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On Some Symplectic Quotients of Schubert Varieties

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Abstract. Let G/P be a generalized flag variety, where G is a complex semisimple connected Lie group and $P \subset G$ a parabolic subgroup. Let also $X \subset G/P$ be a Schubert variety. We consider the canonical embedding of X into a projective space, which is obtained by identifying G/P with a coadjoint orbit of the compact Lie group K, where $G = K^{\mathbb{C}}$. The maximal torus T of K acts linearly on the projective space and it leaves X invariant: let $\Psi : X \to \mathrm{Lie}(T)^*$ be the restriction of the moment map relative to the Fubini-Study symplectic form. By a theorem of Atiyah, $\Psi(X)$ is a convex polytope in $\mathrm{Lie}(T)^*$. In this paper we show that all pre-images $\Psi^{-1}(\mu)$, $\mu \in \Psi(X)$, are connected subspaces of X. We then consider a one-dimensional subtorus $S \subset T$, and the map $f : X \to \mathbb{R}$, which is the restriction of the S moment map to X. We study quotients of the form $f^{-1}(r)/S$, where $r \in \mathbb{R}$. We show that under certain assumptions concerning X, S, and r, these symplectic quotients are (new) examples of spaces for which the Kirwan surjectivity theorem and Tolman and Weitsman's presentation of the kernel of the Kirwan map hold true (combined with a theorem of Goresky, Kottwitz, and MacPherson, these results lead to an explicit description of the cohomology ring of the quotient). The singular Schubert variety in the Grassmannian $G_2(\mathbb{C}^4)$ of 2 planes in $\mathbb{C}^4$ is discussed in detail.
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1. Introduction

Let K be a compact connected semisimple Lie group and $T \subset K$ a maximal torus. We also consider the complexification G of K and $P \subset G$ a parabolic subgroup. In this paper we study Schubert varieties in the flag manifold G/P. More specifically, any such variety X is T invariant and admits canonical T equivariant embeddings into projective spaces with linear T actions: we will be interested in the symplectic quotients of X induced by the action of T. We need to give more details in order to be able to state the results. Let us denote by R the roots of K relative to T and by R^+ the set of all positive roots with respect to a certain choice of a simple root system. Let also $\mathfrak{k}$, $\mathfrak{g}$, $\mathfrak{t}$ denote the Lie algebras of K, G, respectively T. We may assume that the Lie algebra of P is $\mathfrak{t}^{\mathbb{C}} \oplus \bigoplus \mathfrak{g}_\alpha$, where $\alpha \in R^+ \cup (-R^+_P)$. Here $\mathfrak{g}_\alpha$ is the root space of α and R^+_P is a certain subset of R^+ canonically associated to P (cf. [7, Section 23.3]). Let W_P denote the subgroup of W generated by all reflections s_α, with $\alpha \in R^+_P$. The Schubert cells in G/P give a cell decomposition of this space. They are labeled by the quotient W/W_P. Namely, let us consider the Borel subgroup B of G whose Lie algebra is $\mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$. The Schubert cells are the B orbits BwP/P, where $w \in W/W_P$. To any such w corresponds the Schubert variety

$$X(w) = \overline{BwP/P}.$$

In the equation above, the closure is taken relative to the differential topology of G/P: we will see that both G/P and $X(w)$ are (Zariski closed) subvarieties of the same projective space. Such projective embeddings are constructed as follows. Let us first pick a dominant weight $\lambda \in \mathfrak{t}^*$ such that

$$R^+_P = \{\alpha \in R^+ \ : \ \lambda(\alpha^\vee) = 0\}.$$

Here $\{\alpha^\vee \ : \alpha \in R\}$ denotes the root system dual to R. Then we have

$$P = P_\lambda,$$

that is, the Lie subgroup of G whose Lie algebra is $\mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R, \lambda(\alpha^\vee) \geq 0} \mathfrak{g}_\alpha$. Let $\chi_\lambda : P_\lambda \to \mathbb{C}^*$ be the group homomorphism whose differential $d(\chi_\lambda)_e : \mathrm{Lie}(P_\lambda) \to \mathbb{C}$ is the composition of $\lambda \otimes \mathbb{C}$ (regarded as a $\mathbb{C}$-linear function on $\mathfrak{t}^{\mathbb{C}}$) with the natural projection $\mathrm{Lie}(P_\lambda) \to \mathrm{Lie}(T^{\mathbb{C}})$. Consider the line bundle L_λ over G/P_λ whose total space is $G \times \mathbb{C}/P_\lambda$, where $h.(g, z) := (gh^{-1}, \chi_\lambda^{-1}(h)z)$, for $h \in P_\lambda$, $(g, z) \in G \times \mathbb{C}$. One can show that L_λ is very ample. More concretely, one can embed G/P_λ into $\mathbb{P}(V_\lambda)$ as a G orbit, in such a way that L_λ is the restriction to G/P_λ of the hyperplane bundle over $\mathbb{P}(V_\lambda)$. Here $V_\lambda = \Gamma_{hol}(G/P_\lambda, L_\lambda)^*$ is the irreducible representation of K of highest weight λ: this is the content of the Borel-Weil theorem (see e.g. [5, Section 4.12], [17, Chapter V]). The action of B on $\mathbb{P}(V_\lambda)$ is linear, thus the Schubert variety $X(w)$ defined above is a (Zariski closed) subvariety of $\mathbb{P}(V_\lambda)$. Note that $X(w)$ is in general not smooth. The singularities of Schubert varieties have been intensively investigated (see for instance the survey [3]). For instance, by a theorem of Ramanathan [25], $X(w)$ has rational singularities.

We equip $\mathbb{P}(V_\lambda)$ with the Fubini-Study symplectic form. A moment map of the T action is $\Psi : \mathbb{P}(V_\lambda) \to \mathfrak{t}^*$, whose component along $\xi \in \mathfrak{t}$ is given by

$$\Psi_\xi([v]) = \frac{i}{2\pi}\frac{(\xi v)\cdot v}{v \cdot v},$$

for all $v \in V_\lambda \setminus \{0\}$. Here "$\cdot$" is a K-invariant Hermitian inner product on V_λ, which is uniquely determined up to a non-zero factor; the vector $\xi v \in V_\lambda$ arises from the infinitesimal automorphism of V_λ induced by ξ. As already mentioned, we will study symplectic quotients of $X(w)$ relative to $\Psi|_{X(w)}$. Of course this map is the same as the restriction of $\Psi|_{G/P_\lambda}$ to $X(w)$. This observation is useful to us, since $\Psi|_{G/P_\lambda}$ is well understood. Namely, we identify

$$G/P_\lambda = \mathcal{O}_\lambda, \tag{1}$$

where $\mathcal{O}_\lambda := \mathrm{Ad}^*(K)\lambda$ is the coadjoint orbit of λ. More precisely, there is a diffeomorphism $\mathcal{O}_\lambda \to G/P_\lambda$ with the property that the pull-back of the Fubini-Study symplectic form on G/P_λ (see above) is just Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}_\lambda$. We deduce that we have $\Psi|_{G/P_\lambda} = \Phi$, where Φ is given by

$$\Phi : \mathcal{O}_\lambda \hookrightarrow \mathfrak{k}^* \to \mathfrak{t}^*,$$

which is the composition of the inclusion map with the restriction map. The details of this discussion can be found for instance in [17, Chapter V].

The identification (1) leads us to the following approach: the coadjoint orbit $\mathcal{O}_\lambda$ admits an action of G and the Schubert cells are the orbits of the induced B action. To be more precise, we consider the natural action of the Weyl group $W = N_K(T)/T$ on $\mathfrak{t}^*$ and the orbit $W\lambda = W/W_\lambda$, which is contained in $\mathcal{O}_\lambda$. Then the Schubert cells are $Bw\lambda$ and the Schubert varieties are

$$X(w) = \overline{Bw\lambda},$$

for all $w \in W$ (the closure is relative to the standard topology of $\mathcal{O}_\lambda$). Thus, what we actually study in this paper are symplectic quotients of $X(w)$ relative to the T moment map $\Phi : \mathcal{O}_\lambda \to \mathfrak{t}^*$.

We also consider a circle $S \subset T$ and the moment map of the S action on $\mathcal{O}_\lambda$: this is just the composition of Φ with the restriction map $\mathfrak{t}^* \to \mathrm{Lie}(S)^*$. More precisely, let us consider $a \in \mathfrak{t}$ such that $S = \exp(\mathbb{R}a)$. In fact, a is an element of the integral lattice of T, that is, $\ker(\exp : \mathfrak{t} \to T)$. We can also assume that a is not an integer multiple of any other integral element. Denote by ν the element of the dual space $(\mathbb{R}a)^*$ determined by $\nu(a) = 1$. Then the S moment map of $\mathcal{O}_\lambda$ is $\Phi_a\nu : \mathcal{O}_\lambda \to (\mathbb{R}a)^* = \mathbb{R}\nu$, where Φ_a denotes the evaluation of Φ at a.

Let us fix a Schubert variety $X = X(w)$, where $w \in W$. We study the topology of the quotients

$$(\Phi^{-1}(\mu) \cap X)/T \quad \text{and} \quad (\Phi_a^{-1}(r_0) \cap X)/S,$$

where $\mu \in \Phi(X)$ and $r_0 \in \Phi_a(X)$. In the case where X is non-singular, these spaces are symplectic quotients of X. If X does have singularities, we can choose

μ and r_0 such that $\Phi^{-1}(\mu)$ and $\Phi_a^{-1}(r_0)$ are contained in $X \setminus \mathrm{Sing}(X)$. The two quotients are symplectic quotients of the latter space, which is a (non-compact) Kähler manifold. The assumption above will *not* be in force everywhere in this paper.

Remark. Such quotients could be relevant for the study of the Demazure (B)-module $V_w(\lambda)$ (cf. [4]). Guillemin and Sternberg used "quantization commutes with reduction" to prove the following result: if a weight vector μ occurs among the weights of the K representation V_λ (irreducible of highest weight λ, see above), then μ is contained in $\Phi(\mathcal{O}_\lambda)$; moreover, the multiplicity of μ is equal to the dimension of the space of sections of the line bundle induced by L_λ on the symplectic quotient $\Phi^{-1}(\mu)/T$ (cf. [11], [28]). Now the Demazure module $V_w(\lambda)$ is the dual of the space of sections $H^0(X, L_\lambda|_X)$ with the canonical B action (see [4, Corollary 3.3.11]). We can expect again that if the weight vector μ is a weight of $V_w(\lambda)$, then μ is in $\Phi(X)$ (see also Remark 1 below) and the multiplicity of μ is equal to the dimension of the space of sections of the line bundle induced by L_λ on $(\Phi^{-1}(\mu) \cap X)/T$. This is certainly not an obvious result. First, because X is not smooth: however, we could use Teleman's [29] "quantization commutes with reduction" theorem, which holds for linear group actions on projective varieties which have rational singularities (Schubert varieties do have this property). Second, the Borel subgroup B is not reductive: Ion [18] was able to overcome this and obtain geometric formulas for the multiplicities of the weights of $V_w(\lambda)$ by extending methods which had been used previously by Mirković and Vilonen in the context of representations of reductive Lie groups. We will not explore such phenomena in this paper.

Our first result states (or rather implies) that the quotients defined above are connected. Note that the result holds without any assumption on μ or r_0.

Theorem 1.1. *All preimages $\Phi^{-1}(\mu) \cap X$, $\mu \in \Phi(X)$, and $\Phi_a^{-1}(r_0) \cap X$, $r_0 \in \Phi_a(X)$, are connected.*

Remark 1. 1. This theorem is related to the convexity theorem for Hamiltonian torus actions on symplectic manifolds of Atiyah and Guillemin-Sternberg. Namely, Atiyah's proof of the latter result uses the fact that all preimages of the moment map are connected (cf. [1], see also [22, Section 5.5]). In the same paper [1], he shows that if X and Φ are as above, then $\Phi(X)$ is the convex hull of the set $\Phi(X^T)$ in $\mathfrak{t}^*$. Since X is in general not smooth, the argument involving the connectivity of the preimages cannot be used: instead, Atiyah uses a convexity result for closures of $T^{\mathbb{C}}$ orbits on Kähler manifolds. It would be interesting to find a proof of the convexity of $\Phi(X)$ which uses Theorem 1.1.

Remark 2. Here is a simpler proof of the connectivity of $\Phi^{-1}(\mu)$, under the assumption that μ is a weight vector in $\mathfrak{t}^*$. Let us consider the character $\chi_\mu : T^{\mathbb{C}} \to \mathbb{C}^*$ induced by μ and the twisted action of $T^{\mathbb{C}}$ on the line bundle L_λ (see above), which is defined as follows:

$$t.[(g,z)] := [(gt^{-1}, \chi_\mu(t)^{-1}z)],$$

for all $t \in T^{\mathbb{C}}, g \in G$, and $z \in \mathbb{C}$. Let $X^{ss}(L_\lambda)/T^{\mathbb{C}}$ be the corresponding Geometric Invariant Theory (shortly GIT) quotient (here $X^{ss}(L_\lambda)$ is the space of semistable points of the pair (X, L_λ)). It is homeomorphic to the symplectic quotient $(\Phi^{-1}(\mu) \cap X)/T$, by a theorem of Kirwan and Ness (see e.g. [15, Section 1] and [16, Section 2]). The GIT quotient is connected, because $X^{ss}(L_\lambda)$ is connected (in turn, this follows from the fact that X is an irreducible projective variety and $X^{ss}(L_\lambda)$ is Zariski open in X). Thus $(\Phi^{-1}(\mu) \cap X)/T$ is connected. Consequently, $\Phi^{-1}(\mu) \cap X$ is connected as well. Note that this proof works only in the case where μ is a weight vector. We wanted to mention it because the proof of Theorem 1.1 (see Section 2 below) does use these ingredients, together with some others, most importantly a theorem of Heinzner and Migliorini [14]. Everything we said here remains true if we replace T by S and μ by an integer number n.

The cohomology of GIT quotients of *smooth* projective varieties has been extensively investigated during the past two decades, starting with the seminal work of Kirwan [19]. By contrary, little seems to be known in this respect about quotients of *singular* varieties equipped with algebraic group actions. Except the results of [20] (where the *intersection* cohomology is discussed), we are not aware of any other approaches concerning this topic. The next two theorems give a description of the *ordinary* (i.e. singular) cohomology[1] ring of the symplectic quotients

$$X//_\lambda S(r_0) := (\Phi_a^{-1}(r_0) \cap X)/S.$$

We can only do that under certain restrictions on a, r_0, λ, and w.

Assumption 1. (concerning a) The vector $-a$ is in the (interior of the) fundamental Weyl chamber of $\mathfrak{t}$. The numbers $\Phi_a(v\lambda)$, $v\lambda \in W\lambda$, are any two distinct.

One consequence of this is that the fixed point set $\mathcal{O}_\lambda^S$ is given by

$$\mathcal{O}_\lambda^S = \mathcal{O}_\lambda^T = W\lambda.$$

Thus, for our Schubert variety $X = X(w)$ we have

$$X^S = X^T = W\lambda \cap X.$$

Assumption 1 also implies that the unstable manifolds of Φ_a relative to the Kähler metric on $\mathcal{O}_\lambda$ are just the Bruhat cells (cf. e.g. [6]). This will allow us to use Morse theory for the restriction of Φ_a to $X \setminus \mathrm{Sing}(X)$, which is one of the main tools we will be employing in our proofs: for instance, we will show that the critical points of this function are in $W\lambda$ (see Lemma 3.1).

An important instrument will be the Kirwan map

$$\kappa : H_S^*(X) \to H_S^*(\Phi_a^{-1}(r_0) \cap X).$$

[1]All cohomology rings will be with coefficients in $\mathbb{R}$.

The domain of this map can be described as follows. We start with a Goresky-Kottwitz-MacPherson [8] type presentation of the ring $H^*_T(\mathcal{O}_\lambda)$ (cf. [8]): the map

$$H^*_T(\mathcal{O}_\lambda) \to H^*_T(\mathcal{O}_\lambda^T) = \bigoplus_{v\lambda\in W\lambda} H^*_T(\mathrm{pt})$$

induced by the inclusion $\mathcal{O}_\lambda^T \subset \mathcal{O}_\lambda$ is injective and we know exactly its image (see for instance [9, Section 2.3] or the discussion preceding Lemma 3.4 below). The action of T on X is equivariantly formal, by [8, Section 1.2]: indeed, any of the Bruhat cells in the CW decomposition of X is T invariant. Since X is a closed $T^{\mathbb{C}}$ invariant subvariety of $\mathcal{O}_\lambda$, the GKM description of $H^*_T(\mathcal{O}_\lambda)$ yields readily a similar description of the ring $H^*_T(X)$, as the image of the (injective) map

$$\jmath^* : H^*_T(X) \to H^*_T(X^S) = \bigoplus_{v\lambda\in X^S} H^*_T(\mathrm{pt}) \tag{2}$$

induced by the inclusion $\jmath : X^S \to X$. The image of $\jmath^*$ is described in terms of the moment graph of X. The vertices of this graph are the elements of the set $X^S = W\lambda \cap X$. If $\gamma \in \Phi^+$ and $v \in W$ such that $v\lambda \neq s_\gamma v\lambda$ and both $v\lambda$ and $s_\gamma v\lambda$ are in X, then we join the vertices $v\lambda$ and $s_\gamma v\lambda$ by an edge, which is labeled with γ. Denote by Γ the resulting graph. The image of $\jmath^*$ consists of all ordered sets $(p_{v\lambda})_{v\lambda\in X^S}$ where $p_{v\lambda} \in S(\mathfrak{t}^*)$, which are *admissible* relative to Γ: by this we mean that if $v\lambda$ and $u\lambda$ are joined by an edge with label γ, then the difference $p_{v\lambda} - p_{u\lambda}$ is divisible by γ (cf. e.g. [9], see also the discussion preceding Lemma 3.4 below). Now because the action of T on X is equivariantly formal, the map $H^*_T(X) \to H^*_S(X)$ is surjective. The image of the injective map

$$H^*_S(X) \to H^*_S(X^S) = \bigoplus_{v\lambda\in X^S} H^*_S(\mathrm{pt})$$

is obtained from the image of the map given by (2) by projecting it via the canonical map

$$H^*_T(\mathrm{pt}) = \mathrm{Symm}(\mathfrak{t}^*) \to \mathrm{Symm}((\mathbb{R}a)^*) = H^*_S(\mathrm{pt}).$$

If X were smooth and r_0 a regular value of $\Phi_a|_X$, we could determine the ring $H^*(X//_\lambda S(r_0))$ as follows: use that the action of S on $\Phi_a^{-1}(r_0)$ is locally free, which implies that we have the ring isomorphism $H^*_S(\Phi_a^{-1}(r_0) \cap X) \simeq H^*(X//_\lambda S(r_0))$; then use that κ is surjective (cf. Kirwan [19]), and the Tolman-Weitsman [27] description of $\ker\kappa$. In the second part of this paper we give examples of non-smooth Schubert varieties for which this program still works. Namely, they must satisfy the following assumptions.

Assumption 2. (concerning X) The singular set of X consists of one single point, that is, we have

$$\mathrm{Sing}(X) = \{\lambda\}.$$

For example, let us consider the Grassmannian $G_2(\mathbb{C}^n)$ of 2-planes in $\mathbb{C}^n$. Take p an integer such that $4 \leq p \leq n$. The Schubert variety

$$X = \{V \in G_2(\mathbb{C}^n) \ : \ \dim(V \cap \mathbb{C}^2) \geq 1, \dim(V \cap \mathbb{C}^p) \geq 2\}$$

has one singular point (to prove this, we use [3, Theorem 9.3.1]). The case $n = p = 4$ will be discussed in detail in the last section of the paper.

We will also need an assumption concerning r_0. This is expressed in terms of the moment graph of X (see above). We note that, by Assumption 1, the points λ and $w\lambda$ are the global minimum point, respectively the global maximum point of $\Phi_a|_X$. Let us remove from the graph Γ all vertices $v\lambda$ with $\Phi_a(v\lambda) < r_0$, as well as any edge with at least one endpoint at such a vertex. Denote by Γ_{r_0} the resulting graph.

Assumption 3. (concerning r_0)

(i) The number r_0 is in $\Phi_a(X) \setminus \Phi_a(X^S)$.

(ii) If the ordered set $(p'_{u\lambda})_{u\lambda \in X^S, \Phi_a(u\lambda) > r_0}$ is admissible relative to Γ_{r_0}, then there exists an ordered set $(p_{v\lambda})_{v\lambda \in X^S}$ which is admissible relative to Γ and $p_{u\lambda} = p'_{u\lambda}$ whenever $\Phi_a(u\lambda) > r_0$.

The second point of this assumption seems to be hard to verify. We can always find r_0 for which this condition is satisfied. This happens for instance when r_0 is "high enough", such that Γ_{r_0} consists of only one point, that is, $\Gamma_{r_0} = \{w\lambda\}$: an extension $(p_{v\lambda})_{v\lambda \in X^S}$ of the polynomial $p'_{w\lambda}$ is given by $p_{v\lambda} = p'_{w\lambda}$ for all $v\lambda \in W\lambda$. However, we will see in the last section an example where there exist numbers r_0 such that Γ_{r_0} has more than one vertex and Assumption 3 (ii) is satisfied.

These assumptions will allow us to prove the Kirwan surjectivity theorem for κ.

Theorem 1.2. (Kirwan surjectivity) *If Assumptions* 1, 2, *and* 3 *are satisfied, then the map* κ *is surjective.*

To determine the cohomology of our quotient, we first notice that, by Assumption 3, point (i), r_0 is a regular value of the map Φ_a restricted to $X \setminus \{\lambda\}$ (see also Lemma 3.1 below). This space is a Kähler S invariant submanifold of $\mathcal{O}_\lambda$ and $\Phi_a|_{X\setminus\{\lambda\}}$ is a moment map. Thus the action of S on the level $\Phi_a^{-1}(r_0) \cap X$ is locally free and we have

$$H^*_S(\Phi_a^{-1}(r_0) \cap X) \simeq H^*(X//_\lambda S(r_0)).$$

We also deduce that our quotient $X//_\lambda S(r_0)$ has at most orbifold singularities. A complete description of the ring $H^*(X//_\lambda S(r_0))$ will be obtained after finding the kernel of κ. This is done by the following theorem.

Theorem 1.3. (The Tolman-Weitsman kernel) *If Assumptions* 1, 2, *and* 3 *are satisfied, then the kernel of* κ *is equal to* $K_- + K_+$. *Here* K_- *consists of all* $\alpha \in H^*_S(X)$ *such that*

$$\alpha|_{v\lambda} = 0, \ \textit{for all} \ v\lambda \in X^S \ \textit{with} \ \Phi_a(v\lambda) < r_0,$$

and K_+ *is defined similarly (the last condition is* $\Phi_a(v\lambda) > r_0$*).*

Here, by $\alpha|_{v\lambda}$ we have denoted the image of α under the map $H^*_S(X) \to H^*_S(\{v\lambda\})$ induced by the inclusion $\{v\lambda\} \to X$.

Remark. The space $X//_\lambda S(r_0)$ is a symplectic quotient of the Kähler manifold $X \setminus \{\lambda\}$ with respect to the S action. However, Theorems 1.2 and 1.3 are not direct consequences of the classical results known in this context. For example, the surjectivity of κ is not a consequence of Kirwan's surjectivity theorem (cf. [19], [27]): indeed, the restriction of the moment map Φ_a to $X \setminus \{\lambda\}$ is not a proper map (because it is bounded, as $\Phi_a(\mathcal{O}_\lambda)$ is a bounded subset of $\mathbb{R}$). Thus we cannot use Morse theory for $(\Phi_a - r_0)^2$ on $X \setminus \{\lambda\}$. We use instead the restriction of the function Φ_a and its properties, like the fact that its critical set is $X^S \setminus \{\lambda\}$ and its unstable manifolds are Bruhat cells (see Section 3 below). To understand exactly why is each of the three assumptions necessary for our development, one can see the remark at the end of Section 3.

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2. Connectivity of the levels of Φ

In this section we will be concerned with Theorem 1.1. We mention that a similar connectivity result (for Schubert varieties in loop groups) has been proved in [13, Section 3], by using essentially the same arguments as here.

We start with some general considerations. Let Y be a Kähler manifold acted on holomorphically by a compact torus T. Assume that T preserves the Kähler structure and the action of T is Hamiltonian. Let $\Psi : Y \to \mathrm{Lie}(T)^*$ be any moment map. A point $y \in Y$ is called Ψ-semistable if the intersection $\overline{T^{\mathbb{C}}y} \cap \Psi^{-1}(0)$ is non-empty, where $T^{\mathbb{C}}$ is the complexification of T and $\overline{T^{\mathbb{C}}y}$ is the closure of the orbit of y. We denote by $Y^{ss}(\Psi)$ the space of all Ψ-semistable points of Y. We also choose an inner product on $\mathfrak{t}$, denote by $\|\cdot\|$ the corresponding norm, and consider the function $\|\Psi\|^2 : Y \to \mathbb{R}$. We denote by $Y^{\min}(\|\Psi\|^2)$ the minimum stratum of $\|\Psi\|^2$, that is, the space of all points $y \in Y$ with the property that the ω-limit of the integral curve through y of the negative gradient vector field $-\mathrm{grad}(\|\Psi\|^2)$ is contained in $\Psi^{-1}(0)$. The following result is a direct consequence of [19, Theorem 6.18].

Theorem 2.1. (Kirwan) *We have*

$$Y^{ss}(\Psi) = Y^{\min}(\|\Psi\|^2).$$

There is another version of the notion of semistability, which is defined as follows. Let us assume that Y is a smooth projective variety. We endow Y with the Kähler structure induced by its projective embedding. Let L be a $T^{\mathbb{C}}$ equivariant ample line bundle on Y. A point $y \in Y$ is called L-semistable if there exists an integer number $n \geq 1$ and a $T^{\mathbb{C}}$ equivariant section s of $L^{\otimes n}$ such that $s(y) \neq 0$. We denote by $Y^{ss}(L)$ the set of all L-semistable points in Y. The following theorem has been proved by Heinzner and Migliorini [14].

Theorem 2.2. (Heinzner and Migliorini) *If Y is a smooth projective variety with a holomorphic action of T which preserves the Kähler form and $\Psi : Y \to \mathfrak{t}^*$ is a moment map, then there exists a very ample $T^{\mathbb{C}}$ equivariant line bundle L on Y such that*

$$Y^{ss}(\Psi) = Y^{ss}(L).$$

The two theorems above will allow us to prove Theorem 1.1, as follows.

Proof of Theorem 1.1. We take $\mu \in \Phi(X)$ and show t hat $\Phi^{-1}(\mu) \cap X$ is connected. To this end, we consider the function $g : \mathcal{O}_\lambda \to \mathbb{R}$, $g(x) = \|\Phi(x) - \mu\|^2$. By Theorem 2.1, we have

$$\mathcal{O}_\lambda^{\min}(g) = \mathcal{O}_\lambda^{ss}(\Phi - \mu).$$

Since the action of T on $\mathbb{P}(V_\lambda)$ is linear, it is holomorphic and it leaves the Fubini-Study symplectic form invariant. By Theorem 2.2, there exists a $T^{\mathbb{C}}$-equivariant very ample line bundle L on $\mathcal{O}_\lambda$ such that

$$\mathcal{O}_\lambda^{ss}(\Phi - \mu) = \mathcal{O}_\lambda^{ss}(L).$$

The semistable set of $X \subset \mathcal{O}_\lambda$ with respect to the line bundle $L|_X$ is

$$X^{ss}(L|_X) = \mathcal{O}_\lambda^{ss}(L) \cap X.$$

This is a Zariski open subspace of X. The Schubert variety $X = \overline{Bw\lambda}$ is irreducible: indeed, since B is connected, the orbit $Bw\lambda$ is an irreducible locally closed projective variety. Consequently, $X^{ss}(L|_X)$ is a connected topological subspace of X relative to the differential topology (by [24, Corollary 4.16]). We will need the following claim.

Claim. The space $X^{ss}(L|_X) = \mathcal{O}_\lambda^{\min}(g) \cap X$ contains $g^{-1}(0) \cap X$ as a deformation retract.

To prove this, let us first consider the flow σ_t, $t \in \mathbb{R}$, on $\mathcal{O}_\lambda$ induced by the vector field $-\mathrm{grad} g$ (the gradient is relative to the Kähler metric on $\mathcal{O}_\lambda$). By a theorem of Duistermaat (see for instance [21, Theorem 1.1]), the map

$$[0, \infty] \times \mathcal{O}_\lambda^{\min}(g) \to \mathcal{O}_\lambda^{\min}(g), \ (t, x) \mapsto \sigma_t(x)$$

is a deformation retract of $\mathcal{O}_\lambda^{\min}(g)$ to $g^{-1}(0)$. The claim follows from the fact that for any $t \in [0, \infty)$, the automorphism σ_t of $\mathcal{O}_\lambda$ leaves $Bw\lambda$, hence also $X = X(w) = \overline{Bw\lambda}$, invariant. Indeed, for any $x \in Bw\lambda$, we have

$$(\mathrm{grad} g)_x = 2J_x((\Phi(x) - \mu).x), \tag{3}$$

where J_x denotes the complex structure of $\mathcal{O}_\lambda$ at x. In the equation above, we use the inner product on $\mathfrak{t}$ to identify $\Phi(x) - \mu$ with an element of $\mathfrak{t}$; this induces the infinitesimal tangent vector $(\Phi(x) - \mu).x$ at x (cf. [19, Lemma 6.6.]). Equation (3) implies that the vector $(\mathrm{grad} g)_x$ is tangent to $Bw\lambda$, as this space is a complex $T^{\mathbb{C}}$ invariant submanifold of $\mathcal{O}_\lambda$. The claim is proved.

The claim implies that the space $g^{-1}(0) \cap X = \Phi^{-1}(\mu) \cap X$ is connected.

The fact that $\Phi_a^{-1}(r_0) \cap X$ is connected can be proved in a similar way. $\square$

3. Kirwan surjectivity and the Tolman-Weitsman kernel via Morse theory

Assumptions 1, 2, and 3 are in force throughout this section. We will prove Theorems 1.2 and 1.3. To this end, let us consider the function

$$f := \Phi_a|_X : X \to \mathbb{R}.$$

The main instrument of our proofs will be Morse theory for the function f restricted to $X \setminus \{\lambda\}$. This space is a smooth, non-compact submanifold of the orbit $\mathcal{O}_\lambda$. Its tangent space at any of its points is a complex vector subspace of the tangent space to $\mathcal{O}_\lambda = G/P_\lambda$ at that point (since X is a complex subvariety of G/P_λ). Thus the restriction of the canonical Kähler structure of $\mathcal{O}_\lambda$ (cf. e.g. [1, Section 4]) to $X \setminus \{\lambda\}$ makes the latter space into a Kähler manifold. We denote by $\langle\ ,\ \rangle$ the corresponding Riemannian metric. We start with the following lemma, which is a consequence of Assumption 1.

Lemma 3.1. *The critical set of f restricted to $X \setminus \{\lambda\}$ is $W\lambda \cap X \setminus \{\lambda\}$. All critical points are non-degenerate.*

Proof. Let $\psi_t : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$, $t \in \mathbb{R}$, denote the flow on $\mathcal{O}_\lambda$ determined by the gradient vector field $\mathrm{grad}\Phi_a$ with respect to the Kähler metric. The fixed points of this flow are the critical points of Φ_a, that is, the elements of $W\lambda$. If $v\lambda$ is such a point, we consider the unstable manifold $\{x \in \mathcal{O}_\lambda \mid \lim_{t\to\infty} \psi_t(x) = v\lambda\}$. This is the same as the Bruhat cell $Bv\lambda$ (see [1, Section 4]). Consequently, for any $t \in \mathbb{R}$, the automorphism ψ_t of $\mathcal{O}_\lambda$ leaves each Bruhat cell invariant. Thus it leaves $X \setminus \{\lambda\}$ invariant (since this space is a union of Bruhat cells). We deduce that the vector field $\mathrm{grad}\Phi_a$ is tangent to $X \setminus \{\lambda\}$ at any of its points. Its value at any such point x must be the same as $(\mathrm{grad} f)_x$. In conclusion, the critical points mentioned in the lemma are those points $x \in X \setminus \{\lambda\}$ with the property that $(\mathrm{grad}\Phi_a)_x = 0$. This condition is equivalent to $x \in W\lambda$.

The last assertion in the lemma follows from the fact that f is a moment map of the S action on $X \setminus \{\lambda\}$. Thus, it is a Morse function (cf. e.g. [19, p. 39]). □

For any number r we denote

$$X_r^- := f^{-1}((-\infty, r)), \quad X_r^+ := f^{-1}((r, \infty)).$$

Since the function Φ_a and the subspace X of $\mathcal{O}_\lambda$ are S invariant, X^- and X^+ are S invariant subspaces of X. If $\alpha \in H_S^*(X)$ and A is an S invariant subspace of X, we denote by $\alpha|_A$ the image of α under the map $H_S^*(X) \to H_S^*(A)$ induced by the inclusion $A \to X$. We are now ready to state our next lemma.

Lemma 3.2. *Take $\epsilon > 0$ such that $f(\lambda) < r_0 - \epsilon$ and the intersection $f^{-1}([r_0 - \epsilon, r_0 + \epsilon]) \cap X^S$ is empty. Then we have*

$$\ker\kappa = K'_- + K'_+ \tag{4}$$

where we have denoted

$$K'_- = \{\alpha \in H^*_S(X) \ : \ \alpha|_{X^-_{r_0+\epsilon/3}} = 0\},$$

and

$$K'_+ = \{\alpha \in H^*_S(X) \ : \ \alpha|_{X^+_{r_0-\epsilon/3}} = 0\}.$$

Proof. To simplify notations, put $X^- := X^-_{r_0+\frac{\epsilon}{3}}$, $X^+ := X^+_{r_0-\frac{\epsilon}{3}}$. Both K'_- and K'_+, hence also their sum, are evidently contained in $\ker\kappa$. We show that $\ker\kappa \subset K'_- + K'_+$. To this end, we need the following claim.

Claim. The space $X^- \cap X^+ = f^{-1}((r_0 - \frac{\epsilon}{3}, r_0 + \frac{\epsilon}{3}))$ contains $f^{-1}(r_0)$ as an S-equivariant deformation retract.

The idea of the proof is to deform $f^{-1}((r_0 - \frac{\epsilon}{3}, r_0 + \frac{\epsilon}{3}))$ onto $f^{-1}(r_0)$ in $X \setminus \{\lambda\}$ along the gradient lines of the function $f|_{X\setminus\{\lambda\}}$. This is possible since the preimage $f^{-1}([r_0-\epsilon, r_0+\epsilon]) \cap X \setminus \{\lambda\} = f^{-1}([r_0-\epsilon, r_0+\epsilon])$ is compact and does not contain any critical points of f (by Lemma 3.1). The arguments we will employ in what follows are standard (see for instance [23, Proof of Theorem 3.1]). Here are the details of the construction. We start with a smooth function $F : \mathbb{R} \to \mathbb{R}$ such that:

- $F(s) = 1$ for $s \in (r_0 - \frac{2\epsilon}{3}, r_0 + \frac{2\epsilon}{3})$
- $F(s) = 0$ for s outside the interval $[r_0 - \epsilon, r_0 + \epsilon]$.

We then consider the function $\rho : X \setminus \{\lambda\} \to \mathbb{R}$ given by $\rho(x) = \frac{F(f(x))}{\|(\mathrm{grad} f)_x\|^2}$, for all $x \in X \setminus \{\lambda\}$ (here $\|\cdot\|$ is the norm induced by the Kähler metric). The vector field $\rho\mathrm{grad} f$ on $X \setminus \{\lambda\}$ vanishes outside the compact set $f^{-1}([r_0 - \epsilon, r_0 + \epsilon])$. By [23, Lemma 2.4], it generates a flow ϕ_t, $t \in \mathbb{R}$, on $X \setminus \{\lambda\}$. For any $x \in X \setminus \{\lambda\}$ and any $t \in \mathbb{R}$ we have

$$\frac{d}{dt} f(\phi_t(x)) = \langle (\mathrm{grad} f)_{\phi_t(x)}, \rho(\phi_t(x))(\mathrm{grad} f)_{\phi_t(x)} \rangle = F(f(\phi_t(x))).$$

Assume that $x \in f^{-1}((r_0 - \frac{\epsilon}{3}, r_0 + \frac{\epsilon}{3}))$. Then we have

$$f(\phi_t(x)) = t + f(x)$$

for all $t \in (-\frac{\epsilon}{3}, \frac{\epsilon}{3})$ (the reason is that both sides of the equation represent solutions of the same initial value problem). Then

$$R_\tau(x) := \phi_{\tau(r_0 - f(x))}(x),$$

$\tau \in [0, 1]$, $x \in f^{-1}((r_0 - \frac{\epsilon}{3}, r_0 + \frac{\epsilon}{3}))$, defines a deformation retract of $f^{-1}((r_0 - \frac{\epsilon}{3}, r_0 + \frac{\epsilon}{3}))$ onto $f^{-1}(r_0)$. It only remains to show that for any $\tau \in [0, 1]$, the map R_τ from $f^{-1}((r_0 - \frac{\epsilon}{3}, r_0 + \frac{\epsilon}{3}))$ to itself is S-equivariant. This follows from the fact that for any $t \in \mathbb{R}$, the automorphism ϕ_t of $X \setminus \{\lambda\}$ is S-equivariant. Indeed, the function $f : X \setminus \{\lambda\} \to \mathbb{R}$ is S-invariant; since S acts isometrically on $\mathcal{O}_\lambda$, the vector field $\mathrm{grad} f$, hence also $\rho\mathrm{grad} f$, is S equivariant. The claim is now completely proved.

The claim implies that the pair $(X^- \cap X^+, f^{-1}(r_0))$ is S-equivariantly homotopy equivalent to $(f^{-1}(r_0), f^{-1}(r_0))$. Hence we have $H_S^*(X^- \cap X^+, f^{-1}(r_0)) = \{0\}$. From the long exact sequence of the triple $(X, X^- \cap X^+, f^{-1}(r_0))$ we deduce that the canonical map

$$\psi : H_S^*(X, X^- \cap X^+) \to H_S^*(X, f^{-1}(r_0))$$

is an isomorphism.

Let us now focus on the proof of the inclusion $\ker \kappa \subset K'_- + K'_+$. Take $\alpha \in \ker \kappa$, that is, $\alpha \in H_S^*(X)$ such that $\alpha|_{f^{-1}(r_0)} = 0$. From the long exact sequence of the pair $(X, f^{-1}(r_0))$ we deduce that there exists $\beta \in H_S^*(X, f^{-1}(r_0))$ whose image via $H_S^*(X, f^{-1}(r_0)) \to H_S^*(X)$ is α. We set

$$\eta = \psi^{-1}(\beta).$$

We use the relative Mayer-Vietoris sequence of the triple (X, X^-, X^+). Let $i_-^* : H_S^*(X, X^-) \to H_S^*(X, X^- \cap X^+)$ and $i_+^* : H_S^*(X, X^+) \to H_S^*(X, X^- \cap X^+)$ be the maps induced by the obvious inclusions. Because $X^- \cup X^+ = X$, the exactness of the Mayer-Vietoris sequence implies that the map

$$H_S^*(X, X^-) \oplus H_S^*(X, X^+) \to H_S^*(X, X^- \cap X^+)$$

defined by

$$(\eta_1, \eta_2) \mapsto i_-^*(\eta_1) - i_+^*(\eta_2)$$

is an isomorphism. This map is in the top of the following commutative diagram.

$$\begin{tikzcd} H_S^*(X, X^-) \oplus H_S^*(X, X^+) \arrow[r] \arrow[d] & H_S^*(X, X^- \cap X^+) \arrow[d] \\ H_S^*(X) \oplus H_S^*(X) \arrow[r] & H_S^*(X) \end{tikzcd} \tag{5}$$

There exists $(\eta_1, \eta_2) \in H_S^*(X, X^-) \oplus H_S^*(X, X^+)$ such that

$$i_-^*(\eta_1) - i_+^*(\eta_2) = \eta.$$

The image of η via the right-hand side map in the diagram is α. Let (α_1, α_2) be the image of (η_1, η_2) via the left-hand side map in the diagram. The classes α_1, α_2 have the property that $\alpha_1|_{X^-} = 0$ and $\alpha_2|_{X^+} = 0$. From the commutativity of the diagram we have

$$\alpha = \alpha_1 - \alpha_2.$$

This finishes the proof. □

Remark. In the general context of circle actions on compact symplectic manifolds, Tolman and Weitsman gave a description of the kernel of the Kirwan map similar to equation (4) above (see [27, Theorem 1]). Their proof is different from the one above. However, they do mention that their theorem can be proved by using the Mayer-Vietoris sequence of the triple (X, X^-, X^+) (see [27, Remark 3.5]). We have used this idea to prove Lemma 3.2 above.

We will characterize K'_+ and K'_- separately. In the next lemma we describe K'_+.

Lemma 3.3. (i) *For any $r_2 > r_1 > f(\lambda)$, the space $f^{-1}([r_1, r_2]) \cap (X \setminus \{\lambda\})$ is compact.*

(ii) *Take $r \in f(X)$, $r > f(\lambda)$, $r \notin f(X^S)$. Then the restriction map*

$$H^*_S(X^+_r) \to H^*_S(X^+_r \cap X^S)$$

is injective and the canonical map

$$H^*_T(X^+_r) \to H^*_S(X^+_r)$$

is surjective.

(iii) *We have $K'_+ = K_+$.*

(iv) *The restriction map*

$$\kappa_1 : H^*_S(X^+_{r_0 - \frac{\epsilon}{3}}) \to H^*_S(f^{-1}(r_0))$$

is surjective.

Proof. Point (i) follows from the fact that $f^{-1}([r_1, r_2])$ is contained in $X \setminus \{\lambda\}$, thus

$$f^{-1}([r_1, r_2]) \cap (X \setminus \{\lambda\}) = f^{-1}([r_1, r_2]).$$

The latter space is compact, because it is closed in X.

(ii) We use the Morse theoretical arguments of [26] and [27] (see also [12, Section 2]) for the function $f|_{X \setminus \{\lambda\}}$. More precisely, let us first note that this function has a maximum at the point $w\lambda$ (where w is given by $X = X(w)$): this follows from the fact that λ and $w\lambda$ are the minimum, respectively maximum points of f on X (cf. [6, Section 4]). To prove the first assertion it is sufficient to take $r_1, r_2 \in \mathbb{R} \setminus f(X^S)$ such that $r < r_1 < r_2$ and note that:

- if $f^{-1}([r_1, r_2]) \cap X^S = \{w\lambda\}$ then $H^*_S(X^+_{r_1}) \to H^*_S(\{w\lambda\})$ is injective (since $\{w\lambda\}$ is an S-equivariant deformation retract of $X^+_{r_1}$).
- if $f^{-1}([r_1, r_2]) \cap X^S$ is empty, then the map $H^*_S(X^+_{r_1}) \to H^*_S(X^+_{r_2})$ induced by the inclusion $X^+_{r_2} \to X^+_{r_1}$ is an isomorphism (since $X^+_{r_2}$ is an S-equivariant deformation retract of $X^+_{r_1}$).
- if the map $H^*_S(X^+_{r_2}) \to H^*_S(X^+_{r_2} \cap X^S)$ is injective and $f^{-1}([r_1, r_2]) \cap X^S = \{v\lambda\}$ for some $v \in W$, then the map $H^*_S(X^+_{r_1}) \to H^*_S(X^+_{r_1} \cap X^S)$ is injective as well.

To prove the last item, let us consider the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^*_S(X^+_{r_1}, X^+_{r_2}) \arrow[r, "②"] \arrow[d, "\simeq"] & H^*_S(X^+_{r_1}) \arrow[r] \arrow[d, "①"] & H^*_S(X^+_{r_2}) \arrow[r] & \cdots \\
& H^{*-k}_S(\{v\lambda\}) \arrow[r, "\cup e"] & H^*_S(\{v\lambda\}) & &
\end{tikzcd} \tag{6}$$

Here k is the dimension of the positive space of the Hessian of f at the point $v\lambda$, call it $T^+_{v\lambda}X$. Also, $\simeq$ denotes the isomorphism obtained by composing the excision map $H^*_S(X^+_{r_1}, X^+_{r_2}) \simeq H^*_S(D^k, S^{k-1})$ (where D^k, S^{k-1} are the unit disk, respectively unit sphere in $T^+_{v\lambda}X$), with the Thom isomorphism $H^*_S(D^k, S^{k-1}) \simeq H^{*-k}_S(\{v\lambda\})$. The map ① is induced by the inclusion of $\{v\lambda\}$ in $X^+_{r_1}$. The cohomology class $e \in H^k_S(\{v\lambda\}) = H^k(BS)$ is the S equivariant Euler class of $T^+_{v\lambda}X$. Let us note that the group S acts linearly without fixed points on the latter space (since $v\lambda$ is an isolated fixed point of the S action). By the Atiyah-Bott lemma (cf. [2]), e is a non-zero element of $H^*(BS)$. Thus, the multiplication by e is an injective endomorphism of $H^*(BS)$. We deduce that the map ② is injective and consequently the long exact sequence of the pair $(X^+_{r_1}, X^+_{r_2})$ splits into short exact sequences of the form

$$0 \longrightarrow H^*_S(X^+_{r_1}, X^+_{r_2}) \longrightarrow H^*_S(X^+_{r_1}) \longrightarrow H^*_S(X^+_{r_2}) \longrightarrow 0. \qquad (7)$$

Let us consider now the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & H^*_S(X^+_{r_1}, X^+_{r_2}) \arrow[r] \arrow[d, "③"'] & H^*_S(X^+_{r_1}) \arrow[r] \arrow[d, "\imath_1^*"'] & H^*_S(X^+_{r_2}) \arrow[r] \arrow[d, "\imath_2^*"'] & 0 \\
0 \arrow[r] & H^*_S(\{v\lambda\}) \arrow[r] & H^*_S(X^+_{r_1} \cap X^S) \arrow[r] & H^*_S(X^+_{r_2} \cap X^S) \arrow[r] & 0
\end{tikzcd}$$

where we have identified $H^*_S(X^+_{r_1} \cap X^S, X^+_{r_2} \cap X^S) = H^*_S(\{v\lambda\})$. By hypothesis, the map $\imath_2^*$ is injective. The map ③ is the same as the composition of ① and ② (see diagram (6)), thus it is injective as well. By a diagram chase we deduce that $\imath_1^*$ is injective.

The second assertion is proved by the same method as before, by induction over the sublevels of $-f|_{X\setminus\{\lambda\}}$. This time we confine ourselves to show that if $r_1, r_2 \in \mathbb{R} \setminus f(X^S)$ satisfy $r_2 > r_1 > f(\lambda)$, $f^{-1}([r_1, r_2]) \cap X^S = \{v\lambda\}$, and the map $H^*_T(X^+_{r_2}) \to H^*_S(X^+_{r_2})$ is surjective, then the map $H^*_T(X^+_{r_1}) \to H^*_S(X^+_{r_1})$ is surjective too. To this end we first note that both restriction maps $H^*_T(X^+_{r_1}) \to H^*_T(X^+_{r_2})$ and $H^*_S(X^+_{r_1}) \to H^*_S(X^+_{r_2})$ are surjective: we use the exact sequence (7) and its analogue for T equivariant cohomology. Let us consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^*_T(X^+_{r_1}, X^+_{r_2}) \arrow[r] \arrow[d, "④"] & H^*_T(X^+_{r_1}) \arrow[r] \arrow[d, "⑤"] & H^*_T(X^+_{r_2}) \arrow[r] \arrow[d, "⑥"] & 0 \\
0 \arrow[r] & H^*_S(X^+_{r_1}, X^+_{r_2}) \arrow[r] & H^*_S(X^+_{r_1}) \arrow[r] & H^*_S(X^+_{r_2}) \arrow[r] & 0
\end{tikzcd}$$

The map ④ is surjective: indeed, as before, we have $H^*_T(X^+_{r_1}, X^+_{r_2}) \simeq H^{*-k}_T(\{v\lambda\})$, and similarly if we replace T by S; thus the map ④ is just the canonical (restriction) map $\mathrm{Symm}(\mathrm{Lie}(T)^*) \to \mathrm{Symm}(\mathrm{Lie}(S)^*)$. A diagram chase shows that if ⑥ is surjective, then ⑤ is surjective as well.

Point (iii) is a straightforward consequence of the first assertion of (ii), by taking $r = r_0 - \frac{\epsilon}{3}$.

(iv) We prove the surjectivity of κ_1 inductively, along the sublevels of the function $f|_{X^+_{r_0-\frac{\epsilon}{3}}}$. At the first induction step, we note that for any number r such that

$$r_0 + \frac{\epsilon}{3} < r < \min\{f(v\lambda) \ : \ v\lambda \in X^S, f(v\lambda) > r_0\}$$

the space $f^{-1}((r_0-\frac{\epsilon}{3}, r))$ contains $f^{-1}(r_0)$ as an S-equivariant deformation retract. This can be proved exactly like the claim in the proof of Lemma 3.2. Then we consider $r_1, r_2 \in \mathbb{R} \setminus f(X^S)$ such that $r_0 < r_1 < r_2$ and show that the map

$$H^*_S(f^{-1}((r_0 - \frac{\epsilon}{3}, r_2))) \to H^*_S(f^{-1}((r_0 - \frac{\epsilon}{3}, r_1)))$$

is surjective, in each of the following two situations:

- the intersection $f^{-1}([r_1, r_2]) \cap X^S$ is empty.
- the intersection $f^{-1}([r_1, r_2]) \cap X^S$ consists of exactly one point, say $v\lambda$, where $v \in W$.

We use the argument exposed in the proof of point (ii) above. In the second situation (when $f^{-1}([r_1, r_2]) \cap X^S = \{v\lambda\}$) we use the analogue of the short exact sequence (7). □

The following lemma is the final step towards the proof of Theorem 1.2. Assumption 3, (ii) is essential in the proof of this lemma. We will use the 1-skeleton of the T action on $\mathcal{O}_\lambda$. By definition (cf. [26]), this consists of all points in $\mathcal{O}_\lambda$ whose stabilizer have codimension at most 1. In the case at hand one can describe the 1-skeleton as follows. For any $v \in W$ and any $\gamma \in R^+$ such that $s_\gamma v\lambda \neq v\lambda$, there exists a subspace $\mathcal{S}_\gamma(v\lambda) \subset \mathcal{O}_\lambda$, which is a metric sphere in Euclidean space $\mathfrak{k}^*$ relative to an $\mathrm{Ad}^*(K)$ invariant inner product on the latter space. Moreover, it contains $v\lambda$ and $s_\gamma v\lambda$ as antipodal points. The 1-skeleton is the union of all these spheres. The torus T leaves $\mathcal{S}_\gamma(v\lambda)$ invariant; in fact, the sphere is left pointwise fixed by the kernel of the character $T \to S^1$ induced by γ (cf. [9, Section 2.2]). From the Goresky-Kottwitz-MacPherson theorem (cf. [8], [26]), we deduce that the map $H^*_T(\mathcal{O}_\lambda) \to H^*_T(W\lambda)$ is injective and its image consists of all ordered sets $(p_{v\lambda})_{v\lambda \in W\lambda}$ with the property that

$$p_{v\lambda} - p_{s_\gamma v\lambda} \text{ is divisible by } \gamma, \tag{8}$$

for all $v \in W$ and $\gamma \in R^+$. If both $v\lambda$ and $s_\gamma v\lambda$ are in X, then the whole $\mathcal{S}_\gamma(v\lambda)$ is contained in X. Thus, the T equivariant cohomology ring of X is isomorphic to the ring of all ordered sets $(p_{v\lambda})_{v\lambda \in X^S}$ with the property that the condition (8) holds for all $v \in W$ and $\gamma \in R^+$ such that $v\lambda$ and $s_\gamma v\lambda$ are in X. Finally, we note that the space $\mathcal{S}_\gamma(v\lambda)$ is an orbit of the complex subgroup $SL_2(\mathbb{C})_\gamma$ of G (cf. [15, Chapter V, Section 6]), thus it is a Kähler submanifold of $\mathcal{O}_\lambda$.

Lemma 3.4. *The map*

$$\kappa_2 : H^*_S(X) \to H^*_S(X^+_{r_0 - \frac{\epsilon}{3}})$$

is surjective. Thus, $\kappa = \kappa_1 \circ \kappa_2$ is surjective (the map κ_1 was defined in Lemma 3.3, (iv)).

Proof. It is sufficient to prove that the map $\kappa_2^T : H_T^*(X) \to H_T^*(X^+_{r_0-\frac{\epsilon}{3}})$ is surjective. This is because in the commutative diagram

$$\begin{array}{ccc} H_T^*(X) & \longrightarrow & H_S^*(X) \\ \downarrow{\scriptstyle \kappa_2^T} & & \downarrow{\scriptstyle \kappa_2} \\ H_T^*(X^+_{r_0-\frac{\epsilon}{3}}) & \longrightarrow & H_S^*(X^+_{r_0-\frac{\epsilon}{3}}) \end{array}$$

the horizontal maps are surjective (here we have used that X is T equivariantly formal, respectively Lemma 3.3, (ii)).

Take $\alpha \in H_T^*(X^+_{r_0-\frac{\epsilon}{3}})$. Take $\gamma \in R^+$ and $u \in W$ such that $u\lambda$ and $s_\gamma u\lambda$ are in X, $u\lambda \neq s_\gamma u\lambda$. We will need the following claim.

Claim. If $u\lambda$ and $s_\gamma u\lambda$ are in $X^+_{r_0-\frac{\epsilon}{3}}$, then the whole $\mathcal{S}_\gamma(u\lambda)$ is contained in $X^+_{r_0-\frac{\epsilon}{3}}$.

Indeed, the sphere $\mathcal{S}_\gamma(u\lambda)$ is contained in X. Moreover, the restriction of Φ_a to $\mathcal{S}_\gamma(u\lambda)$ is just the moment map of the S action. Thus its critical points are the fixed points of the S action, namely $u\lambda$ and $s_\gamma u\lambda$. One of these two points is a global minimum point. The claim is now proved.

The class α restricted to $\mathcal{S}_\gamma(u\lambda)$ is an element of its T equivariant cohomology. From the discussion preceding the lemma we deduce that

$$\alpha|_{u\lambda} - \alpha|_{s_\gamma u\lambda} \text{ is divisible by } \gamma.$$

Here we have denoted by $\alpha|_{u\lambda}$ the image of α under the map induced by the inclusion $\{u\lambda\} \to X^+_{r_0-\frac{\epsilon}{3}}$, and similarly for $\alpha|_{s_\gamma u\lambda}$. Consequently, the ordered set $(\alpha|_{u\lambda})_{u\lambda\in X^S, \Phi_a(u\lambda)>r_0}$ is admissible relative to Γ_{r_0}. By Assumption 3, (ii), there exists an ordered set $(p_{v\lambda})_{v\lambda\in X^S}$ which is admissible relative to Γ, such that

$$p_{v\lambda} = \alpha|_{v\lambda}, \text{ whenever } v\lambda \in X^+_{r_0-\epsilon}. \tag{9}$$

The collection $(p_{v\lambda})$ represents a cohomology class, call it β, in $H_T^*(X)$. By equation (9) and Lemma 3.3, (ii), we have $\kappa_2(\beta) = \alpha$. This finishes the proof. □

The only piece of information which is still missing is the fact that K'_- equals K_-. This is the content of the following lemma.

Lemma 3.5. *We have $K'_- = K_-$.*

Proof. We only need to show that $K_- \subset K'_-$. We actually show that for any $r > f(\lambda)$, $r \notin f(X^S)$, the map $H_S^*(X_r^-) \to H_S^*(X_r^- \cap X^S)$ is injective. The idea we will use is the same as in the proof of Lemma 3.3, (ii). Some adjustments are necessary, though, since X_r^- and all the other sublevels involved in the argument contain the singular point λ. We proceed by proving the following claims.

Claim 1. Take $\delta > 0$ such that $\Phi_a(\lambda) < \delta < \Phi_a(v\lambda)$ for any $v\lambda \in W\lambda \setminus \{\lambda\}$. Then the restriction map $H_S^*(X_\delta^-) \to H_S^*(\{\lambda\})$ is injective.

Claim 2. If $r_1, r_2 \in \mathbb{R} \setminus f(X^S)$ such that $f(\lambda) < r_1 < r_2$ and $f^{-1}([r_1, r_2]) \cap X^S$ is empty, then the map $H^*_S(X^-_{r_2}) \to H^*_S(X^-_{r_1})$ induced by the inclusion $X^-_{r_1} \to X^-_{r_2}$ is an isomorphism.

Claim 3. If $r_1, r_2 \in \mathbb{R} \setminus f(X^S)$ such that $f(\lambda) < r_1 < r_2$, the map $H^*_S(X^-_{r_1}) \to H^*_S(X^-_{r_1} \cap X^S)$ is injective, and $f^{-1}([r_1, r_2]) \cap X^S = \{v\lambda\}$ for some $v \in W$, then the map $H^*_S(X^-_{r_2}) \to H^*_S(X^-_{r_2} \cap X^S)$ is injective as well.

To prove Claim 1, we note that X^-_δ is a subset of the stable manifold at λ of the function $\Phi_a : \mathcal{O}_\lambda \to \mathbb{R}$ relative to the Kähler metric on $\mathcal{O}_\lambda$. It is also invariant under any of the automorphisms ψ_t of $\mathcal{O}_\lambda$, where $t \le 0$ (by ψ_t, $t \in \mathbb{R}$, we denote the flow of $\mathcal{O}_\lambda$ induced by gradΦ_a, like in the proof of Lemma 3.1). Consequently, it is contractible to $\{\lambda\}$. Moreover, since X^-_δ is S invariant, the retract $X^-_\delta \to \{\lambda\}$ is S equivariant. Thus the inclusion map $\{\lambda\} \to X^-_\delta$ induces an isomorphism $H^*_S(X^-_\delta) \overset{\simeq}{\to} H^*_S(\{\lambda\})$. The claim is now proved.

To prove Claim 2, we take into account that $f^{-1}([r_1, r_2])$ is a compact subset of the smooth manifold $X \setminus \{\lambda\}$. Consequently, there exists a family of maps $Q_\tau : X^-_{r_2} \setminus \{\lambda\} \times [0,1] \to X^-_{r_2} \setminus \{\lambda\}$, $\tau \in [0,1]$, which is a strong deformation retract of $X^-_{r_2} \setminus \{\lambda\}$ on $X^-_{r_1} \setminus \{\lambda\}$, that is, $Q_1(X^-_{r_2} \setminus \{\lambda\}) \subset X^-_{r_1} \setminus \{\lambda\}$, Q_0 is the identity map of $X^-_{r_2} \setminus \{\lambda\}$, and Q_τ restricted to $X^-_{r_1} \setminus \{\lambda\}$ is the identity map, for all $\tau \in [0,1]$. The concrete expression of Q_τ can be found for instance in [23, p. 13]: it can be seen from there that for any $\tau \in [0,1]$, the map Q_τ is S equivariant. We extend Q_τ to a continuous map $X^-_{r_2} \to X^-_{r_2}$ by setting $Q_\tau(\lambda) = \lambda$. This gives a (continuous) deformation retract of $X^-_{r_2}$ on $X^-_{r_1}$, which is S equivariant. In this way we have proved Claim 2.

We now prove Claim 3. Like for the previous claim, we use Morse theory for the function f on $X \setminus \{\lambda\}$. By Lemma 3.1, $v\lambda$ is a non-degenerate critical point of this function. Denote by m its index. There exists a closed m-cell $e^m \subset X^-_{r_2} \setminus \{\lambda\}$ which can be attached to $X^-_{r_1}$, and a strong deformation retract Q'_τ, $\tau \in [0,1]$, of $X^+_{r_2} \setminus \{\lambda\}$ onto $(X^-_{r_1} \cup e^m) \setminus \{\lambda\}$. From the exact expression of e^m and Q'_τ (see [23, proof of Theorem 3.2]), we can see that the former is S invariant and the latter is S equivariant. By setting $Q'_\tau(\lambda) = \lambda$, for all $\tau \in [0,1]$, we obtain an S equivariant deformation retract of $X^-_{r_2}$ onto $X^-_{r_1} \cup e^m$. We repeat the argument in the proof of Lemma 3.3, (ii). □

Theorems 1.2 and 1.3 are now completely proved.

Remark. At this point we can understand exactly where each of the assumptions made in the introduction has been used. Namely, Lemmas 3.1, 3.2, and 3.3 use only Assumption 1 and the regularity hypothesis on r_0 given in Assumption 3, (i). Lemma 3.5 needs Assumption 2 (see especially Claim 1 in the proof of this lemma). Lemma 3.4 needs Assumption 3, (ii).

4. An example

The Grassmannian $G_2(\mathbb{C}^4)$ of 2-dimensional vector subspaces in $\mathbb{C}^4$ can be identified with a coadjoint orbit of $SU(4)$. Let us be more specific. Denote by T the space of all diagonal matrices in $SU(4)$, which is a maximal torus. Its Lie algebra $\mathfrak{t}$ can be described as

$$\mathfrak{t} = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_1 + x_2 + x_3 + x_4 = 0\},$$

which is a hyperplane in $\mathbb{R}^4$. We equip $\mathbb{R}^4$, and also its subspace $\mathfrak{t}$, with the Euclidean metric $(\ ,\ )$. To each $x \in \mathbb{R}^4$ we assign the element $x^* \in \mathfrak{t}^*$ given by

$$x^*(y) := (x, y),$$

for all $y \in \mathfrak{t}$. A simple root system of $SU(4)$ relative to T is

$$\alpha_1 := e_2^* - e_1^*, \ \alpha_2 := e_3^* - e_2^*, \ \alpha_3 := e_4^* - e_3^*,$$

where e_1, e_2, e_3, e_4 denotes the standard basis of $\mathbb{R}^4$. The fundamental weights are $e_1^*, e_1^* + e_2^*, e_1^* + e_2^* + e_3^*$. The polyhedral cone generated by them is the fundamental Weyl chamber. The coadjoint orbit of any λ which is a positive integer multiple of $e_1^* + e_2^*$ can be identified with the Grassmannian $G_2(\mathbb{C}^4)$ of all 2-planes V in $\mathbb{C}^4$ (see [10, Section 5.3]). As usual, we denote by s_i the reflection of $\mathfrak{t}^*$ corresponding to the root α_i, where $i \in \{1, 2, 3\}$. They generate the Weyl group W. To each element of $W\lambda$ corresponds a Schubert variety in $G_2(\mathbb{C}^4)$. In this example we discuss the Schubert variety X corresponding to $s_3 s_1 s_2 \lambda$. This can be described explicitly as

$$X = \{V \in G_2(\mathbb{C}^4) \ : \ \dim(V \cap \mathbb{C}^2) \geq 1\}.$$

It has an isolated singularity at $\mathbb{C}^2$, which is the same as the point λ on the coadjoint orbit (see the paragraph following Assumption 2 in the introduction). So Assumption 2 is satisfied. The image of X under the moment map Φ is the polytope generated by the elements of X^T, which are λ, $s_2\lambda$, $s_1 s_2 \lambda$, $s_3 s_2 \lambda$, and $s_3 s_1 s_2 \lambda$.

We identify $H_T^*(\mathrm{pt})$ with the polynomial ring $S(\mathfrak{t}^*) = \mathbb{R}[\alpha_1, \alpha_2, \alpha_3]$. According to the GKM theorem (see the introduction) the T equivariant cohomology ring $H_T^*(X)$ consists of all elements $(f_1, f_2, f_3, f_4, f_5)$ of $H_T^*(\{\lambda\}) \oplus H_T^*(\{s_2\lambda\}) \oplus H_T^*(\{s_1 s_2 \lambda\}) \oplus H_T^*(\{s_3 s_2 \lambda\}) \oplus H_T^*(\{s_3 s_1 s_2 \lambda\}) = S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*)$ such that

$$\begin{aligned}
&f_5 - f_4 \text{ is divisible by } \alpha_1 \\
&f_5 - f_3 \text{ is divisible by } \alpha_3 \\
&f_5 - f_1 \text{ is divisible by } \alpha_1 + \alpha_2 + \alpha_3 \\
&f_4 - f_2 \text{ is divisible by } \alpha_3 \\
&f_4 - f_1 \text{ is divisible by } \alpha_2 + \alpha_3 \\
&f_3 - f_2 \text{ is divisible by } \alpha_1 \\
&f_3 - f_1 \text{ is divisible by } \alpha_1 + \alpha_2 \\
&f_2 - f_1 \text{ is divisible by } \alpha_2.
\end{aligned} \tag{10}$$

We compute the cohomology ring of the symplectic quotient of X with respect to $S = \exp(\mathbb{R}a)$, where a is an integral element of $\mathfrak{t}$. The situation is described in

Figure 1 (a), where the identification $\mathfrak{t}^* = \mathfrak{t}$ induced by $(\ ,\ )$ is in force. We also consider the element $\nu \in \mathfrak{t}^*$ such that $\nu(a) = 1$ and ν is identically zero on the orthogonal complement of a in $\mathfrak{t}$. Then ν is identified with the vector $a/(a,a)$. For any $x \in \mathfrak{t}$, the orthogonal projection of x^* on the line $\mathbb{R}\nu$ is $(x,a)\nu$, which is the same as $\Phi_a(x^*)\nu$.

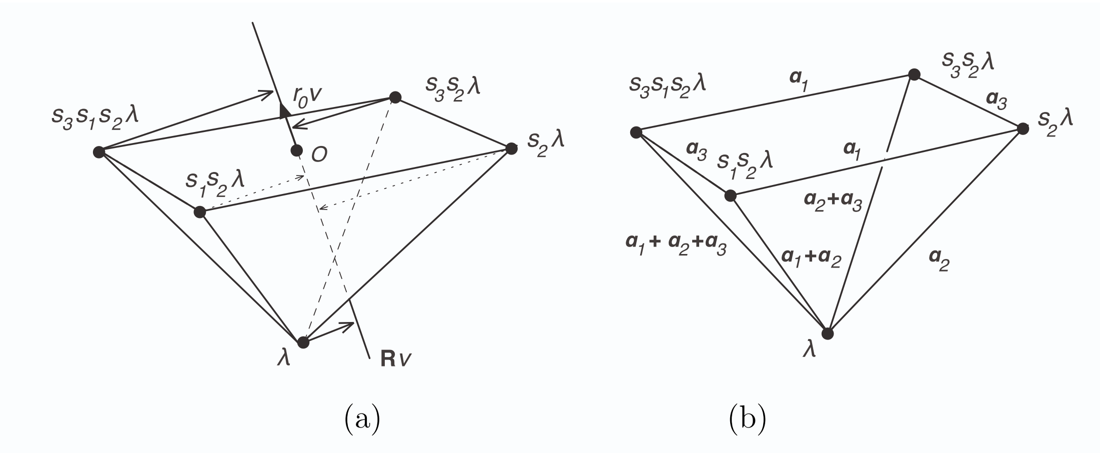

Figure 1. The upside-down pyramid in figure (a) is the image under the moment map of th e Schubert variety $X(s_3s_1s_2)$ in the Grassmannian $G_2(\mathbb{C}^4)$. Its corners λ, $s_2\lambda$, $s_1s_2\lambda$, $s_3s_2\lambda$ and $s_3s_1s_2\lambda$ are projected orthogonally onto the line $\mathbb{R}\nu$, where ν is in the Weyl chamber opposite to the fundamental one. We can also see the vector $r_0\nu$, where $r_0 > 0$ satisfies equation (11). Figure (b) describes the moment graph of $X(s_3s_1s_2)$.

The figure shows that r_0 has been chosen such that

$$\Phi_a(\lambda) < \Phi_a(s_2\lambda) < \Phi_a(s_1s_2\lambda) < \Phi_a(s_3s_2\lambda) < r_0 < \Phi_a(s_3s_1s_2\lambda). \tag{11}$$

Assumption 3 is obviously satisfied in this case (see the paragraph following Assumption 3 in the introduction). This is the case where we will compute the cohomology ring. Before doing this, we would like to point out that if r_0 satisfies

$$\Phi_a(\lambda) < \Phi_a(s_2\lambda) < \Phi_a(s_1s_2\lambda) < r_0 < \Phi_a(s_3s_2\lambda) < \Phi_a(s_3s_1s_2\lambda)$$

then Assumption 3 is also satisfied. Indeed, the moment graph is described in Figure 1 (b). Thus, Γ_{r_0} consists of the vertices $s_3s_2\lambda$ and $s_3s_1s_2\lambda$ which are joined by an edge with label α_1. Let f_5' and f_4' be two polynomials such that $f_5' - f_4'$ is divisible by α_1, that is, $f_4' = f_5' + \alpha_1 g$, for some $g \in \mathbb{R}[\alpha_1, \alpha_2, \alpha_3]$. The polynomials

$$f_5 = f_5',\ f_4 = f_4' = f_5 + \alpha_1 g,\ f_3 = f_5 + \alpha_3 g,$$
$$f_2 = f_5 + (\alpha_1 + \alpha_3)g,\ f_1 = f_5 + (\alpha_1 + \alpha_2 + \alpha_3)g$$

satisfy the equations (10).

Let us now return to the case where r_0 satisfies equation (11) and compute the cohomology of $X//_\lambda S(r_0)$. To this end we first note that equations (10) yield the following description of $H_T^*(X)$: it consists of all elements of $H_T^*(\{\lambda\}) \oplus H_T^*(\{s_2\lambda\}) \oplus H_T^*(\{s_1s_2\lambda\}) \oplus H_T^*(\{s_3s_2\lambda\}) \oplus H_T^*(\{s_3s_1s_2\lambda\})$ which are of the form

$$\begin{aligned}&(p_1, p_1+\alpha_2 p_2, p_1+(\alpha_1+\alpha_2)p_2+\alpha_1(\alpha_1+\alpha_2)p_3,\ p_1+(\alpha_2+\alpha_3)p_2+\alpha_3(\alpha_2+\alpha_3)p_4,\\ &p_1+(\alpha_1+\alpha_2+\alpha_3)p_2+\alpha_1(\alpha_1+\alpha_2+\alpha_3)p_3+\alpha_3(\alpha_1+\alpha_2+\alpha_3)p_4\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\alpha_1\alpha_3(\alpha_1+\alpha_2+\alpha_3)p_5),\end{aligned}$$

where p_1, p_2, p_3, p_4, p_5 are in $\mathbb{R}[\alpha_1, \alpha_2, \alpha_3]$. The restriction map $\mathrm{Symm}(\mathfrak{t}^*) = \mathbb{R}[\alpha_1, \alpha_2, \alpha_3] \to \mathrm{Symm}((\mathbb{R}a)^*) = \mathbb{R}[\nu]$ is given by $\alpha_i \mapsto a_i\nu$, $1 \le i \le 3$, where $a_i := \alpha_i(a)$. We deduce that $H_S^*(X)$ consists of the elements of $H_S^*(\{\lambda\}) \oplus H_S^*(\{s_2\lambda\}) \oplus H_S^*(\{s_1s_2\lambda\}) \oplus H_S^*(\{s_3s_2\lambda\}) \oplus H_S^*(\{s_3s_1s_2\lambda\})$ which are of the form

$$\begin{aligned}&(q_1, q_1 + a_2\nu q_2, q_1 + (a_1 + a_2)\nu q_2 + a_1(a_1 + a_2)\nu^2 q_3,\\ &q_1 + (a_2 + a_3)\nu q_2 + a_3(a_2 + a_3)\nu^2 q_4,\\ &q_1 + (a_1 + a_2 + a_3)\nu q_2 + a_1(a_1 + a_2 + a_3)\nu^2 q_3 + a_3(a_1 + a_2 + a_3)\nu^2 q_4 + \nu^3 q_5),\end{aligned} \tag{12}$$

where $q_1, q_2, q_3, q_4, q_5 \in \mathbb{R}[\nu]$. By Theorem 1.3, the ring $H^*(X/\!/_\lambda S(r_0))$ is the quotient of $H_S^*(X)$ by the ideal $K_- + K_+$. Here K_- is the ideal generated by $(0, 0, 0, 0, \nu^3)$ and K_+ consists of all ordered sets of the form (12) where

$$q_1 + (a_1 + a_2 + a_3)\nu q_2 + a_1(a_1 + a_2 + a_3)\nu^2 q_3 + a_3(a_1 + a_2 + a_3)\nu^2 q_4 + \nu^3 q_5 = 0.$$

To achieve a better understanding of this ring, we first note that it is graded, with the graduation given by $\deg\nu = 2$. Thus, we have $H^{2k+1}(X/\!/_\lambda S(r_0)) = \{0\}$, for all $k \ge 0$. We also have as follows:

- The elements of $H^0(X/\!/_\lambda S(r_0))$ are the cosets of (c, c, c, c, c), where $c \in \mathbb{R}$. Thus, $\dim H^0(X/\!/_\lambda S(r_0)) = 1$, with a basis consisting of the coset of $(1, 1, 1, 1, 1)$.
- The elements of $H^2(X/\!/_\lambda S(r_0))$ are the cosets of

  $$(c_1\nu, c_1\nu+c_2a_2\nu, c_1\nu+(a_1+a_2)c_2\nu, c_1\nu+(a_2+a_3)c_2\nu, c_1\nu+(a_1+a_2+a_3)c_2\nu),$$

  where $(c_1, c_2) \in \mathbb{R}^2$. Thus $H^2(X/\!/_\lambda S(r_0))$ can be identified with the quotient of $\mathbb{R}^2$ by the kernel of the function $(c_1, c_2) \mapsto c_1 + (a_1 + a_2 + a_3)c_2$. A basis of this quotient is the coset of $(c_1, c_2) = (1, 0)$. We deduce that $\dim H^2(X/\!/_\lambda S(r_0)) = 1$, with a basis consisting of the coset of $(\nu, \nu, \nu, \nu, \nu)$ in $H_S^*(X)$.
- Similarly, $\dim H^4(X/\!/_\lambda S(r_0)) = 1$, with a basis consisting of the coset of $(\nu^2, \nu^2, \nu^2, \nu^2, \nu^2)$.
- $H^m(X/\!/_\lambda S(r_0)) = \{0\}$ for all $m \ge 5$.

From the description above we can see that $X/\!/_\lambda S(r_0)$ has the same cohomology ring as $\mathbb{C}P^2$. We do not know whether it is actually diffeomorphic to $\mathbb{C}P^2$ or even whether it is smooth.

References

[1] Atiyah, M.-F.: *Convexity and commuting Hamiltonians*. Bull. Lond. Math. Soc. **14** (1982), 1–15. Zbl 0482.58013

[2] Atiyah, M.; Bott, R.: *The Yang-Mills equations over Riemann surfaces*. Philos. Trans. R. Soc. Lond. A **308** (1982), 523–615. Zbl 0509.14014

[3] Billey, S.; Lakshimbai, V.: *Singular Loci of Schubert Varieties*. Progress in Mathematics **182**, Birkhäuser, Boston 2000. Zbl 0959.14032

[4] Brion, M.; Kumar, S.: *Frobenius splitting methods in geometry and representation theory*. Progress in Mathematics **231**, Birkhäuser, Boston 2005. Zbl 1072.14066

[5] Duistermaat, J. J.; Kolk, J. A. C.: *Lie Groups*. Springer-Verlag, Berlin 2000. Zbl 0955.22001

[6] Duistermaat, J. J.; Kolk, J. A. C.; Varadarajan, V. S.: *Functions, flows, and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups*. Compos. Math. **49** (1983), 309–398. Zbl 0524.43008

[7] Fulton, W.; Harris, J.: *Representation Theory*. Graduate Texts in Mathematics **129**, Springer-Verlag, Berlin 2004. Zbl 0744.22001

[8] Goresky, M.; Kottwitz, R.; MacPherson, R.: *Equivariant cohomology, Koszul duality, and the localization theorem*. Invent. Math. **131** (1998), 25–83. Zbl 0897.22009

[9] Guillemin, V.; Holm, T. S.; Zara, C.: *A GKM description of the equivariant cohomology ring of a homogeneous space*. J. Algebr. Comb. **23** (2006), 21–41. Zbl 1096.53050

[10] Guillemin, V.; Lerman, E.; Sternberg, S.: *Symplectic Fibrations and Multiplicity Diagrams*. Cambridge University Press, Cambridge 1996. Zbl 0870.58023

[11] Guillemin, V.; Sternberg, S.: *Geometric quantization and multiplicities of group representations*. Invent. Math. **67** (1982), 515–538. Zbl 0503.58018

[12] Harada, M.; Holm, T.: *The equivariant cohomology of hypertoric varieties and their real loci*. Comm. Anal. Geom. **13** (2005), 645–677. Zbl 1088.53055

[13] Harada, M.; Holm, T.; Jeffrey, L.; Mare, A.-L.: *Connectivity properties of moment maps on based loop groups*. Geom. Topol. **10** (2006), 2001–2028. Zbl 1128.22012

[14] Heinzner, P.; Migliorini, L.: *Projectivity of moment map quotients*. Osaka J. Math. **38** (2001), 167–184. Zbl 0982.32020

[15] Hu, Y.: *Geometry and topology of quotient varieties of torus actions*. Duke Math. J. **68** (1992), 151–183. Zbl 0812.14031

[16] Hu, Y.: *Topological aspects of Chow quotients*. J. J. Differ. Geom. **69** (2005), 399–440. Zbl 1087.14032

[17] Huckleberry, A.: *Introduction to group actions in symplectic and complex geometry*. In: A. Huckleberry (ed.) et al., Infinite Dimensional Kähler Manifolds, D.M.V. Semin. **31**, Birkhäuser, 2001. Zbl 0998.53056

[18] Ion, B.: *A weight multiplicity formula for Demazure modules*. Intern. Math. Res. Not. **5** (2005), 311–323. Zbl 1081.20054

[19] Kirwan, F. C.: *Cohomology of Quotients in Symplectic Cohomology of Quotients in Symplectic and Algebraic Geometry*. Mathematical Notes **31**, Princeton University Press, Princeton 1984. Zbl 0553.14020

[20] Kirwan, F. C.: *Rational intersection cohomology of quotient varieties, II*. Invent. Math. **90** (1987), 153–167. Zbl 0631.14012

[21] Lerman, E.: *Gradient flow of the norm squared of a moment map*. Enseign. Math. (2) **51** (2005), 117–127. Zbl 1103.53051

[22] McDuff, D.; Salamon, D.: *Introduction to Symplectic Topology*. Second Edition, Oxford Mathematical Monographs, Clarendon Press, Oxford 1998. Zbl 1066.53137

[23] Milnor, J.: *Morse Theory*. Annals of Mathematics Studies **51**, Princeton University Press, Princeton 1963. Zbl 0108.10401

[24] Mumford, D.: *Algebraic Geometry I: Complex Projective Varieties*. Grundlehren der mathematischen Wissenschaften **221**, Springer-Verlag, Berlin 1976. Zbl 0356.14002

[25] Ramanathan, A.: *Schubert varieties are arithmetically Cohen-Macaulay*. Invent. Math. **80** (1985), 283–294. Zbl 0541.14039

[26] Tolman, S.; Weitsman, J.: *On the cohomology rings of Hamiltonian T-spaces*. In: Ya. Eliashberg (ed.) et al.: Northern California Symplectic Geometry Seminar, A.M.S. Transl., Ser. 2, Am. Math. Soc. **196**(45) (1999), 251–258. Zbl 0955.57023

[27] Tolman, S.; Weitsman, J.: *The cohomology ring of symplectic quotients*. Commun. Anal. Geom. **11** (2003), 751–773.

[28] Sjamaar, R.: *Holomorphic slices, symplectic reduction and multiplicities of representations*. Ann. Math. (2) **141** (1995), 87–129. Zbl 0827.32030

[29] Teleman, C.: *The quantization conjecture revisited*. Ann. Math. (2) **152** (2000), 1–43. Zbl 0980.53102

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