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On the Depth of Graded Rings Associated to Lex-segment Ideals in $K[x, y]$

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Abstract. In this article, we show that the depths of the associated graded ring and fiber cone of a lex-segment ideal in $K[x, y]$ are equal.

Keywords: lex-segment ideals, associated graded ring, fiber cone, Rees algebra, Cohen-Macaulay

1. Introduction

Let K be a field of characteristic zero and $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K. Let R_i denote the K-vector subspace of all monomials of degree i. We fix the ordering of variables as $x_1 > x_2 > \cdots > x_n$. For monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$, we say that $u <_{Lex} v$ if $\deg u \leq \deg v$ or $\deg u = \deg v$ and $b_i - a_i > 0$ for the first time when it is nonzero. An initial lex-segment in degree d is the set of all monomials of the form $\{m \in R_d \ : \ m \geq u\}$, where $u \in R_d$. A graded ideal I is said to be a *lex-segment ideal* if I_d is generated by initial lex-segments for each d with $I_d \neq 0$. Lex-segment ideals are important due to many reasons. It is well known that among ideals with a given Hilbert function, the lex-segment ideal has the largest number of generators. A. M. Bigatti [1] and H. A. Hulett [7] in characteristic zero and K. Pardue [10] in positive characteristic generalized this to all Betti numbers. They proved that the lex-segment ideals have the largest Betti numbers among all ideals with a given Hilbert function. Lex-segment ideals are of interest also due to classical reasons. O. Zariski used the theory of contracted ideals to study complete ideals in 2-dimensional regular local rings $(R, \mathfrak{m})$. In the graded setting, when K is algebraically closed, Zariski's factorization theorem for homogeneous contracted ideals asserts that any homogeneous contracted ideal I can be written as $I = \mathfrak{m}^c L_1 \cdots L_t$, where each L_i is a lex-segment ideal with respect to an appropriate system of coordinates x_i, y_i which depends on i. [13, Theorem 1, Appendix 5], [3, Theorem 3.8].

In this article, we study the blowup algebras, namely, the associated graded ring and the fiber cone of lex-segment ideals in a two dimensional polynomial ring. Let R be a ring, I any ideal of R and $\mathfrak{m}$ a maximal ideal. Then the associated graded ring and the fiber cone of I are respectively defined as $\operatorname{gr}_I(R) = \oplus_{n\geq 0} I^n/I^{n+1}$ and $F(I) = \oplus_{n\geq 0} I^n/\mathfrak{m}I^n$. In [6], Huckaba and Marley showed that in a regular local ring $(R, \mathfrak{m})$, $\operatorname{depth} \operatorname{gr}_I(R) = \operatorname{depth} R(I) - 1$ for any $\mathfrak{m}$-primary ideal I, where $R(I) = \oplus_{n\geq 0} I^n t^n$ denotes the Rees algebra of I. It is interesting to ask if there is a similar relation between the depths of the fiber cone and the associated graded ring. It is well known that this is not the case in general (cf. Example 11, Example 12). In this article, we prove that the depths of these algebras are equal for lex-segment ideals in $K[x, y]$, where K is a field of characteristic zero.

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2. Equality of depths

Let $R = K[x, y]$, where K is a field of characteristic zero and $\mathcal{M} = (x, y)$. In this case, the lex-segment ideals are easy to describe. If I is a lex-segment ideal in $K[x, y]$, then $I = (x^d, x^{d-1}y^{a_1}, \dots, x^{d-k}y^{a_k})$ for some $0 \leq k \leq d$ and $1 \leq a_1 < a_2 < \cdots < a_k$. Note that if I is a lex-segment ideal, then I^n is also a lex-segment ideal for all $n \geq 1$.

Remark 1. Let $S = K[\![x, y]\!]$ and $\mathfrak{m} = (x, y)$. Then for any ideal $I \subset \mathfrak{m}$, $S/IS \cong R/I$ and $S/\mathfrak{m}S \cong R/\mathfrak{m}$. Therefore $\operatorname{gr}_{IS}(S) \cong \operatorname{gr}_I(R)$ and $F(IS) \cong F(I)$ [3, Lemma 2.1]. Hence, we may use the local techniques to prove the results for $\operatorname{gr}_I S(S)$ and $F(IS)$ and derive the same for $\operatorname{gr}_I(R)$ and $F(I)$.

We first show that the Cohen-Macaulay property of the associated graded ring and the fiber cone are equivalent. The dimension of the fiber cone, denoted by $s(I)$, is called the analytic spread. It is well known that $h(I) \leq s(I)$, where $h(I)$ denotes the height of the ideal I. The difference, $s(I) - h(I)$, is called the analytic deviation. Let $I = (x^d, x^{d-1}y^{a_1}, \dots, x^{d-k}y^{a_k})$. If $k = d$, then I is an $\mathcal{M}$-primary homogeneous contracted ideal. Because of Remark 1, we can use local theory of $\mathcal{M}$-primary contracted ideals in 2-dimensional regular local rings to study the blowup algebras. If $0 < k < d$, then I is a non-$\mathcal{M}$-primary ideal of analytic deviation one. Here we note that if $0 < k < d$, then $I = x^{d-k}(x^k, x^{k-1}y^{a_1}, \dots, xy^{a_{k-1}}, y^{a_k})$, which is of the form $I = zL$, where z is an R-regular element and L an $\mathcal{M}$-primary homogeneous contracted ideal. We show that the depth of $\operatorname{gr}_I(R)$ is at most the depth of $\operatorname{gr}_L(R)$. In particular, when

$\mathrm{gr}_I(R)$ is Cohen-Macaulay, so is $\mathrm{gr}_L(R)$. For an element $a \in I$, let a^* denote its initial form in $\mathrm{gr}_I(R)$ and a^o denote its initial form in $F(I)$.

Let I be an ideal of a ring R. An ideal $J \subseteq I$ is said to be reduction of I if $I^{n+1} = JI^n$ for some $n \geq 0$. A reduction which is minimal with respect to inclusion is called a minimal reduction. For a reduction J of I, the number $r_J(I) = \min\{n \mid I^{n+1} = JI^n\}$, is called the reduction number of I with respect to J.

Proposition 2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and L an $\mathfrak{m}$-primary ideal of R. Let x be a regular element in R and $I = xL$. Then* $\operatorname{depth} \mathrm{gr}_I(R) \leq \operatorname{depth} \mathrm{gr}_L(R)$. *In particular, if* $\mathrm{gr}_I(R)$ *is Cohen-Macaulay, then so is* $\mathrm{gr}_L(R)$.

Proof. Let $\operatorname{depth} \mathrm{gr}_I(R) = t$. Let $a_1, \dots, a_t \in L \setminus L^2$ and $b_i = xa_i$ be such that $b_1^*, \dots, b_t^* \in \mathrm{gr}_I(R)$ is a regular sequence. Then by Valabrega-Valla [12], $(b_1, \dots, b_t) \cap I^n = (b_1, \dots, b_t)I^{n-1}$ for all $n \geq 1$. We show that $(a_1, \dots, a_t) \cap L^n = (a_1, \dots, a_t)L^{n-1}$ for all $n \geq 1$.

Let $p \in (a_1, \dots, a_t) \cap L^n$ for some $n \geq 1$. Then $x^n p \in (b_1, \dots, b_t) \cap I^n = (b_1, \dots, b_t)I^{n-1} = x^n(a_1, \dots, a_t)L^{n-1}$. Therefore $x^n p = x^n q$ for some $q \in (a_1, \dots, a_t)L^{n-1}$. Since x is regular in R, $p = q$ which implies that $p \in (a_1, \dots, a_t)L^{n-1}$. Therefore, by Valabrega-Valla condition, $a_1^*, \dots, a_t^* \in \mathrm{gr}_L(R)$ is a regular sequence. ☐

Remark 3. In the above proposition, we have shown that if $b_1^*, \dots, b_t^*$ is a regular sequence in $\mathrm{gr}_I(R)$, then $a_1^*, \dots, a_t^*$ is a regular sequence in $\mathrm{gr}_L(R)$. The following example shows that the converse is not true in general.

Example 4. Let $R = K[x, y]$. Let $L = \mathcal{M} = (x, y)$ and $I = (x^3, x^2y)$. Then x^*, y^* is a regular sequence in $\mathrm{gr}_L(R)$. It can be easily seen that $I^2 : x^3 = (x^3, x^2y, xy^2) \neq I$. Therefore $(x^3)^* \in \mathrm{gr}_I(R)$ is not regular. However, this does not imply that the $\operatorname{depth} \mathrm{gr}_I(R) < 2$. In fact, in this case, it can be seen (using any of the computational commutative algebra packages) that $\mathrm{gr}_I(R)$ is indeed Cohen-Macaulay.

The following result follows directly from Theorem 2.1 of [4].

Proposition 5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and I be an ideal of R with $s(I) = r$ and*

$$H(F(I), t) = \frac{a + bt}{(1-t)^r}.$$

If $F(I)$ is Cohen-Macaulay, then $r_J(I) \leq 1$ for any minimal reduction J of I.

We show that the Cohen-Macaulay property of the associated graded ring and the fiber cone are equivalent:

Theorem 6. *Let I be a lex-segment ideal in $K[x, y]$. Then $F(I)$ is Cohen-Macaulay if and only if* $\mathrm{gr}_I(R)$ *is Cohen-Macaulay.*

Proof. Let $I = (x^d, x^{d-1}y^{a_1}, \dots, x^{d-k}y^{a_k})$ for some $0 \leq k \leq d$ and $1 \leq a_1 < a_2 < \cdots < a_k$. If $k = 0$, then $I = (x^d)$ and both $\mathrm{gr}_I(R)$ and $F(I)$ are Cohen-Macaulay. We deal the cases $k = d$ and $0 < k < d$ separately. Note that because of Remark 1, we may assume that I is an ideal in a two dimensional regular local ring $(R, \mathcal{M})$.

Let $k = d$. In this case, I is $\mathcal{M}$-primary. Suppose $\mathrm{gr}_I(R)$ is Cohen-Macaulay. Since I is contracted, by Theorem 5.1 of [8], for any minimal reduction $J \subset I$, $I^2 = JI$. By [11], $F(I)$ is Cohen-Macaulay.

Conversely, suppose that $F(I)$ is Cohen-Macaulay. Note that for all $n \geq 0$, $\mu(I^n) = nd + 1$ so that the Hilbert series of $F(I)$ is given by

$$H(F(I), t) = \frac{1 + (d-1)t}{(1-t)^2}.$$

Since $F(I)$ is Cohen-Macaulay, by Proposition 5, $r_J(I) \leq 1$ for any minimal reduction J of I. Therefore, $\mathrm{gr}_I(R)$ is Cohen-Macaulay [12].

Now let $0 < k < d$. In this case, $I = x^{d-k}L$, where $L = (x^k, x^{k-1}y^{a_1}, \dots, y^{a_k})$. Suppose $\mathrm{gr}_I(R)$ is Cohen-Macaulay. Then by Proposition 2, $\mathrm{gr}_L(R)$ is Cohen-Macaulay. By Proposition 2.6 of [6], $R(L)$ is Cohen-Macaulay. Hence the reduction number $r(L)$ is at most one, by Goto-Shimoda theorem [5]. Therefore $r(I) \leq 1$. Therefore by [11], $F(I)$ is Cohen-Macaulay.

Suppose now that $F(I)$ is Cohen-Macaulay. Since $\mu(I^n) = nk + 1$,

$$H(F(I), t) = \frac{1 + (k-1)t}{(1-t)^2}.$$

Therefore by Proposition 5, $I^2 = JI$ for any minimal reduction J of I. Hence, Valabrega-Valla condition implies that $\operatorname{depth} \mathrm{gr}_I(R) \geq s(I) = 2$ so that $\mathrm{gr}_I(R)$ is Cohen-Macaulay. $\square$

Using Proposition 2, we give a simple proof of the fact that for lex-segment ideals the Cohen-Macaulayness of the Rees algebra and the associated graded rings are equivalent. This has been proved for $\mathfrak{m}$-primary ideals in a regular local ring $(R, \mathfrak{m})$. Since we could not find a generalization of this result for the non-$\mathfrak{m}$-primary ideals, we use this opportunity to present a simple proof in the case of lex-segment ideals.

Theorem 7. *Let $R = K[x, y]$ and I a lex-segment ideal. Then $R(I)$ is Cohen-Macaulay if and only if $\mathrm{gr}_I(R)$ is Cohen-Macaulay.*

Proof. Let $I = (x^d, x^{d-1}y^{a_1}, \dots, x^{d-k}y^{a_k})$. If $k = d$, then I is $\mathcal{M}$-primary and hence it follows from Proposition 2.6 of [6]. If $k = 0$, then I is a parameter ideal and hence both the graded algebras are Cohen-Macaulay. Suppose $0 < k < d$. Then $I = x^{d-k}L$, where $L = (x^k, x^{k-1}y^{a_1}, \dots, y^{a_k})$. If $\mathrm{gr}_I(R)$ is Cohen-Macaulay, then by Proposition 2 $\mathrm{gr}_L(R)$ is Cohen-Macaulay. Since L is $\mathcal{M}$-primary by Proposition 2.6 of [6], $R(L)$ is Cohen-Macaulay. Since x^{d-k} is a regular element, $R(L) \cong R(I)$ and hence $R(I)$ is Cohen-Macaulay.

Conversely, suppose $R(I)$ is Cohen-Macaulay. Hence $R(L)$ is Cohen-Macaulay. By Goto-Shimoda theorem, $r(L) \leq 1$. Therefore $r(I) \leq 1$ and hence $\mathrm{gr}_I(R)$ is Cohen-Macaulay. $\square$

Remark 8. The above result together with Theorem 3.4 of [9] implies that for any lex-segment ideal I in $K[x, y]$, $\operatorname{depth} \mathrm{gr}_I(R) = \operatorname{depth} R(I) - 1$.

Now we proceed to prove that the fiber cone has positive depth if and only if the associated graded ring has positive depth. We begin with some properties of lex-segment ideals.

Lemma 9. *Let $I = (x^d, x^{d-1}y^{a_1}, \ldots, x^{d-k}y^{a_k})$ be a lex-segment ideal in $R = K[x, y]$ and $\mathcal{M} = (x, y)$. Then,*

1. *$\mathcal{M}I^n : y = I^n$ for all $n \geq 0$;*
2. *$\mathcal{M}I^{n+1} : I = \mathcal{M}(I^{n+1} : I)$ for all $n \geq 0$.*

Proof. (1) Note that $\mathcal{M}I^n = (x^{nd+1}) + yI^n$ for all $n \geq 0$. Since $\mathcal{M}I^n : y$ is a monomial ideal, it is enough to show that the monomials in $\mathcal{M}I^n : y$ are in I^n. For a polynomial $p \in K[x, y]$, let $\deg_x p$ denote the degree of the polynomial with respect to x, considering it as a polynomial in x with coefficients in $K[y]$ and $\deg_y p$ denote the degree of the polynomial p with respect to y, considering it as a polynomial in y with coefficients in $K[x]$. Let $p \in \mathcal{M}I^n : y$. If $\deg_x p \geq nd$, then clearly $p \in I^n$. Therefore, we may assume that $\deg_x p < nd$. Set $p = x^{nd-r}y^s$ for some $r, s \geq 1$. Since I^n is also a lex-segment ideal, for each $nd - nk \leq t \leq nd$, there exists a unique minimal generator p_t such that $\deg_x p_t = t$. Let $u = x^{nd-r}y^b$ be the minimal generator of I^n with $\deg_x u = nd - r$. Then, $py \in \mathcal{M}I^n$ implies that $s + 1 \geq b + 1$. Hence $s \geq b$. Therefore, $p = x^{nd-r}y^s \in I^n$.

(2) Let $p \in \mathcal{M}I^{n+1} : I$. If $p = x^r$ for some r, then $x^r.x^d \in \mathcal{M}I^{n+1}$. Since any term which is a pure power in x in $\mathcal{M}I^{n+1}$ has degree at least $(n+1)d+1$, we get that $r + d \geq (n + 1)d + 1$. Therefore $r \geq nd + 1$ so that $x^r \in \mathcal{M}I^n \subseteq \mathcal{M}(I^{n+1} : I)$. Now assume that y divides p. Write $p = yp'$. Then $yp'f \in \mathcal{M}I^{n+1}$ for all $f \in I$. Therefore $p'f \in \mathcal{M}I^{n+1} : y = I^{n+1}$ for all $f \in I$. Hence $p' \in I^{n+1} : I$ so that $p \in \mathcal{M}(I^{n+1} : I)$. $\square$

Theorem 10. *Let I be a lex-segment ideal in R. Then* $\operatorname{depth} \mathrm{gr}_I(R) > 0$ *if and only if* $\operatorname{depth} F(I) > 0$.

Proof. Let $\operatorname{depth} \mathrm{gr}_I(R) > 0$. Then, $I^{n+1} : I = I^n$ for all $n \geq 0$. Therefore, $\mathcal{M}I^{n+1} : I = \mathcal{M}(I^{n+1} : I) = \mathcal{M}I^n$ for all $n \geq 0$. Hence $\operatorname{depth} F(I) > 0$.

Conversely, assume that $\operatorname{depth} F(I) > 0$. Then, $(\mathcal{M}I^{n+1} : I) \cap I^n = \mathcal{M}I^n$ for all $n \geq 0$. We need to prove that $I^{n+1} : I = I^n$ for all $n \geq 0$. Suppose that there exists an n such that $I^n \subsetneq I^{n+1} : I$. Since I is a monomial ideal, $I^{n+1} : I$ is generated by monomials and hence there exists a monomial generator p of $I^{n+1} : I$ such that $p \notin I^n$. Since $I^{n+1} : I$ is also a lex-segment ideal, we can write $p = x^{nd-t}y^s$ for some s. Let $q \in I^n$ be the minimal generator of I^n such that $\deg_x q = nd - t$. Then $\deg_y q > s$, since $p \notin I^n$. Therefore $qI \subseteq \mathcal{M}I^{n+1}$. Hence $q \in (\mathcal{M}I^{n+1} : I) \cap I^n = \mathcal{M}I^n$. This contradicts the fact that q is a minimal generator of I^n. Therefore $I^{n+1} : I = I^n$ for all $n \geq 0$. Hence $\operatorname{depth} \mathrm{gr}_I(R) > 0$. $\square$

3. Examples

In this section we give some examples to show that depths of fiber cone and the associated graded rings are not related in general.

Example 11. Let $I = (x^5, x^3y^3, xy^7, y^9) \subset R = K[x, y]$. Then it can be seen that $x^2y^6 \in I^2 : I$, but not in I. Therefore $\operatorname{depth} \operatorname{gr}_I(R) = 0$. It can also be seen that $\mathcal{M}I^{n+1} : I = \mathcal{M}I^n$ for all $n \geq 1$. Therefore $\operatorname{depth} F(I) > 0$.

Example 12. Let $A = K[\![t^6, t^{11}, t^{15}, t^{31}]\!]$, $I = (t^6, t^{11}, t^{31})$ and $J = (t^6)$. Then, it can easily be verified that $\ell(I^2/JI) = 1$ and $I^3 = JI^2$. Since $I^2 \cap J = JI$, $G(I)$ is Cohen-Macaulay. It can also be seen that $t^{37} \in \mathcal{M}I^2$, but $t^{37} \notin \mathcal{M}JI$. Therefore $F(I)$ is not Cohen-Macaulay.

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