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Some Small-Centralizer Properties for Rings

Howard E. Bell[*] **Abraham A. Klein**

Department of Mathematics, Brock University
St. Catharines, Ontario, Canada L2S 3A1
e-mail: hbell@brocku.ca

School of Mathematical Sciences, Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
e-mail: aaklein@post.tau.ac.il

Abstract. We characterize rings R in which certain elements x have the property that $C_R(x)$ (resp. the set of zero divisors in $C_R(x)$) is finite. We also explore the consequences of an assumption that certain x satisfy $C_R(x) = \langle x \rangle$.

1. Introduction

Let R be a ring with center Z, and let D be the set of zero divisors of R. For $x \in R$, let $C_R(x)$ be the centralizer of x in R. We study rings in which $C_R(x)$ is finite for all $x \in R \backslash Z$ and rings in which $C_R(x) \cap D$ is finite for all $x \in D \backslash Z$. In the first case we show that R is either finite or commutative; in the second case we show that either R is finite or $D \subseteq Z$.

As in [2], we call an element $x \in R$ extremely noncommutative if $C_R(x) = x\mathbb{Z}[x]$ – i.e. if $C_R(x)$ is the subring generated by x. Our most difficult result deals with rings such that each element of $D \backslash Z$ is extremely noncommutative.

Let us fix some additional notation and terminology. Let $N = N(R)$ denote the set of nilpotent elements of R, and $T = T(R)$ the set of elements of finite additive

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order. For $x \in R$, let $\langle x \rangle$ and $A(x)$ be respectively the subring generated by x and the two-sided annihilator of x. For a subring S of R, let $[R : S]$ denote the index of $(S, +)$ in $(R, +)$; and for a subset X of R, let $|X|$ denote the cardinality of X. An element $x \in R$ is called periodic if there exist distinct positive integers m, n for which $x^m = x^n$, and the ring R is called periodic if each of its elements is periodic.

The following lemmas will be useful.

Lemma 1.1. [6] *If R is a periodic ring with $N \subseteq Z$, then R is commutative.*

Lemma 1.2. [3] *Let R be a ring such that for each $x \in R$ there exist a positive integer m and a polynomial $p(X) \in \mathbb{Z}[X]$ for which $x^m = x^{m+1}p(x)$. Then R is periodic.*

Lemma 1.3. [7] *If R is infinite and $x \in N$, then $|A(x)| = |R|$. In particular, $A(x)$ is infinite.*

2. Finite-centralizer conditions

Theorem 2.1. *If R is a ring such that $C_R(x)$ is finite for all $x \in R \backslash Z$, then R is either finite or commutative.*

Proof. Suppose that R is infinite. Since $A(x) \subseteq C_R(x)$, it follows by Lemma 1.3 that $N \subseteq Z$. Suppose also that R is not commutative and $x \in R \backslash Z$. Since $\langle x \rangle \subseteq C_R(x)$, $\langle x \rangle$ is finite and hence x is periodic; and since Z is clearly finite, central elements are periodic as well. Thus, R is a noncommutative periodic ring with $N \subseteq Z$, contrary to Lemma 1.1. Therefore R must be commutative. $\square$

Theorem 2.2. *Let R be a ring such that $C_R(x) \cap D$ is finite for all $x \in D \backslash Z$. Then either R is finite or $D \subseteq Z$.*

Proof. Note that if S is any infinite subring of R such that $C_S(x) = C_R(x) \cap S$ is finite for all $x \in S \backslash Z$, S is commutative by Theorem 2.1 and therefore $S \subseteq Z$. In particular, if S is any infinite subring contained in D, $S \subseteq Z$.

Suppose that $D \backslash Z \neq \phi$, and assume without loss of generality that $xy = 0$ with $x \in D \backslash Z$ and $y \neq 0$. If $A_l(y)$ is infinite, we have $x \in A_l(y) \subseteq Z$ – a contradiction; therefore $A_l(y)$ is finite. For each $w \in A_l(y)$, consider the map $f_w : R \to A_l(y)$ given by $f_w(r) = rw$. By applying the first isomorphism theorem for additive groups, we see that $ker(f_w) = A_l(w)$ is of finite index in R; hence $S = A_l(A_l(y))$ is of finite index and therefore is infinite. Thus $S \subseteq Z$; and since $S \subseteq A_l(x)$, we see that $A(x)$ is an infinite subset of $C_R(x) \cap D$ – a contradiction. $\square$

3. An extreme non-commutativity condition

In [2], the following theorem is proved.

Theorem 3.1. *If R is a ring in which all noncentral elements are extremely noncommutative, then R is either finite or commutative.*

In [8], we were led to consider an infinite noncentral subring A with subring $B = A \cap Z$ such that $A^2 \subseteq Z$, $A = \langle a \rangle$ for all $a \in A \backslash B$, and $[A : B]$ is a prime. In the sections headed *Proof of Theorem* 2.1 and *Completion of proof of Theorem* 2.1, we showed that such a subring cannot exist. Thus, we proved, but did not explicitly state, the following lemma.

Lemma 3.2. *Let R be an infinite noncommutative ring. Then R contains no infinite noncentral subring A such that $A^2 \subseteq Z$, $A = \langle a \rangle$ for all $a \in A \backslash Z$, and $[A : A \cap Z]$ is a prime.*

The principal theorem of this section, which we now state, is obtained by weakening the extreme noncommutativity hypothesis in Theorem 3.1.

Theorem 3.3. *Let R be a ring in which every element of $D \backslash Z$ is extremely noncommutative. Then either R is finite or $D \subseteq Z$.*

The proof will be presented as a series of lemmas, the first of which is almost obvious. In each lemma, it will be assumed without explicit mention that R is a ring in which every element of $D \backslash Z$ is extremely noncommutative.

Lemma 3.4. *If $D \nsubseteq Z$, then R is indecomposable. Hence R has no nonzero central idempotent zero divisors.*

Lemma 3.5. *If $N \nsubseteq Z$, then R is finite.*

Proof. Since Z centralizes $N \backslash Z$, $Z \subseteq N$. We show first that all zero divisors are periodic. This is clearly true for nilpotent elements, so we consider $d \in D \backslash N$. Then $d^2 \notin N$, so $d^2 \notin Z$ and hence $d \in \langle d^2 \rangle$. Thus there exists $p(X) \in \mathbb{Z}[X]$ such that $d = d^2 p(d)$. Since each element of D is in some subring of zero divisors, Lemma 1.2 shows that zero divisors are periodic.

Next we show that $D \subseteq T(R)$. Let $d \in D$ and $D' = \langle d \rangle$. By [1, Lemma 1(c)], $d = a + u$ with $u \in N$ and a a power of d such that $a^n = a$ for some $n > 1$. Now $e = a^{n-1}$ is an idempotent such that $a = ae$; and since e is in the periodic ring D' , $2e$ is periodic, hence $e \in T(R)$ and $a \in T(R)$. We now need to show that $N \subseteq T(R)$; and since $Z = Z \cap N \subseteq (N \backslash Z) - (N \backslash Z)$, it suffices to show that $N \backslash Z \subseteq T(R)$. Let $u \in N \backslash Z$ and suppose $u^k \in T(R)$ for $k \geq 2$. Since $u \notin Z$, there exists $n \geq 2$ such that $nu \notin Z$; and it follows that $u \in \langle nu \rangle$, so that there exist $c_1, c_2, \ldots, c_t \in \mathbb{Z}$ such that $u = c_1(nu) + c_2(nu)^2 + \cdots + c_t(nu)^t$. Multiplying by u^{k-2} gives $(1 - c_1 n)u^{k-1} \in T(R)$ and hence $u^{k-1} \in T(R)$. By backward induction, $u \in T(R)$.

We now know that if $d \in D \backslash Z$, $\langle d \rangle$ is finite and consequently $C_R(d)$ is finite. Thus, R is finite by Theorem 2.2. □

Lemma 3.6.

(i) *If $D \nsubseteq Z$ and $N \subseteq Z$, then $d^n \in Z$ for all $d \in D$ and $n \geq 2$.*

(ii) *If $D \nsubseteq Z$, there exists a prime p such that $pD \subseteq Z$.*

Proof. By Lemma 3.4, R has no nonzero idempotent zero divisors. Hence, we need only adapt in an obvious way the proof of Lemma 2.8 of [2]. □

Lemma 3.7. *If $N \subseteq Z$, then every subring of zero divisors is commutative.*

Proof. Let H be any subring of zero divisors, and let $h \in H \backslash Z(H)$. Then $C_H(h) = \langle h \rangle$, so H is either finite or commutative by Theorem 3.1. Moreover, if H is finite, it is commutative by Lemma 1.1. □

Lemma 3.8. *Let R be infinite with $D \not\subseteq Z$ and $N \subseteq Z$. Then*

(i) *D is infinite;*

(ii) *D is a commutative ideal and hence $D^2 \subseteq Z$;*

(iii) *$D = \langle d \rangle$ for every $d \in D \backslash Z$;*

(iv) *$[D : D \cap Z] = p$ for some prime p.*

Proof. (i) follows immediately from an old theorem of Ganesan [4,5], which asserts that any ring R with $1 \leq |D \backslash \{0\}| < \infty$ must be finite.

(ii) Use the proof of Lemma 2.4 of [8], which employs Lemma 3.7.

(iii) Let $d \in D \backslash Z$. By (ii), $D \subseteq C_R(d) = \langle d \rangle$; and obviously $\langle d \rangle \subseteq D$.

(iv) Since we now know that D is an additive subgroup, the result follows from (iii) and Lemma 3.6.

Proof of Theorem 3.3. Assume that $D \not\subseteq Z$. By Lemmas 3.8 and 3.2, we cannot have $N \subseteq Z$; hence R is finite by Lemma 3.5. □

Theorem 3.3 does not provide a characterization of rings such that all $d \in D \backslash Z$ are extremely noncommutative, since we do not have complete information about the finite examples. We do, however, have partial information.

Theorem 3.9. *Let R be a finite ring with $D \not\subseteq Z$ such that each $d \in D \backslash Z$ is extremely noncommutative. Then either R is isomorphic to a matrix ring of form $GF(p)e_{11} + GF(p)e_{12}$ or $GF(p)e_{11} + GF(p)e_{21}$, or R is nil.*

Proof. If $R = D$, the result follows by Theorem 2.11 of [2]. Otherwise, if $x \in R \backslash D$, some power of x is a regular idempotent, necessarily 1. Now by Lemma 1.1, there exists $u \in N \backslash Z$; and since $1 + u \in C_R(u)$, $1 + u \in \langle u \rangle$. But this is not possible, since $\langle u \rangle$ is a nil ring and $1 + u$ is invertible. □

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