideal, subring) B of an involution ring A is called a *-ideal (respectively: *-biideal, *-subring) of A if B is closed under involution; that is, $B^* = \{a^* \in A : a \in B\} \subseteq B$. An involution ring A is semiprime if and only if, for any *-ideal I of A, $I^2 = 0$ implies $I = 0$. An involution ring A is called *-subdirectly irreducible if the intersection of all nonzero *-ideals of A (called the *-heart of A) is nonzero.

2. Involution rings whose nonzero *-subrings have nonzero intersection

In this section the involutive version of ([10], Theorem 4) shall be proved. We shall make use of the following results.

Lemma 1. *Let B be a minimal *-biideal of an involution ring A. Then $B^2 = B$ if and only if B is a semiprime involution ring.*

Proof. Suppose I is a nonzero *-ideal of B such that $I^2 = 0$. The biideal of A generated by I is $\overline{I} = I + IAI$, which is clearly a *-biideal of A contained in B. The minimality of B implies $B = I + IAI$. Then we have $0 \neq B = B^2 = (I + IAI)^2 = 0$, a contradiction. The converse is obvious. □

Proposition 2. *A *-subring S of an involution ring A is a minimal *-subring of A if and only if, for some prime p, either*

(i) *$S \simeq GF(p)$; or*

(ii) *$S \simeq Z(p)$ where $Z(p)$ is the zero-ring on the cyclic additive group $C(p)$ of order p.*

Proof. If S is a minimal *-subring of A, then S is a *-simple ring. According to ([2], Proposition 2.1), either S is simple or S contains an ideal K such that $S = K \oplus K^*$ and $S^2 \neq 0$. However, the latter case cannot occur. Indeed, suppose $S = K \oplus K^*$. By the assumption on S, K cannot contain nonzero proper left ideals, whence it is a division ring. Now if P denotes the prime field of K, then $\{a + a^* : a \in P\}$ is a proper *-subring of A, properly contained in S, which contradicts the minimality of S. Therefore S must be a simple involution ring. If $S^2 = S$, then S is obviously a minimal *-biideal of itself. Therefore, by the previous lemma, S is semiprime and, by ([6], Proposition 4), S is a minimal biideal of itself and hence a division ring. By the assumption on S, $S \simeq GF(p)$ for some prime p. Finally, if $S^2 = 0$, then it is clear that the additive group of S is a cyclic group of prime order, say p, and hence $S \simeq Z(p)$. □

It is well-known that if R is any ring and $A = R \oplus R^{op}$, where R^{op} is the anti-isomorphic image of R, then A is a ring with involution defined by $(a, b)^* = (b, a)$ for every $a, b \in R$. This involution is called the exchange involution. Let A be an involution ring such that either A is a division ring or $A = D \oplus D^{op}$ where D is a division ring. In the first case, A has a unique smallest subfield, which we shall call the prime *-field of A. In the second case, if F is the smallest subfield of D, we shall call $\{(a, a) : a \in F\}$ the prime *-field of A. We say that A is *-algebraic if, for any nonzero *-subring B of A, there exists a nonzero element $b \in B$ which is algebraic over the prime *-field of A. Now we are in a position to prove the following:

Theorem 3. *If the intersection S of the nonzero *-subrings of an involution ring A is nonzero, then A is one of the following rings:*

(i) *A is a *-algebraic division ring with prime *-field $S \simeq GF(p)$ of finite characteristic p;*

(ii) *$A \simeq D \oplus D^{op}$, where D is a division ring and A is *-algebraic with prime *-field $S \simeq GF(2)$;*

(iii) *A is a *-subdirectly irreducible ring with *-heart $S \simeq Z(p)$ for some prime p;*

(iv) *A is a *-subdirectly irreducible ring with *-heart $H \simeq K \oplus K^*$ where $K \simeq Z(2) \simeq K^*$.*

Proof. If A has a unique smallest *-subring S, then S generates a *-ideal H in A. Obviously, A is *-subdirectly irreducible with *-heart H.

Case 1. ($H^2 \neq 0$). Either H is a simple prime ring or $H = K \oplus K^*$ where each of the ideals K and K^* of A are simple prime rings. Since H has d.c.c. on *-biideals, we know by ([1], Corollary 4), that H is an artinian ring and so H has a minimal left ideal, say L.

If H is simple prime, $L = He$ for some idempotent $e \in L$. Then $L^* = e^*H$ is a minimal right ideal of H. Now $L^*L = (e^*H)(He) = e^*He \neq 0$. Furthermore, by ([8], Theorem 4), L^*L is a minimal *-biideal of H. So, $B = L^*L \subseteq L$. The *-ideal H does not contain other minimal left ideals, besides L. Indeed, if L_1 is a minimal left ideal of H, then $B = L_1^*L_1 \subseteq L_1$. Now, $0 \neq B \subseteq L \cap L_1 \subseteq L_1$ and the minimality of L and L_1 implies that $L_1 = L$. Thus $H = L$ and H is a division ring with prime field of finite characteristic p. Since H has a unity and is a *-essential *-ideal, we have, by ([7], Lemma 8), that $A = H$. Hence A is a division ring.

Now we consider the case when $H = K \oplus K^*$. Clearly K is artinian with a unique minimal left ideal, which implies that K is a division ring. Consequently, $H = B = K \oplus K^*$ where K is a division ring with prime field $GF(p)$, for some prime p. Consequently, we have that $A = H$. If $p \neq 2$, then $\{a + a^* : a \in GF(p)\}$ and $\{a - a^* : a \in GF(p)\}$ are two distinct *-subrings of A.

We notice that, in either case, the ring A is *-algebraic, since S is the prime *-field of A and S is contained in every nonzero *-subring of A.

Case 2. ($H^2 = 0$). By Proposition 2, the *-subring S is a minimal subring of A. By ([3], Proposition 6.2), H^+, the additive group of H, is an elementary abelian p-group and hence is a direct sum of cyclic groups of order p. By our assumption on A, it is clear that either $H \simeq Z(p)$ or $H = K \oplus K^*$, where $K \simeq Z(p) \simeq K^*$. If $p \neq 2$, then the case $H = K \oplus K^*$ cannot occur, for then $\{a + a^* : a \in K\}$ and $\{a - a^* : a \in K\}$ would be two distinct minimal *-subrings of A. □

As an immediate consequence, we have:

Corollary 4. *If A is a semiprime involution ring, then the intersection S of the nonzero *-subrings of A is nonzero if, and only if, A is one of the following rings:*

(i) *A is a *-algebraic division ring with prime *-field $S \simeq GF(p)$ of finite characteristic p;*

(ii) *$A \simeq D \oplus D^{op}$, where D is a division ring and A is *-algebraic with prime *-field $S \simeq GF(2)$.*

3. Involution rings with one atom in their lattice of *-subrings

An involution ring in which every idempotent element is central shall be called a CI-involution ring. In this section, our main aim is to determine the structure of CI-involution rings with descending chain condition (d.c.c.) on *-biideals which have exactly one atom S in their lattice of *-subrings (admitting also nonzero *-subrings T such that $S \cap T = 0$), namely, the involutive version of ([4], Theorem 1). The problem shall also be considered in a more general context. First, however, we need some preliminary results. In an involution ring A, an element a is symmetric if $a^* = a$ and b in A is skew-symmetric if $b^* = -b$. We shall call an element in A a *-element if it is either symmetric or skew-symmetric.

Lemma 5. *If an involution ring A satisfies the condition*

$$\text{for any nonzero *-elements } a, b \in A, \; ab \neq 0 \tag{$\diamond$}$$

then A is semiprime.

Proof. If I is a nonzero *-ideal of A such that $I^2 = 0$, then, for any $0 \neq a \in I$, we have $a^2 = 0$. So, if a is a *-element, we immediately have a contradiction with condition ($\diamond$). If a is not a *-element, then $a^* + a$ is a nonzero *-element and $(a + a^*)^2 = 0$, again contradicting condition ($\diamond$). □

Proposition 6. *A finite involution ring A satisfies condition ($\diamond$) if, and only if, either A is a field, or $A = F \oplus F^*$ where F is a field.*

Proof. Suppose that the ring A satisfies condition ($\diamond$). Then every nonzero left ideal L of A is *-essential in A; that is, $L \cap I \neq 0$ for any nonzero *-ideal I of A. Indeed, suppose that $L \cap I = 0$ for some nonzero *-ideal I of A. Then, for any nonzero *-element $a \in I$ and nonzero $b \in L$, we have $ab \in IL \subseteq I \cap L = 0$. If b is also a *-element, we have a contradiction with condition ($\diamond$); otherwise, $b + b^*$ is a nonzero *-element and $ba^* \in LI = 0$, whence $ab^* = 0$. Consequently, $a(b + b^*) = 0$, contradicting condition ($\diamond$). Now since A is finite, A has a finite number of *-essential minimal left ideals and, consequently, a finite number of *-essential minimal *-biideals. Finally, by ([7], Corollary 10), the fact that A satisfies condition ($\diamond$) and that a finite division ring is a field, we have that either A is a field or $A = F \oplus F^*$, where F is field. □

In what follows, $[a]$ denotes the subring of A generated by a. We consider the following condition:

($\triangle$) If a, b are nonzero *-elements of an involution ring A such that $ab = 0$, then either $[a]$ or $[b]$ is infinite.

Proposition 7. *If A is a CI-involution ring, then the following conditions are equivalent:*

(i) *A has a unique minimal *-subring S and S is a field;*

(ii) *A has a nonzero finite *-subring and A satisfies condition* ($\triangle$).

Proof. (i) implies (ii). Let a, b be nonzero *-elements of A such that $[a]$ and $[b]$ are finite. Then the field S is contained in both of $[a]$ and $[b]$ and so $S \subseteq [a]\,[b]$. Thus we have $ab \neq 0$.

(ii) implies (i). If A has a nonzero finite *-subring, then this *-subring contains a minimal *-subring S. Condition ($\triangle$) implies that the *-subring S is a field. It remains to show that S is the unique minimal $*$-subring of A. Let e be the identity element of S and let f be any other idempotent symmetric element in A such that $[f]$ is a minimal *-subring of A. Then $[e, f]$, the subring of A generated by e and f, is clearly a finite *-subring of A in which every element is symmetric. Hence it has no zero-divisors, by condition ($\triangle$). Thus it is a field and so $e = f$. $\square$

Clearly, (i) implies (ii) in the previous proposition is valid for an arbitrary involution ring A.

Let us recall that a ring A is π-regular if for every element a in A there exists a positive integer n (depending on a) and an element x in A such that $a^n x a^n = a^n$. It is easy to see that if a is a *-element of an involution ring A, then there exists a *-element in A, namely $y = x^* a^n x$, such that $a^n y a^n = a^n$. As usual, A is said to be torsion-free if it does not have nonzero elements of finite order.

Clearly, if $A = A_1 \oplus A_2$ is a direct sum of rings A_1 and A_2 with involutions $*_1$ and $*_2$, respectively, then we may define an involution $*$ on A by $(a_1, a_2)^* = (a_1^{*_1}, a_2^{*_2})$, for every $a_1 \in A_1$ and $a_2 \in A_2$.

Proposition 8. *Let A be a π-regular CI-involution ring. Then A has a unique minimal *-subring S and $S \simeq GF(p)$, for some prime p, if and only if, either:*

(i) *$A \simeq T \oplus D$ where T is a torsion-free involution ring and D is a division involution ring of finite characteristic p; or;*

(ii) *$A \simeq T \oplus (D \oplus D^{op})$, where T is a torsion-free involution ring, D is a division ring of characteristic 2 and the ring $D \oplus D^{op}$ is endowed with the exchange involution.*

Proof. We prove the direct implication. Let e be the identity element of S. Then, by assumption, e is a central element. First consider the *-subring

$$T = \{a \in A : a = c - ec \text{ for some } c \in A\ \}$$

of A. By ($\triangle$), any nonzero symmetric element in T generates an infinite subring. Suppose a is a *-element in T and a has finite order. Since T is also π-regular, there exist a positive integer n and a *-element $x \in T$ such that $a^n x a^n = a^n$ holds. If $a^n \neq 0$, then $a^n x$ is a nonzero idempotent element and hence is central. We have $a^n x \in T$ and $a^n x = x a^n$, since $a^n x x a^n = ((a^n x)\, x)\, a^n = (x\,(a^n x))\, a^n = x\,(a^n x a^n) = x a^n$ and $a^n x x a^n = a^n\,(x\,(x a^n)) = a^n\,((x a^n)\, x) = (a^n x a^n)\, x = a^n x$. So, the *-subring generated by $a^n x$ is a nonzero finite *-subring of T. This contradiction shows that $a^n = 0$. Therefore the *-subring generated by a is finite and so $a = 0$. Hence any nonzero *-element in T is torsion-free. If b is a nonzero element in T and b is not a *-element, then $0 \neq b^* + b \in T$ and, reasoning as above, we conclude that $b^* + b$ has infinite order. From $\mathbb{Z}(b^* + b) \subseteq \mathbb{Z}b^* + \mathbb{Z}b$ (where $\mathbb{Z}$ denotes the set of integers), it is clear that b must have infinite order. Therefore T is torsion-free.

For any $a \in A$, $a = (a - ea) + ea \in T \oplus eA$. Next we show that either eA is a division ring or the direct sum of two division rings. First, however, we notice that since $pe = 0$, eA has no nonzero nilpotent *-elements, neither does it have nonzero idempotent *-elements other than e.

Let a be a nonzero element in eA. If $aa^* \neq 0$, then also $a^*a \neq 0$. Now there exist *-elements $y, z \in A$ and positive integers k, m such that $(aa^*)^k y (aa^*)^k = (aa^*)^k$ and $(a^*a)^m z (a^*a)^m = (aa^*)^m$. The *-subring generated by the idempotent *-element $(aa^*)^k y$ is finite and hence $(aa^*)^k y = e$. Similarly $z (a^*a)^m = e$. Therefore a is invertible. So, if for every $0 \neq a \in eA$, $aa^* \neq 0$, eA is a division ring. Suppose, now, that there exists an element a in eA such that $aa^* = 0$. Then $a^*a = 0$ and a is not a *-element. Moreover, a cannot be nilpotent, otherwise, $a + a^*$ would be a nonzero nilpotent *-element in eA. Since A is π-regular, there exist $x \in eA$ and a positive integer n such that $a^n x a^n = a^n$. Now $a^n x + (a^n x)^*$ is a nonzero idempotent *-element and hence $a^n x + (a^n x)^* = e$. Let $e_1 = a^n x$. We now assert that $eA = e_1 A \oplus e_1^* A$. Since $e = e_1 + e_1^*$, it is clear that $eA = e_1 A + e_1^* A$. Now let $b \in e_1 A \cap e_1^* A$. Then $b = e_1 c = e_1^* d$, for some $c, d \in A$. Thus $e_1 b = e_1 c = e_1 e_1^* d = e_1 (e - e_1) d = 0$; that is $b = 0$. Finally, we notice that the ideals $e_1 A$ and $e_1^* A$ cannot contain nonzero nilpotent elements, for, otherwise eA would contain nonzero nilpotent *-elements. Similarly, these ideals cannot contain nontrivial idempotent elements. Thus $e_1 A$ and $e_1^* A$ are division rings. Moreover, $e_1^* A \simeq (e_1 A)^{op}$. As was noticed in the proof of Theorem 3, the direct sum of division rings, $e_1 A \oplus e_1^* A$, has exactly one minimal *-subring only if the division rings $e_1 A$ and $e_1^* A$ have characteristic 2. □

Lemma 9. *If A is a nil involution ring with unique minimal *-subring S of order p, then S is an ideal of A.*

Proof. S is a zero ring of order p, for some prime p. Let $S = [s]$ where s is a *-element and let a be an arbitrary nonzero element of A. If $s (a + a^*) s \neq 0$, then $[s (a + a^*) s] = [s]$ and therefore $is (a + a^*) s = s$ for some integer i, $(0 < i < p)$. But then $is (a + a^*)$ is a nonzero idempotent element, which contradicts our assumption. Hence $s (a + a^*) s = 0$ and so $sa^*s = -sas$. Arguing as above, we conclude that $sas = 0$. Finally, we show that $sb = 0$ for any $b \in A$. Suppose that $sb \neq 0$. If sb is a *-element, then $[sb] = [s]$, which implies that $s (jb) = jsb = s$ for some integer j $(0 < j < p)$, which is impossible in a nil ring. If sb is not a *-element, then $[sb + b^*s] = [s]$ and $k (sb + b^*s) = s$ for some integer k $(0 < k < p)$. This implies that $ksb^2 + kb^*sb = sb$; that is $ksb^2 - sb = -kb^*sb$, which is a *-element. So, if $ksb^2 - sb \neq 0$, then $[ksb^2 - sb] = [s]$ and $l (ksb^2 - sb) = s$ for some integer l $(0 < l < p)$; that is, $s (lkb^2 - lb) = s$, which is impossible. Therefore $ksb^2 - sb = 0$ and this implies that $sb (kb) = sb$, which is again impossible. Thus we must have $sb = 0$. Similarly, we can prove that $bs = 0$. Thus $SA = AS = 0$ and S is a *-ideal of A. □

Lemma 10. *Let A be a nilpotent involution p-ring (p prime). If A has a unique minimal *-subring, then the intersection of all nonzero *-subrings of A is nonzero.*

Proof. Let A have a unique minimal *-subring S. Then $S^2 = 0$ and S is an ideal of A of order p. Let S_1 be any nonzero *-subring of A. There exists a nonzero *-element s_1 in S_1, of order p and such that $s_1^2 = 0$. Hence $[s_1] = S$ and so $S \subseteq S_1$. $\square$

In what follows, if S is a *-subring of the involution ring A and p is a prime, then we put $A_S = \{a \in A : pa = 0 = a^2 \text{ and } a \notin S\}$.

Proposition 11. *Let A be a nilpotent involution p-ring ($p \neq 2$ and p prime). The following conditions are equivalent:*

(i) *A has a unique minimal *-subring S;*

(ii) *A is subdirectly irreducible with heart $S \simeq Z(p)$ and either $aa^* \neq 0$ or $a^*a \neq 0$ for each $a \in A_S$.*

Proof. Suppose (i) holds. From the previous lemma, we have that S is contained in every nonzero *-biideal of A. By Theorem 3, A is *-subdirectly irreducible with *-heart $S \cong Z(p)$. Finally, we show that A is, in fact, subdirectly irreducible. Let K be any nonzero ideal of A such that $K \neq K^*$ and let $0 \neq k \in K$ such that $k^2 = 0 = pk$. If $KK^* = 0 = K^*K$, then for $K_1 = [k]$, we have two distinct *-subrings of A of order p, $\{k_1^* + k_1 : k_1 \in K_1\}$ and $\{k_1^* - k_1 : k_1 \in K_1\}$; a contradiction with our assumption. Thus either $KK^* \neq 0$ or $K^*K \neq 0$ and hence either $S \subseteq KK^* \subseteq K$ or $S \subseteq K^*K \subseteq K$. Therefore A is a subdirectly irreducible ring with heart S. Suppose there exists $a \in A_S$ such that $a^*a = aa^* = 0$. For $T = [a]$, we have two distinct minimal *-subrings of A, namely, $\{t + t^* : t \in T\}$ and $\{t - t^* : t \in T\}$, contradicting our assumption.

Suppose (ii) holds and let $S_1 = [s_1]$ be a minimal *-subring of A such that $S_1 \neq S$. Clearly $s_1 \in A_s$ and $s_1^* s_1 = s_1^* s_1 = 0$, contradicting (ii). $\square$

We notice that a nilpotent involution ring A having a unique minimal *-subring of order 2 does not necessarily have a unique minimal subring. In fact, the ring $A = Z(2) \oplus Z(2)$, with the exchange involution, has a unique minimal *-subring of order 2, but three minimal subrings.

As usual, a ring A with identity is called a *local ring* if $A/\mathcal{J}(A)$ is a division ring, where $\mathcal{J}(A)$ denotes the Jacobson radical of A.

Proposition 12. *Let A be a local involution ring of characteristic p^n ($p \neq 2$ a prime and $n \geq 2$) and with nilpotent Jacobson radical. Then the following conditions are equivalent:*

(i) *A has a unique minimal *-subring;*

(ii) *$\mathcal{J}(A)$ is subdirectly irreducible with heart $S \simeq Z(p)$ and $aa^* \neq 0$ or $a^*a \neq 0$, for each $a \in \mathcal{J}(A)_S$.*

Proof. If S is a minimal *-subring of A, then it is clear that $S \subseteq \mathcal{J}(A)$. Notice that if a is a nonzero element which does not belong to $\mathcal{J}(A)$, then a does not generate a subring of order p. In fact, since a is invertible, $pa \neq 0$, for otherwise $paa^{-1} = p1 = 0$, which is a contradiction with the fact that the identity 1 has order p^n. Taking into account the previous proposition, the result is clear. □

It is well-known (see [1], Theorem 3 and Corollary 4) that an involution ring has d.c.c. on *-biideals if and only if it is an artinian ring with artinian Jacobson radical and that the Jacobson radical of such an involution ring is nilpotent. For any prime p, A_p shall denote, as usual, the p-component of the ring A and $T(A) = \{a \in A : na = 0$ for some nonzero integer $n\}$. We are now in a position to investigate the structure of CI-rings with involution and with d.c.c. on *-biideals, which have a unique minimal *-subring.

Theorem 13. *Let A be a CI-ring with involution and with d.c.c. on *-biideals. Then A has a unique minimal *-subring if and only if A is one of the following rings:*

(i) *$A \simeq T \oplus D$ where T is a torsion-free involution ring with identity and D is a division ring with involution of finite characteristic p;*

(ii) *$A \simeq T \oplus (D \oplus D^{op})$ where T is a torsion-free involution ring with identity, D is a division ring of characteristic 2 and $D \oplus D^{op}$ is endowed with the exchange involution;*

(iii) *$A \simeq T \oplus L$ where T is a torsion-free involution ring with identity and L is a nonzero local involution ring of characteristic p^n (p prime and $n \geq 2$) with unique minimal *-subring;*

(iv) *$A \simeq T \oplus (L \oplus L^{op})$ where T is a torsion-free involution ring with identity, each of L and L^{op} is a nonzero local ring of characteristic 2^n ($n \geq 2$) with unique minimal subring and $L \oplus L^{op}$ is endowed with the exchange involution;*

(v) *$A \simeq T \oplus N$ where T is a torsion-free involution ring with identity and N is a nonzero nilpotent involution p-ring (p a prime) having a unique minimal *-subring.*

Proof. We shall first prove the only if part. By hypothesis, there exists a prime p such that $A_p \neq 0$ and A/A_p is torsion free. By ([9], Theorem 5), A is a direct sum of A_p and a torsion-free ring with right identity. Clearly, the right identity is also a left identity. Since A_p is artinian, either A_p has a non-zero idempotent or A_p is nilpotent. First we consider the case when A_p has a nonzero idempotent e. If e is a *-element, then e must be the identity of A_p and A_p is a local ring of characteristic p^n for some integer $n \geq 1$. If $n = 1$, then A_p has the minimal *-subring $S \cong GF(p)$ generated by the identity of A_p. In this case, A satisfies (i), according to Proposition 8. If $n \geq 2$, then A satisfies (iii). If e is not a *-element, then $e + e^*$ is the identity of A_p. Furthermore, $A_p = eA_p \oplus e^*A_p$, where eA_p and e^*A_p are local rings of characteristic p^n for some integer $n \geq 1$. If $n = 1$ and $p = 2$, then (ii) holds. If, on the other hand, $n \geq 2$ and $p = 2$, then (iv) holds. Notice that, if $p \neq 2$ and S is the unique minimal subring of eA_p, then $\{a + a^* : a \in S\}$

and $\{a - a^* : a \in S\}$ are two distinct *-subrings of A_p. If A_p is nilpotent, then (v) holds.

Conversely, since artinian rings are π-regular (see [4]), we have, by Proposition 8, that the involution rings in (i) and (ii) have a unique minimal *-subring. Regarding the involution ring $A = T \oplus (L \oplus L^{op})$ in (iv), if S denotes the unique minimal subring in L, then it is clear that S is a zero-ring on a cyclic group of order 2, $S = S^{op}$ and $\{(0, (a, a)) : a \in S\}$ is the unique minimal *-subring in A. $\square$

Theorem 14. *Let A be an involution ring with unique minimal *-subring S.*

(i) *If $S \simeq GF(p)$ and every idempotent element in A is central, then A decomposes as follows: $A = A' \oplus D$ where A' is an involution ring all of whose nonzero *-subrings are infinite, D has finite characteristic p and any finite *-subring of D is either a field or of the form $F \oplus F^*$ (where F is a field) and contains S.*

(ii) *If $S \simeq Z(p)$ and the prime radical of A_p is nonzero, then S is a *-ideal of A, $T(A)/A_p$ has no nonzero nilpotent *-elements and the prime radical of A_p is a nil p-ring having the unique minimal *-subring S.*

Proof. (i) $S \simeq GF(p)$. Let e be the identity of S. If every idempotent element in A is central, then, by Proposition 7, any non-zero *-subring of $A' = \{x - ex : x \in A\}$ is infinite. By Propositions 6 and 7, any finite *-subring of eA is either a field or of the form $F \oplus F^*$ (where F is a field) and contains S.

(ii) $S \simeq Z(p)$. In this case, S is an ideal of A. To see this, let P denote the prime radical of A_p. As is well-known, P is locally nilpotent and hence any finitely generated *-subring of P is a nilpotent p-ring with unique minimal *-subring. Let $S = [s]$ and consider any $a \in A$. Since P is an ideal of A, we have $sa \in P$. Arguing as in the first part of Lemma 9, we can show that $sa = as = 0$. $\square$

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