Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 51 (2010), No. 1, 209-227.

# Projectivity and Flatness over the Colour Endomorphism Ring of a Finitely Generated Graded Comodule

**T. Guédénon**

*110 Penworth Drive S.E., Calgary, AB, T2A 5H4, Canada*
*e-mail: guedenth@yahoo.ca*

**Abstract.** Let $k$ be a field, $G$ an abelian group with a bicharacter, $A$ a colour algebra; i.e., an associative graded $k$-algebra with identity, $\mathcal{C}$ a graded $A$-coring that is projective as a right $A$-module, $\mathcal{C}^*$ the graded dual ring of $\mathcal{C}$ and $\Lambda$ a left graded $\mathcal{C}$-comodule that is finitely generated as a graded right $\mathcal{C}^*$-module. We give necessary and sufficient conditions for projectivity and flatness of a graded module over the colour endomorphism ring ${}^{\mathcal{C}}END(\Lambda)$.

## 0. Introduction

The notion of graded corings (except graded algebras and graded coalgebras) rarely appears in the literature on corings. The only paper we know where this notion appears is [8]. In the present paper we will give some conditions to test projectivity or flatness over the colour endomorphism ring of a finitely generated graded $\mathcal{C}$-comodule, where $\mathcal{C}$ is a graded coring. Let $k$ be a field, $A$ a $k$-algebra, $\mathcal{C}$ an $A$-coring, ${}^*\mathcal{C}$ the left dual ring of $\mathcal{C}$ and $\Lambda$ a right $\mathcal{C}$-comodule that is finitely generated as a left ${}^*\mathcal{C}$-module. In [11], we gave necessary and sufficient conditions for projectivity and flatness over the endomorphism ring $End^{\mathcal{C}}(\Lambda)$ of $\Lambda$. In the present paper, we will extend these results to a $G$-graded $A$-coring $\mathcal{C}$, where $G$ is an abelian group with a bicharacter and $A$ is a colour algebra; i.e., a graded associative $k$-algebra with identity. More precisely, let us denote by $\mathcal{C}^* = HOM_A(\mathcal{C}_A, A_A)$ the largest graded vector space contained in $Hom_A(\mathcal{C}_A, A_A)$. It has a colour algebra structure. Let $\Lambda$ be a graded left $\mathcal{C}$-comodule that is finitely generated as a graded right $\mathcal{C}^*$-module. We give necessary and sufficient conditions for projectivity and flatness over the colour endomorphism ring ${}^{\mathcal{C}}END(\Lambda)$ of $\Lambda$. The presence of the bicharacter makes the difference with the classical gradation. These results are interesting when $\mathcal{C} = A$, or when $\mathcal{C}$ contains a grouplike element, or when $\mathcal{C}$ comes from a graded entwining structure with respect to a bicharacter. If $\mathcal{C} = A$ then ${}^{gr-A}\mathcal{M}$ is the category of graded left $A$-modules and $A^*$ is isomorphic to the opposite algebra $A^{op}$ of $A$. If $\mathcal{C}$ contains a grouplike element $X$, then $A$ is a graded left $\mathcal{C}$-comodule that is finitely generated as a graded right $\mathcal{C}^*$-module. In this case, ${}^{\mathcal{C}}END(A)$ is the colour subring of $(\mathcal{C}, X)$-coinvariants of $A$. Our techniques and methods are inspired from [10], [7] and [11].

## 1. Preliminary results

Throughout the paper, $k$ is a field, $G$ is an abelian group and $(./.)$ is a bicharacter on $G$; i.e., a map from $G \times G$ into $k^\times$ satisfying:

$$(x/y) = (y/x)^{-1} \quad \text{and} \quad (x/y+z) = (x/y)(x/z).$$

These two relations imply that $(x+y/z) = (x/z)(y/z)$. If $M$ and $N$ are vector spaces $Hom(M,N)$ is the vector space of $k$-linear maps from $M$ to $N$.

A vector space $A$ is $G$-graded or graded if $A = \oplus_{x \in G} A_x$, where the $A_x$ are vector subspaces of $A$. An algebra $A$ (not necessarily associative with identity) is said to be graded if $A$ is a graded vector space as above and the $A_x$ satisfy $A_x A_y \subseteq A_{x+y}$. According to [8, Section 1], a colour algebra is an associative graded algebra. In what follows we assume that all colour algebras are unital. We will consider $k$ as a colour algebra with the trivial gradation. Given colour algebras $A$ and $B$, a morphism of colour algebras $A \to B$ is a morphism of algebras which is homogeneous of degree 0. Let $m$ be an element of a graded vector space $M$. If $m$ is homogeneous, we denote by $|m|$ its degree. If $|m|$ occurs in some expression, this means that we regard $m$ as a homogeneous element and that the expression extends to the other elements by linearity. Let $M$ and $N$ be graded vector spaces. An element of $Hom(M,N)$ is homogeneous of degree $x$ if $f(M_y) \subseteq N_{x+y}$ for all $y \in G$. We denote by $HOM(M,N)_x$ the vector subspace of $Hom(M,N)$ whose elements are homogeneous of degree $x$ and we will set $HOM(M,N) = \oplus_{x \in G} HOM(M,N)_x$. Clearly, $HOM(M,N)$ is the largest graded vector space contained in $Hom(M,N)$. The space $HOM(M,N)$ is denoted $Hom_k(M,N)_G$ in [9]. By [13, Corollary 1.2.11], $HOM(M,N) = Hom(M,N)$ if $G$ is finite or if $M$ is finite-dimensional. By [9], $HOM(M,M)$ is a colour algebra. If $M$, $N$, $M'$ and $N'$ are graded vector spaces and if $f : M \to M'$ and $g : N \to N'$ are homogeneous linear maps then $(f \otimes g)(m \otimes n) = (|g|/|m|) f(m) \otimes g(n)$. We will denote by ${}_{gr-k}\mathcal{M}$ the category of graded $k$-vector spaces. The morphisms of ${}_{gr-k}\mathcal{M}$ are the homogeneous $k$-linear maps of degree 0; we call them the graded $k$-linear maps. Let $N$ be a graded vector space. For every $x$ in $G$, the $x$-suspension of $N$ is the graded vector space $N(x)$ obtained from $N$ by a shift of the gradation by $x$. As vector spaces, $N$ and $N(x)$ coincide but the gradations are related by $N(x)_y = N_{x+y}$ for all $y \in G$.

Let $A$ be a colour algebra. A left $A$-module $M$ is called a graded left $A$-module if $M$ admits a decomposition as a direct sum of vector spaces $M = \oplus_{x\in G} M_x$ such that $A_x M_y \subseteq M_{x+y}$; $\forall x,\, y \in G$.

**Definition 1.1.** *Let $M$, $N$ be graded left $A$-modules. A homogeneous element $f$ of $Hom(M, N)$ is colour left $A$-linear if $f(am) = (|f|/|a|)af(m)$ for all $a \in A$.*

If $M$, $N$ are graded left $A$-modules, we let ${}_AHOM(M,N)_x$ denote the vector subspace of $Hom(M,N)$ whose elements are colour $A$-linear of degree $x$. So the colour left $A$-linear maps of degree 0 are exactly the left $A$-linear maps of degree 0; i.e., ${}_AHOM(M,N)_0 = {}_AHom(M,N) \cap HOM(M,N)_0$. We define ${}_AHOM(M,N)$ to be the sum of these subspaces; the sum is direct: ${}_AHOM(M,N) = \oplus_{x\in G}\, {}_AHOM(M,N)_x$. We call ${}_AHOM(M,N)$ the subspace of colour left $A$-linear maps of $Hom(M,N)$. Contrary to the classical gradation, if $A \neq k$ and if the bicharacter is not trivial, there is no comparison relation between ${}_AHOM(M,N)$ and ${}_AHom(M,N)$ even if $M$ is finitely generated as an $A$-module or if $G$ is finite. If $G = \mathbb{Z}/2\mathbb{Z}$, colour $A$-linear maps are called $A$-superlinear in [16]. We will denote by ${}_{gr-A}\mathcal{M}$ the category of graded left $A$-modules. The morphisms of ${}_{gr-A}\mathcal{M}$ are the colour left $A$-linear maps of degree 0; we call them the graded left $A$-linear maps. It is well known that ${}_{gr-A}\mathcal{M}$ is a Grothendieck category. We can define in a similar way a graded right $A$-module and a graded $A$-bimodule. A colour right $A$-linear map of degree $x$ is just a homogeneous right $A$-linear map of degree $x$. To establish our main results we will need the following well-known results of graded ring theory.

- If $N$ is a graded left (right) $A$-module, $N(x)$ is a graded left (right) $A$-module which coincides with $N$ as a graded left (right) $A$-module.

- An object of ${}_{gr-A}\mathcal{M}$ is projective (resp. flat) in ${}_{gr-A}\mathcal{M}$ if and only if it is projective (resp. flat) in ${}_A\mathcal{M}$, the category of left $A$-modules.

- An object of ${}_{gr-A}\mathcal{M}$ is free in ${}_{gr-A}\mathcal{M}$ if it has an $A$-basis consisting of homogeneous elements or equivalently, if it is isomorphic to some $\oplus_{i\in I} A(x_i)$, where $(x_i, i \in I)$ is a family of elements of $G$.

- An object of ${}_{gr-A}\mathcal{M}$ is called finitely generated if it is a quotient of a free graded module of finite rank $\oplus_{i\leq m} A(x_i)$, where the $x_i \in G$ and $m$ is a natural integer.

- Any object of ${}_{gr-A}\mathcal{M}$ is a quotient of a free object in ${}_{gr-A}\mathcal{M}$, and any projective object in ${}_{gr-A}\mathcal{M}$ is isomorphic in ${}_{gr-A}\mathcal{M}$ to a direct summand of a free object.

- An object of ${}_{gr-A}\mathcal{M}$ is flat in ${}_{gr-A}\mathcal{M}$ if and only if it is the inductive limit of finitely generated free objects in ${}_{gr-A}\mathcal{M}$.

- An object $\Lambda$ of ${}_{gr-A}\mathcal{M}$ is called finitely presented if there is an exact sequence $\oplus_{i\leq m} A(x_i) \to \oplus_{j\leq n} A(y_j) \to \Lambda \to 0$ for $x_i,\, y_j \in G$ and some natural integers $m$ and $n$. A finitely presented graded module is finitely generated.

**Lemma 1.2.** *Let $A$ be a colour algebra and $M$ a graded left $A$-module which is generated as $A$-module by a homogeneous element $m$ of degree 0. Then $M$ is finitely generated as a graded left $A$-module.*

*Proof.* We have $M = Am$. The $k$-linear map $f : A \to M$; $a \mapsto am$ is surjective, homogeneous of degree 0 and left $A$-linear. So $f$ is an epimorphism in ${}_{gr-A}\mathcal{M}$. $\square$

An $A$-coring $\mathcal{C}$ is an $A$-bimodule together with two $A$-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ such that the usual coassociativity and counit properties hold. Let $\mathcal{C}$ be an $A$-coring. A left $\mathcal{C}$-comodule is a left $A$-module $M$ together with a left $A$-linear map $\rho_{M,\mathcal{C}} : M \to \mathcal{C} \otimes_A M$ such that

$$(\epsilon_{\mathcal{C}} \otimes_A id_M) \circ \rho_{M,\mathcal{C}} = id_M, \quad \text{and} \quad (\Delta_{\mathcal{C}} \otimes_A id_M) \circ \rho_{M,\mathcal{C}} = (id_{\mathcal{C}} \otimes_A \rho_{M,\mathcal{C}}) \circ \rho_{M,\mathcal{C}}.$$

For more details on corings, we refer to [1], [2], [3], [4] and [5].

An $A$-coring $\mathcal{C}$ is called a graded $A$-coring if $\mathcal{C}$ admits a decomposition as a direct sum of vector spaces $\mathcal{C} = \oplus_x \mathcal{C}_x$ such that $\mathcal{C}$ is a graded $A$-bimodule, and $\Delta_{\mathcal{C}}$ and $\epsilon_{\mathcal{C}}$ are graded left and right $A$-linear maps. Note that $\epsilon_{\mathcal{C}}(c) = 0$ if $c$ is homogeneous of degree $|c| \neq 0$. We use the notation-type of Sweedler-Heyneman for $\Delta_{\mathcal{C}}$ but we will omit the parentheses on subscripts. So for every homogeneous element $c \in \mathcal{C}$ we will write $\Delta_{\mathcal{C}}(c) = \sum_{|c|} c_1 \otimes_A c_2$; where $\sum_{|c|} = \sum_{|c_1|+|c_2|=|c|}$. We have $\sum_{|c|} \sum_{|c_1|} c_{11} \otimes_A c_{12} \otimes_A c_2 = \sum_{|c|} \sum_{|c_2|} c_1 \otimes_A c_{21} \otimes_A c_{22}$. Note that $\epsilon_{\mathcal{C}}(c) = 0$ if $|c| \neq 0$. A left $\mathcal{C}$-comodule $M$ is called a graded left $\mathcal{C}$-comodule if $M$ admits a decomposition as a direct sum of vector spaces $M = \oplus_x M_x$ such that $\rho_{M,\mathcal{C}}$ is homogeneous of degree 0; i.e., $\rho_{M,\mathcal{C}}$ is a graded left $A$-linear map. We will write $\rho_{M,\mathcal{C}}(m) = \sum_{|m|} m_{(-1)} \otimes_A m_{(0)}$, where $\sum_{|m|} = \sum_{|m_{(-1)}|+|m_{(0)}|=|m|}$.

Any colour algebra $A$ is a graded $A$-coring called the trivial $A$-coring, and a graded $k$-coalgebra is a graded $k$-coring. A morphism of graded left $\mathcal{C}$-comodules $f : M \to N$ is a morphism in ${}_{gr-A}\mathcal{M}$ such that

$$\rho_{N,\mathcal{C}} \circ f = (id_{\mathcal{C}} \otimes_A f) \circ \rho_{M,\mathcal{C}}, \quad \text{that is}$$

$$\sum_{|m|} f(m)_{(-1)} \otimes_A f(m)_{(0)} = \sum_{|m|} m_{(-1)} \otimes_A f(m_{(0)}) \quad \forall m \in M.$$

A morphism of graded left $\mathcal{C}$-comodule will be called a graded left $\mathcal{C}$-colinear map. We denote by ${}^{gr-\mathcal{C}}\mathcal{M}$ the category of graded left $\mathcal{C}$-comodules. The morphisms of ${}^{gr-\mathcal{C}}\mathcal{M}$ are the graded left $\mathcal{C}$-colinear maps. The category ${}^{gr-\mathcal{C}}\mathcal{M}$ has direct sums. If $\mathcal{C}$ is projective as a right $A$-module, then ${}^{gr-\mathcal{C}}\mathcal{M}$ is a Grothendieck category ([4] for the ungraded case).

**Definition 1.3.** *Let $\mathcal{C}$ be a graded $A$-coring and $M$, $N$ be objects of ${}^{gr-\mathcal{C}}\mathcal{M}$. A homogeneous element $f \in Hom(M, N)$ is colour left $\mathcal{C}$-colinear if $f$ is colour left $A$-linear and $\rho_{N,\mathcal{C}} \circ f = (id_{\mathcal{C}} \otimes_A f) \circ \rho_{M,\mathcal{C}}$.*

It follows from Definition 1.3 that a graded left $\mathcal{C}$-colinear map is a colour left $\mathcal{C}$-colinear map of degree 0. If $M$ and $N$ are objects of ${}^{gr-\mathcal{C}}\mathcal{M}$ and $x \in G$, we will denote by ${}^{\mathcal{C}}HOM(M, N)_x$ the vector subspace of $Hom(M, N)$ whose elements are colour left $\mathcal{C}$-colinear of degree $x$. So we have

$${}^{\mathcal{C}}HOM(M, N)_x = \{f \in {}_AHOM(M, N)_x, \sum_{|m|} f(m)_{(-1)} \otimes_A f(m)_{(0)} =$$

$$\sum_{|m|}(|f|, |m_{(-1)}|)m_{(-1)} \otimes_A f(m_{(0)})\}.$$

We will set ${}^{\mathcal{C}}HOM(M, N) = \oplus_{x\in G}{}^{\mathcal{C}}HOM(M, N)_x$. We call ${}^{\mathcal{C}}HOM(M, N)$ the subspace of colour left $\mathcal{C}$-colinear maps of $Hom(M, N)$. We can define in a similar way a graded right $\mathcal{C}$-comodule. A homogeneous colour right $\mathcal{C}$-colinear map is just a homogeneous right $\mathcal{C}$-colinear.

If $N$ is a graded left $\mathcal{C}$-comodule, then for every $x$ in $G$, the $x$-suspension $N(x)$ is a graded left $\mathcal{C}$-comodule which coincides with $N$ as a $\mathcal{C}$-comodule. By [17], the linear map $i_{-x} : N \to N(x)$ defined by $i_{-x}(n) = (-x/|n|)n$ is bijective and homogeneous of degree $-x$. It is obviously colour left $\mathcal{C}$-colinear.

**Lemma 1.4.** *Let $\mathcal{C}$ be a graded $A$-coring, and $M$, $N$ be graded left $\mathcal{C}$-comodules. For every $x \in G$, the linear map ${}^{\mathcal{C}}HOM(M, N)_x \to {}^{\mathcal{C}}HOM(M, N(x))_0; f \mapsto i_{-x} \circ f$, where $i_{-x}$ is defined above is an isomorphism of vector spaces.*

**Lemma 1.5.** *Let $P$ be an object of ${}^{gr-\mathcal{C}}\mathcal{M}$. Then the functor ${}^{\mathcal{C}}HOM(P, -)$ : ${}^{gr-\mathcal{C}}\mathcal{M} \to_{gr-k} \mathcal{M}$ is left exact.*

*Proof.* Let $0 \to L \to M \to N \to 0$ be an exact sequence in ${}^{gr-\mathcal{C}}\mathcal{M}$; so $0 \to L(x) \to M(x) \to N(x) \to 0$ is exact in ${}^{gr-\mathcal{C}}\mathcal{M}$ for every $x \in G$. By [13, Corollary 1.2.2], $P$ is projective in ${}_{gr-k}\mathcal{M}$. So the sequence

$$0 \to HOM(P, L(x))_0 \to HOM(P, M(x))_0 \to HOM(P, N(x))_0 \to 0$$

is exact for every $x \in G$. It follows from Lemma 1.4 that

$$0 \to HOM(P, L)_x \to HOM(P, M)_x \to HOM(P, N)_x \to 0$$

is an exact sequence for every $x \in G$. We know that $i \circ f \in {}^{\mathcal{C}}HOM(P, M)_x$ for all $f \in {}^{\mathcal{C}}HOM(P, L)_x$. This means that the sequence

$$0 \to {}^{\mathcal{C}}HOM(P, L)_x \to {}^{\mathcal{C}}HOM(P, M)_x \to {}^{\mathcal{C}}HOM(P, N)_x$$

is exact for every $x \in G$; i.e.,

$$0 \to {}^{\mathcal{C}}HOM(P, L) \to {}^{\mathcal{C}}HOM(P, M) \to {}^{\mathcal{C}}HOM(P, N)$$

is an exact sequence. So the functor ${}^{\mathcal{C}}HOM(P, -)$ is left exact. $\square$

We say that an object $P$ of ${}^{gr-\mathcal{C}}\mathcal{M}$ is projective if the functor ${}^{\mathcal{C}}HOM(P, -)_0$ is exact.

**Lemma 1.6.** *Let $\mathcal{C}$ be a graded $A$-coring. An object $P$ of ${}^{gr-\mathcal{C}}\mathcal{M}$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$ if and only if the functor ${}^{\mathcal{C}}HOM(P, -)$ is exact.*

*Proof.* Assume that ${}^{\mathcal{C}}HOM(P, -)$ is exact in ${}^{gr-\mathcal{C}}\mathcal{M}$. Let $0 \to L \to M \to N \to 0$ be an exact sequence in ${}^{gr-\mathcal{C}}\mathcal{M}$. So the sequence $0 \to {}^{\mathcal{C}}HOM(P, L) \to {}^{\mathcal{C}}HOM(P, M) \to {}^{\mathcal{C}}HOM(P, N) \to 0$ is exact. It follows that the sequence $0 \to {}^{\mathcal{C}}HOM(P, L)_0 \to {}^{\mathcal{C}}HOM(P, M)_0 \to {}^{\mathcal{C}}HOM(P, N)_0 \to 0$ is exact. This means that $P$ is a projective object in ${}^{gr-\mathcal{C}}\mathcal{M}$. Assume that $P$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$. Let $0 \to L \to M \to N \to 0$ be an exact sequence in ${}^{gr-\mathcal{C}}\mathcal{M}$. Clearly, $0 \to L(x) \to M(x) \to N(x) \to 0$ is an exact sequence in ${}^{gr-\mathcal{C}}\mathcal{M}$ for every $x \in G$. By the projectivity of $P$, the sequence

$$0 \to {}^{\mathcal{C}}HOM(P, L(x))_0 \to {}^{\mathcal{C}}HOM(P, M(x))_0 \to {}^{\mathcal{C}}HOM(P, N(x))_0 \to 0$$

is exact for every $x \in G$. Using Lemma 1.4, we get that the sequence $0 \to {}^{\mathcal{C}}HOM(P, L) \to {}^{\mathcal{C}}HOM(P, M) \to {}^{\mathcal{C}}HOM(P, N) \to 0$ is exact. □

Let us consider $A$ as a graded right $A$-module. By [13], $HOM_A(\mathcal{C}_A, A_A) = \oplus_x HOM_A(\mathcal{C}_A, A_A)_x$ is a graded vector space: it is the largest graded vector space contained in $Hom_A(\mathcal{C}_A, A_A)$. We write $\mathcal{C}^*_x = HOM_A(\mathcal{C}_A, A_A)_x$ and $\mathcal{C}^* = HOM_A(\mathcal{C}_A, A_A)$. Then $\mathcal{C}^*$ is a colour algebra called the graded right dual ring of $\mathcal{C}$ (see [5, 17.8] for the ungraded case): the multiplication is defined by $f\#g = (|f|,|g|)g \circ (f \otimes_A id_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}$; i.e., $f\#g(c) = \sum_{|c|}(|f|,|g|)g(f(c_1)c_2)$ for all colour right $A$-linear maps $f$, $g$: $\mathcal{C} \to A$ and homogeneous element $c \in \mathcal{C}$; where $\Delta_{\mathcal{C}}(c) = \sum_{|c|} c_1 \otimes_A c_2$. The unit of $\mathcal{C}^*$ is $\epsilon_{\mathcal{C}}$ and there is a morphism of colour algebras $i : A^{op} \to \mathcal{C}^*$ defined by $i(a)(c) = (|a|,|c|)\epsilon_{\mathcal{C}}(c)a$. We will denote by $\mathcal{M}_{gr-\mathcal{C}^*}$ the category of graded right $\mathcal{C}^*$-modules. Any graded left $\mathcal{C}$-comodule $M$ is a graded right $\mathcal{C}^*$-module: the action is defined by $m.f = \sum_{|m|}(|m_{(-1)}|,|f|)f(m_{(-1)})m_{(0)}$. If $\mathcal{C}$ is projective as a right $A$-module, then ${}^{gr-\mathcal{C}}\mathcal{M}$ is a full subcategory of $\mathcal{M}_{gr-\mathcal{C}^*}$; i.e., ${}^{\mathcal{C}}HOM(M, N) = HOM_{\mathcal{C}^*}(M, N)$ for any $M, N \in {}^{gr-\mathcal{C}}\mathcal{M}$. As a consequence, an object of ${}^{gr-\mathcal{C}}\mathcal{M}$ that is projective in $\mathcal{M}_{gr-\mathcal{C}^*}$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$. Another consequence is that if $M$ and $N$ are objects of ${}^{gr-\mathcal{C}}\mathcal{M}$ with $M$ finitely generated as a right $\mathcal{C}^*$-module, then ${}^{\mathcal{C}}HOM(M, N) = HOM_{\mathcal{C}^*}(M, N) = Hom_{\mathcal{C}^*}(M, N)$.

Given two graded left $\mathcal{C}$-comodules $\Lambda$ and $N$, the graded vector space ${}^{\mathcal{C}}HOM(\Lambda, N)$ is a graded left module over the colour endomorphism ring $B = {}^{\mathcal{C}}END(\Lambda)$ of $\Lambda$: the action is given by $bf = (|b|,|f|)(f \circ b); \forall f \in {}^{\mathcal{C}}HOM(\Lambda, N), b \in B$. This defines a functor $F' = {}^{\mathcal{C}}HOM(\Lambda, -) : {}^{gr-\mathcal{C}}\mathcal{M} \to {}_{gr-B}\mathcal{M}$. Let us consider $\Lambda$ as a graded right $B$-module by $\lambda.b = (|\lambda|/|b|)b(\lambda)$. So $\Lambda$ is a graded $(A, B)$-bimodule. For any $P \in {}_{gr-B}\mathcal{M}$, $\Lambda \otimes_B P$ is a graded left $\mathcal{C}$-comodule with the coaction $\rho_{P,\mathcal{C}} = \rho_{\Lambda,\mathcal{C}} \otimes_B id_P$.

**Lemma 1.7.** *Let $\Lambda$ and $N$ be graded left $\mathcal{C}$-comodules and $P$ be a graded left $B$-module. For every $x \in G$, the canonical linear map*

$$\phi : {}^{\mathcal{C}}HOM(\Lambda \otimes_B P, N)_x \to {}_BHOM(P, {}^{\mathcal{C}}HOM(\Lambda, N))_x$$

*defined by $\phi(f)(p)(\lambda) = (|p|,|\lambda|)f(\lambda \otimes_B p)$ is an isomorphism.*

*Proof.* The inverse of $\phi$ is defined by $\psi(g)(\lambda \otimes_B p) = (|\lambda|,|p|)g(p)(\lambda)$. □

We deduce from Lemma 1.7 that ${}^{\mathcal{C}}HOM(\Lambda \otimes_B P, N)_0 \simeq {}_BHOM(P, {}^{\mathcal{C}}HOM(\Lambda, N))_0$, and this means that the functor $F'$ has the left adjoint $F = \Lambda \otimes_B - :{}_{gr-B}\mathcal{M} \to {}^{gr-\mathcal{C}}\mathcal{M}$. The unit of the adjunction is given by the graded $k$-linear map

$$u_N : N \to {}^{\mathcal{C}}HOM(\Lambda, \Lambda \otimes_B N), n \mapsto [\lambda \mapsto (|n|,|\lambda|)(\lambda \otimes n)]$$

for $N \in {}_{gr-B}\mathcal{M}$, while the counit is given by the graded $k$-linear map (the evaluation map)

$$c_M : \Lambda \otimes_B {}^{\mathcal{C}}HOM(\Lambda, M) \to M; \lambda \otimes f \mapsto (|\lambda|, |f|)f(\lambda)$$

for $M \in {}^{gr-\mathcal{C}}\mathcal{M}$. The adjointness property means that we have

$$F'(c_M) \circ u_{F'(M)} = id_{F'(M)},\ c_{F(N)} \circ F(u_N) = id_{F(N)};\ M \in {}^{gr-\mathcal{C}}\mathcal{M},\ N \in {}_{gr-B}\mathcal{M}.\ (\star)$$

## 2. The main results

Let $A$ be a colour algebra and $\mathcal{C}$ a graded $A$-coring. We keep the notations of the preceding sections.

**Lemma 2.1.** *Let $\Lambda$ and $N$ be graded left $\mathcal{C}$-comodules. Set $B = {}^{\mathcal{C}}END(\Lambda)$. For every $x \in G$, we have*

(1) ${}^{\mathcal{C}}HOM(\Lambda, N(x)) = {}^{\mathcal{C}}HOM(\Lambda, N)(x)$

(2) $\Lambda \otimes_B B(x) = \Lambda(x)$.

An object $\Lambda \in {}^{gr-\mathcal{C}}\mathcal{M}$ is called semi-quasiprojective if the functor ${}^{\mathcal{C}}HOM(\Lambda, -) : {}^{gr-\mathcal{C}}\mathcal{M} \to {}_{gr-k}\mathcal{M}$ sends an exact sequence of the form $\oplus_I \Lambda(x_i) \to \oplus_J \Lambda(x_j) \to N \to 0$ to an exact sequence (see [15]). A projective object in ${}^{gr-\mathcal{C}}\mathcal{M}$ is semi-quasiprojective in ${}^{gr-\mathcal{C}}\mathcal{M}$.

**Lemma 2.2.** *Assume that $\mathcal{C}$ is projective as a right $A$-module. Let $\Lambda$ be a graded left $\mathcal{C}$-comodule and set $B = {}^{\mathcal{C}}END(\Lambda)$. Then the functor ${}^{\mathcal{C}}HOM(\Lambda, -)$ commutes with*

(1) *direct sums if $\Lambda$ is finitely generated as a graded right $\mathcal{C}^*$-module,*

(2) *direct limits if $\Lambda$ is finitely presented as a graded right $\mathcal{C}^*$-module.*

*Proof.* (2) We know that ${}^{gr-\mathcal{C}}\mathcal{M}$ is a Grothendieck category so the functor ${}^{\mathcal{C}}HOM(\Lambda, -)_0$ preserves direct limits. We also know from Lemma 1.4 that $HOM_{\mathcal{C}^*}(\Lambda, N)_x = HOM_{\mathcal{C}^*}(\Lambda, N(x))_0$ for every $x \in G$. Let $(N_i)_{i \in I}$ be a directed system of right graded $\mathcal{C}^*$-modules. It is easy to show that $(\varinjlim N_i)(x) = \varinjlim(N_i(x))$ for every $x \in G$. Now the result follows from the fact direct limit commutes with direct sum. $\square$

**Lemma 2.3.** *Assume that $\mathcal{C}$ is projective as a right $A$-module. Let $\Lambda$ be a graded left $\mathcal{C}$-comodule that is finitely generated as a graded right $\mathcal{C}^*$-module, and let $B = {}^{\mathcal{C}}END(\Lambda)$. For every index set $I$,*

(1) *$c_{\oplus_I \Lambda(x_i)}$ is an isomorphism for every $x_i \in G$;*

(2) *$u_{\oplus_I B(x_i)}$ is an isomorphism for every $x_i \in G$;*

(3) *if $\Lambda$ is semi-quasiprojective in ${}^{gr-\mathcal{C}}\mathcal{M}$, then $u$ is a natural isomorphism; in other words, the induction functor $F = \Lambda \otimes_B (-)$ is fully faithful.*

*Proof.* (1) By Lemma 2.1(1), ${}^{\mathcal{C}}HOM(\Lambda,\Lambda)(x_i) = {}^{\mathcal{C}}HOM(\Lambda,\Lambda(x_i))$ for every $i \in I$. This implies that $\oplus_I B(x_i) = \oplus_I {}^{\mathcal{C}}HOM(\Lambda,\Lambda(x_i))$. By Lemma 2.2(1), the natural map $\kappa : \oplus_I B(x_i) \to {}^{\mathcal{C}}HOM(\Lambda, \oplus_I \Lambda(x_i))$ is an isomorphism. Lemma 2.1(2) implies that $\Lambda \otimes_B (\oplus_I B(x_i)) \simeq \oplus_I \Lambda(x_i)$. It is easy to see that this isomorphism is just $c_{\oplus_I \Lambda(x_i)} \circ (id_\Lambda \otimes \kappa)$. So $c_{\oplus_I \Lambda(x_i)}$ is an isomorphism since $\kappa$ is an isomorphism.

(2) Putting $M = \oplus_I \Lambda(x_i)$ in $(\star)$ and using (1), we find

$$ {}^{\mathcal{C}}HOM(\Lambda, c_{\oplus_I \Lambda(x_i)}) \circ u_{{}^{\mathcal{C}}HOM(\Lambda,\oplus_I \Lambda(x_i))} = id_{{}^{\mathcal{C}}HOM(\Lambda,\oplus_I \Lambda(x_i))}; i.e., $$

$$ {}^{\mathcal{C}}HOM(\Lambda, c_{\oplus_I \Lambda(x_i)}) \circ u_{\oplus_I B(x_i)} = id_{\oplus_I B(x_i)}. $$

From (1), ${}^{\mathcal{C}}HOM(\Lambda, c_{\oplus_I \Lambda(x_i)})$ is an isomorphism, hence $u_{\oplus_I B(x_i)}$ is an isomorphism.

(3) Take a graded free resolution $\oplus_J B(x_j) \to \oplus_I B(x_i) \to N \to 0$ of a graded left $B$-module $N$. Since $u$ is natural, we have a commutative diagram

$$
\begin{array}{ccccccc}
\oplus_{\mathbf{J}} \mathbf{B}(\mathbf{x_j}) & \longrightarrow & \oplus_{\mathbf{I}} \mathbf{B}(\mathbf{x_i}) & \longrightarrow & \mathbf{N} & \longrightarrow & 0 \\
{\scriptstyle u_{\oplus_J B(x_j)}}\Big\downarrow & & {\scriptstyle u_{\oplus_I B(x_i)}}\Big\downarrow & & {\scriptstyle u_N}\Big\downarrow & & \\
\mathbf{F'F}(\oplus_{\mathbf{J}} \mathbf{B}(\mathbf{x_j})) & \longrightarrow & \mathbf{F'F}(\oplus_{\mathbf{I}} \mathbf{B}(\mathbf{x_i})) & \longrightarrow & \mathbf{F'F}(\mathbf{N}) & \longrightarrow & 0\,.
\end{array}
$$

The top row is exact; the bottom row is exact, since

$$ F'F(\oplus_I B(x_i)) = {}^{\mathcal{C}}HOM(\Lambda, \Lambda \otimes_B \oplus_I B(x_i)) = {}^{\mathcal{C}}HOM(\Lambda, \oplus_I \Lambda(x_i)) $$

and $\Lambda$ is semi-quasiprojective. By (2), $u_{\oplus_I B(x_i)}$ and $u_{\oplus_J B(x_j)}$ are isomorphisms; and it follows from the five lemma that $u_N$ is an isomorphism. □

We can now give equivalent conditions for the projectivity and flatness of $P \in {}_{gr-B}\mathcal{M}$.

**Theorem 2.4.** *Assume that $\mathcal{C}$ is projective as a right $A$-module. Let $\Lambda$ be a graded left $\mathcal{C}$-comodule that is finitely generated as a graded right $\mathcal{C}^*$-module, and let $B = {}^{\mathcal{C}}END(\Lambda)$. For $P \in {}_{gr-B}\mathcal{M}$, we consider the following statements.*

1. *$\Lambda \otimes_B P$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$ and $u_P$ is injective;*
2. *$P$ is projective as a graded left $B$-module;*
3. *$\Lambda \otimes_B P$ is a direct summand in ${}^{gr-\mathcal{C}}\mathcal{M}$ of some $\oplus_I \Lambda(x_i)$, and $u_P$ is bijective;*
4. *there exists $Q \in {}^{gr-\mathcal{C}}\mathcal{M}$ such that $Q$ is a direct summand of some $\oplus_I \Lambda(x_i)$, and $P \cong {}^{\mathcal{C}}HOM(\Lambda, Q)$ in ${}_{gr-B}\mathcal{M}$;*
5. *$\Lambda \otimes_B P$ is a direct summand in ${}^{gr-\mathcal{C}}\mathcal{M}$ of some $\oplus_I \Lambda(x_i)$.*

*Then* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

*If* $\Lambda$ *is semi-quasiprojective in* ${}^{gr-\mathcal{C}}\mathcal{M}$, *then* (5) $\Rightarrow$ (3); *if* $\Lambda$ *is projective in* ${}^{gr-\mathcal{C}}\mathcal{M}$, *then* (3) $\Rightarrow$ (1).

*Proof.* (2) $\Rightarrow$ (3): If P is projective as a graded left $B$-module, then we can find an index set $I$ and $P' \in {}_{gr-B}\mathcal{M}$ such that $\oplus_I B(x_i) \cong P \oplus P'$. Then obviously

$$\oplus_I \Lambda(x_i) \cong \Lambda \otimes_B (\oplus_I B(x_i)) \cong (\Lambda \otimes_B P) \oplus (\Lambda \otimes_B P').$$

Since $u$ is a natural transformation, we have a commutative diagram:

$$\begin{array}{ccc}
\oplus_{\mathbf{I}}\mathbf{B}(\mathbf{x_i}) & \xrightarrow{\cong} & \mathbf{P} \oplus \mathbf{P'} \\
\Big\downarrow {\scriptstyle u_{\oplus_I B(x_i)}} & & \Big\downarrow {\scriptstyle u_P \oplus u_{P'}} \\
{}^{\mathcal{C}}\mathbf{HOM}(\boldsymbol{\Lambda}, \oplus_{\mathbf{I}}\boldsymbol{\Lambda}(\mathbf{x_i})) & \xrightarrow[\cong]{} & {}^{\mathcal{C}}\mathbf{HOM}(\boldsymbol{\Lambda}, \boldsymbol{\Lambda} \otimes_{\mathbf{B}} \mathbf{P}) \oplus {}^{\mathcal{C}}\mathbf{HOM}(\boldsymbol{\Lambda}, \boldsymbol{\Lambda} \otimes_{\mathbf{B}} \mathbf{P'})
\end{array}$$

From the fact that $u_{\oplus_I B(x_i)}$ is an isomorphism, it follows that $u_P$ (and $u_{P'}$) are isomorphisms.

(3) $\Rightarrow$ (4): Take $Q = \Lambda \otimes_B P$.

(4) $\Rightarrow$ (2): Let $f : \oplus_I \Lambda(x_i) \to Q$ be a split epimorphism in ${}^{gr-\mathcal{C}}\mathcal{M}$. Then

$${}^{\mathcal{C}}HOM(\Lambda, f) : {}^{\mathcal{C}}HOM(\Lambda, \oplus_I \Lambda(x_i)) \cong \oplus_I B(x_i) \to {}^{\mathcal{C}}HOM(\Lambda, Q) \cong P$$

is also split surjective, hence $P$ is projective as a graded left $B$-module.

(4) $\Rightarrow$ (5): If (4) is true, we know from the proof of (4) $\Rightarrow$ (2) that $P$ is a direct summand of some $\oplus_I B(x_i)$ in ${}_{gr-B}\mathcal{M}$. So $\Lambda \otimes_B P$ is a direct summand of $\oplus_I \Lambda(x_i)$.

(1) $\Rightarrow$ (2): Take an epimorphism $f : \oplus_I B(x_i) \to P$ in ${}_{gr-B}\mathcal{M}$. Then

$$F(f) = id_\Lambda \otimes_B f : \Lambda \otimes_B (\oplus_I B(x_i)) \cong \oplus_I \Lambda(x_i) \to \Lambda \otimes_B P$$

is an epimorphism in ${}^{gr-\mathcal{C}}\mathcal{M}$, and it splits since $\Lambda \otimes_B P$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$ . Consider the commutative diagram:

$$\begin{array}{ccccc}
\oplus_{\mathbf{I}}\mathbf{B}(\mathbf{x_i}) & \xrightarrow{f} & \mathbf{P} & \longrightarrow & 0 \\
\Big\downarrow {\scriptstyle u_{\oplus_I B(x_i)}} & & \Big\downarrow {\scriptstyle u_P} & & \\
{}^{\mathcal{C}}\mathbf{HOM}(\boldsymbol{\Lambda}, \oplus_{\mathbf{I}}\boldsymbol{\Lambda}(\mathbf{x_i})) & \xrightarrow[F'F(f)]{} & {}^{\mathcal{C}}\mathbf{HOM}(\boldsymbol{\Lambda}, \boldsymbol{\Lambda} \otimes_{\mathbf{B}} \mathbf{P}) & \longrightarrow & 0\,.
\end{array}$$

The bottom row is split exact, since any functor, in particular, ${}^{\mathcal{C}}HOM(\Lambda,-)$ preserves split exact sequences. By Lemma 2.3(2), $u_{\oplus_I B(x_i)}$ is an isomorphism. A diagram chasing tells us that $u_P$ is surjective. By assumption, $u_P$ is injective, so $u_P$ is bijective. We deduce that the top row is isomorphic to the bottom row, and therefore splits. Thus $P$ is projective in $\in {}_{gr-B}\mathcal{M}$.

Under the assumption that $\Lambda$ is semi-quasiprojective in ${}^{gr-\mathcal{C}}\mathcal{M}$, (5) $\Rightarrow$ (3) follows from Lemma 2.3(3).

(3) $\Rightarrow$ (1): By (3), $\Lambda\otimes_B P$ is a direct summand of some $\oplus_I\Lambda(x_i)$. If $\Lambda$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$, then $\oplus_I\Lambda(x_i)$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$. So $\Lambda\otimes_B P$ being a direct summand of a projective object of ${}^{gr-\mathcal{C}}\mathcal{M}$ is projective in ${}^{gr-\mathcal{C}}\mathcal{M}$. $\square$

**Theorem 2.5.** *Assume that $\mathcal{C}$ is projective as a right $A$-module. Let $\Lambda$ be a graded left $\mathcal{C}$-comodule that is finitely presented as a graded right $\mathcal{C}^*$-module, and let $B = {}^{\mathcal{C}}END(\Lambda)$. For $P \in {}_{gr-B}\mathcal{M}$, the following assertions are equivalent.*

(1) *$P$ is flat as a graded left $B$-module;*
(2) *$\Lambda\otimes_B P = \varinjlim Q_i$, where $Q_i \cong \oplus_{j\le n_i}B(x_{ij})$ in ${}^{gr-\mathcal{C}}\mathcal{M}$ for some positive integer $n_i$, and $u_P$ is bijective;*
(3) *$\Lambda\otimes_B P = \varinjlim Q_i$, where $Q_i \in {}^{gr-\mathcal{C}}\mathcal{M}$ is a direct summand of some $\oplus_{j\in I_i}\Lambda$ $(x_{ij})$ in ${}^{gr-\mathcal{C}}\mathcal{M}$, and $u_P$ is bijective;*
(4) *there exists $Q = \varinjlim Q_i \in {}^{gr-\mathcal{C}}\mathcal{M}$, such that $Q_i \cong \oplus_{j\le n_i}\Lambda(x_{ij})$ for some positive integer $n_i$ and ${}^{\mathcal{C}}HOM(\Lambda,Q)\cong P$ in ${}_{gr-B}\mathcal{M}$;*
(5) *there exists $Q = \varinjlim Q_i \in {}^{gr-\mathcal{C}}\mathcal{M}$, such that $Q_i$ is a direct summand of some $\oplus_{j\in I_i}\Lambda(x_{ij})$ in ${}^{gr-\mathcal{C}}\mathcal{M}$, and ${}^{\mathcal{C}}HOM(\Lambda,Q)\cong P$ in ${}_{gr-B}\mathcal{M}$.*

*If $\Lambda$ is semi-quasiprojective in ${}^{gr-\mathcal{C}}\mathcal{M}$, these conditions are also equivalent to conditions (2) and (3), without the assumption that $u_P$ is bijective.*

*Proof.* (1) $\Rightarrow$ (2): $P = \varinjlim N_i$, with $N_i = \oplus_{j\le n_i}B(x_{ij})$ for some positive integer $n_i$. Take $Q_i = \oplus_{j\le n_i}\Lambda(x_{ij})$ , then

$$\varinjlim Q_i \cong \varinjlim(\Lambda\otimes_B N_i)\cong \Lambda\otimes_B \varinjlim N_i \cong \Lambda\otimes_B P.$$

Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{P} = \varinjlim \mathbf{N_i} & \xrightarrow{\lim(u_{N_i})} & \varinjlim {}^{\mathcal{C}}\mathbf{HOM}(\mathbf{\Lambda}, \mathbf{\Lambda}\otimes_{\mathbf{B}} \mathbf{N_i}) \\
u_P \Big\downarrow & & \Big\downarrow f \\
{}^{\mathcal{C}}\mathbf{HOM}(\mathbf{\Lambda}, \mathbf{\Lambda}\otimes_{\mathbf{B}} (\varinjlim \mathbf{N_i})) & \xrightarrow[\cong]{} & {}^{\mathcal{C}}\mathbf{HOM}(\mathbf{\Lambda}, \varinjlim(\mathbf{\Lambda}\otimes_{\mathbf{B}} \mathbf{N_i}))\,.
\end{array}$$

By Lemma 2.3(2), the $u_{N_i}$ are isomorphisms; by Lemma 2.2, the natural homomorphism $f$ is an isomorphism. Hence $u_P$ is an isomorphism.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.
(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5): Put $Q = \Lambda \otimes_B P$. Then $u_P : P \to {}^{\mathcal{C}}HOM(\Lambda, \Lambda \otimes_B P)$ is the required isomorphism.
(5) $\Rightarrow$ (1): We have a split exact sequence $0 \to N_i \to P_i = \oplus_{j \in I_i} \Lambda(x_{ij}) \to Q_i \to 0$ in ${}^{gr-\mathcal{C}}\mathcal{M}$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \mathbf{FF'(N_i)} & \longrightarrow & \mathbf{FF'(P_i)} & \longrightarrow & \mathbf{FF'(Q_i)} & \longrightarrow & \mathbf{0} \\
& & \Big\downarrow{\scriptstyle c_{N_i}} & & \Big\downarrow{\scriptstyle c_{P_i}} & & \Big\downarrow{\scriptstyle c_{Q_i}} & & \\
\mathbf{0} & \longrightarrow & \mathbf{N_i} & \longrightarrow & \mathbf{P_i} & \longrightarrow & \mathbf{Q_i} & \longrightarrow & \mathbf{0}\,.
\end{array}$$

We know from Lemma 2.3(1) that $c_{P_i}$ is an isomorphism. Both rows in the diagram are split exact, so it follows that $c_{N_i}$ and $c_{Q_i}$ are also isomorphisms. Next consider the commutative diagram:

$$\begin{array}{ccc}
\mathbf{\Lambda \otimes_B \varinjlim {}^{\mathcal{C}}HOM(\Lambda, Q_i)} & \xrightarrow{id_\Lambda \otimes f} & \mathbf{\Lambda \otimes_B {}^{\mathcal{C}}HOM(\Lambda, Q)} \\
h \Big\uparrow & & \Big\downarrow{\scriptstyle c_Q} \\
\mathbf{\varinjlim(\Lambda \otimes_B {}^{\mathcal{C}}HOM(\Lambda, Q_i))} & \xrightarrow[\lim c_{Q_i}]{} & \mathbf{Q}
\end{array}$$

where $h$ and $f$ are the natural homomorphisms. $h$ is an isomorphism, because $\Lambda \otimes_B (-)$ preserves inductive limits; by Lemma 2.2, $f$ is an isomorphism; and $limc_{Q_i}$ is an isomorphism because every $c_{Q_i}$ is an isomorphism. It follows that $c_Q$ is an isomorphism, hence ${}^{\mathcal{C}}HOM(\Lambda, c_Q)$ is an isomorphism. From ($\star$), we get

$${}^{\mathcal{C}}HOM(\Lambda, c_Q) \circ u_{{}^{\mathcal{C}}HOM(\Lambda,Q)} = id_{{}^{\mathcal{C}}HOM(\Lambda,Q)}.$$

It follows that $u_{{}^{\mathcal{C}}HOM(\Lambda,Q)}$ is also an isomorphism. Since ${}^{\mathcal{C}}HOM(\Lambda, Q) \cong P$, $u_P$ is an isomorphism. Consider the isomorphisms

$$P \cong {}^{\mathcal{C}}HOM(\Lambda, \Lambda \otimes_B P) \cong {}^{\mathcal{C}}HOM(\Lambda, \Lambda \otimes_B {}^{\mathcal{C}}HOM(\Lambda, Q)) \cong$$
$${}^{\mathcal{C}}HOM(\Lambda, Q) \cong \varinjlim {}^{\mathcal{C}}HOM(\Lambda, Q_i);$$

where the first isomorphism is $u_P$, the third is ${}^{\mathcal{C}}HOM(\Lambda, c_Q)$ and the last one is $f$. It follows from Lemmas 2.1(1)and 2.2 that ${}^{\mathcal{C}}HOM(\Lambda, P_i) \cong \oplus_{j \in I_i} B(x_{ij})$ is projective as a graded left $B$-module, hence ${}^{\mathcal{C}}HOM(\Lambda, Q_i)$ is also projective as a graded left $B$-module, and we conclude that $P$ is flat in $\in {}_{gr-B}\mathcal{M}$. The final statement is an immediate consequence of Lemma 2.3(3). $\square$

## 3. Applications

### 3.1. $\mathcal{C}$ contains a grouplike element

A grouplike element of $\mathcal{C}$ is an element $X \in \mathcal{C}_0$ such that $\Delta_{\mathcal{C}}(X) = X \otimes_A X$ and $\epsilon_{\mathcal{C}}(X) = 1_A$ (see [14]). If $\mathcal{C}$ contains a grouplike element $X$, then $A$ is an object of ${}^{gr-\mathcal{C}}\mathcal{M}$: the $\mathcal{C}$-coaction is defined by $\rho_{A,\mathcal{C}}(a) = aX = aX \otimes_A 1_A; \forall a \in A$. Conversely, if $A$ is an object of ${}^{gr-\mathcal{C}}\mathcal{M}$, then $\rho_{A,\mathcal{C}}(1_A) = X$ is a grouplike element of $\mathcal{C}$.

Assume that $\mathcal{C}$ contains a grouplike element $X$. Then $A$ is an object of $\mathcal{M}_{gr-\mathcal{C}^*}$ and $a.\epsilon_{\mathcal{C}} = a$, that is, $A$ is generated as a right $\mathcal{C}^*$-module by the homogeneous element $\epsilon_{\mathcal{C}}$ of degree 0. Lemma 1.2 implies that $A$ is finitely generated in $\mathcal{M}_{gr-\mathcal{C}^*}$. For any graded left $\mathcal{C}$-comodule $M$, we call ${}^{co\mathcal{C},X}M = \{m \in M, \quad \rho_{M,\mathcal{C}}(m) = X \otimes_A m\}$ the vector space of $(\mathcal{C}, X)$-coinvariants of $M$. Clearly, ${}^{co\mathcal{C},X}A = \{a \in A, \quad Xa = aX\}$ is a colour subalgebra of $A$: the colour subalgebra of $(\mathcal{C}, X)$-coinvariants. For every $f \in {}^{\mathcal{C}}HOM(A, M)$, $f(1) \in {}^{co\mathcal{C},X}M$. The graded $k$-linear map $f \mapsto f(1)$ establishes an isomorphism ${}^{\mathcal{C}}HOM(A, M) \to {}^{co\mathcal{C},X}M$ with inverse the graded $k$-linear map $\psi$ defined by $\psi(m)(a) = (|m|, |a|)am$. We have ${}^{\mathcal{C}}END(A) = {}^{co\mathcal{C},X}A$. Set $B = {}^{co\mathcal{C},X}A$. Then we get from Theorems 2.4 and 2.5 necessary and sufficient conditions for projectivity and flatness over the colour algebra $B = {}^{co\mathcal{C},X}A$.

### 3.2. A colour algebra as a trivial coring

A colour algebra $A$ is a graded $A$-bimodule. Let us define $\Delta_A(a) = a \otimes_A 1_A$ and $\epsilon_A(a) = a$. Then $A$ is an $A$-coring. A graded left $A$-comodule is just a graded left $A$-module. The product on $A^*$ is defined by $f\#g(a) = \sum_{|a|}(|f|, |g|)g(f(a)1_A) = \sum_{|a|}(|f|, |g|)g(1_A)(f(a))$. It is easy to show that the algebra $A^*$ is isomorphic to $A^{op}$, the opposite algebra of $A$: this isomorphism is defined by $f \mapsto f(1_A)$. For graded left $A$-modules $M$ and $N$, we have ${}^{A}HOM(M, N) = {}_{A}HOM(M, N)$. Then Theorems 2.4 and 2.5 give necessary and sufficient conditions for projectivity and flatness over $B ={}_A END(\Lambda)$, where $\Lambda$ is a finitely generated graded left $A$-module. When the gradation is trivial we recover [11]. In many examples, $\Lambda$ will be a colour algebra and $A$ will be a graded $\Lambda$-ring with a graded left grouplike character.

**Definition 3.1.** (see [6], Section 2) *Let $A$ and $\Lambda$ be two colour algebras and $i : \Lambda \to A$ a graded ring morphism. A graded $k$-linear map $\chi : A \to \Lambda$ is called a graded left grouplike character on $A$ if $\chi$ is graded left $\Lambda$-linear and*

$$\chi(a\chi(a')) = \chi(aa') \quad \textit{and} \quad \chi(1_A) = 1_\Lambda \quad \forall \quad a, a' \in A.$$

*We then say that $A$ is a graded $\Lambda$-ring with a graded left grouplike character $\chi$.*

Let $A$ be a graded $\Lambda$-ring with a graded left grouplike character $\chi$. Then $\Lambda$ is a graded left $A$-module: the action is given by $a \rightharpoonup \lambda = \chi(a\lambda)$. Furthermore, $\Lambda$ is cyclic as a left $A$-module, since $\lambda = (\lambda 1_A) \rightharpoonup 1_\Lambda$. But $1_\Lambda$ is homogeneous of degree 0, so $\Lambda$ is a finitely generated as a graded left $A$-module (Lemma 1.2). So we get necessary and sufficient conditions for projectivity and flatness over the colour endomorphism ring ${}_AEND(\Lambda)$ of $\Lambda$.

Now we will give two examples of this situation. There are other examples in the literature.

• Let $H$ be a colour algebra, the colour tensor product $H \otimes H$ is the $G$-graded vector space $H \otimes H = \oplus_{x \in G}(\oplus_{y+z=x} H_y \otimes H_z)$ with multiplication $(h \otimes l)(h' \otimes l') = (|l|/|h'|)hh' \otimes ll'$ for homogeneous elements $h$, $h'$, $l$, $l' \in H$. By [9, Lemma 3.2], $H \otimes H$ is a colour algebra. A Hopf colour algebra is a colour algebra and a graded coalgebra such that $\Delta_H$ and $\epsilon_H$ are morphisms of colour algebras and there exists a graded $k$-linear map $S_H : H \to H$ (called antipode) such that $(S_H \otimes id_H) \circ \Delta_H = \epsilon_H = (id_H \otimes S_H) \circ \Delta_H$ or equivalently, $\sum_{|h|} \epsilon(h_1)h_2 = h = \sum_{|h|} h_1\epsilon(h_2)$ and $\sum_{|h|} S(h_1)h_2 = \epsilon(h) = \sum_{|h|} h_1 S(h_2)$.

Let $H$ be a Hopf colour algebra over $k$ with comultiplication $\Delta_H$, counit $\epsilon_H$ and antipode $S_H$. A colour algebra $\Lambda$ which is a graded left $H$-module such that $h.(\lambda\lambda') = \sum_{|h|}(|h_2|/|\lambda|)(h_1.\lambda)(h_2.\lambda')$ for all $h \in H$ and $\lambda, \lambda' \in \Lambda$ will be called a graded left $H$-module algebra. We denote by $A = \Lambda \# H$ the associated smash product; i.e., the colour algebra generated by $\Lambda$ and $H$ whose multiplication is defined by $(\lambda h)(\lambda' h') = \sum_{|h|}(|h_2|/|\lambda'|)\lambda(h_1.\lambda')(h_2 h')$ (see [12]). A graded vector space $M$ is a graded left $A$-module if and only if it is a graded left $\Lambda$-module and a graded left $H$-module such that $h.(\lambda m) = \sum_{|h|}(|h_2|/|\lambda|)(h_1.\lambda)(h_2 m)$. Define a $k$-linear map $\chi : A \to \Lambda$ by $\chi(\lambda h) = \epsilon_H(h)\lambda$. Since $\epsilon_H(h) = 0$ for $|h| \neq 0$, $\chi$ is homogeneous of degree 0. Clearly, $\chi$ is left $\Lambda$-linear. It follows that $\Lambda \# H$ is a graded $\Lambda$-ring with a graded left grouplike character $\chi$. Note that ${}_{\Lambda \# H}END(\Lambda)$ is exactly the colour subring of invariants of $\Lambda$; i.e., ${}_{\Lambda \# H}END(\Lambda) = \{\lambda \in \Lambda; h.\lambda = \epsilon_H(h)\lambda\}$.

• Assume that $\mathcal{C}$ contains a grouplike element $X$. The linear map $i : A \to \mathcal{C}^*$ defined by $i(a)(c) = a\epsilon_{\mathcal{C}}(c)$ is a morphism of colour algebras. Define $\chi : \mathcal{C}^* \to A$ by $\chi(f) = f(X)$. Then $\chi$ is a graded left grouplike character on $\mathcal{C}^*$. So $\mathcal{C}^*$ is a graded $A$-ring.

### 3.3. $\mathcal{C}$ comes from a graded entwining structure

In this section, $A$ is a colour algebra with multiplication $\mu$ and unit $\iota$, and $C$ is a graded coalgebra with comultiplication $\Delta_C$ and counit $\epsilon_C$. We denote by $\tau : A \otimes C \to C \otimes A$ the twist map; that is $\tau(a \otimes c) = (|a|/|c|)c \otimes a$. If $M$ is a left (resp. right $C$-comodule, we write $\rho_{M,C}(m) = m_{-1} \otimes m_0$ (resp. $\rho_{M,C}(m) = m_0 \otimes m_1$). We remind that $C$ is a graded $k$-coring. Interesting examples of graded corings come from graded entwining structures. We will often refer to [5] for the ungraded case.

• Graded left-left entwined modules.

A graded left-left entwining structure over $k$ is a triple $(A, C, \psi)$ with a graded $k$-linear map $\psi : A \otimes C \to C \otimes A$; $a \otimes c \mapsto (|a_\alpha|/|c|)c^\alpha \otimes a_\alpha$ satisfying the following conditions [5, 32.1]:

$$\psi \circ (\mu \otimes id_C) = (id_C \otimes \mu) \circ (\psi \otimes id_A) \circ (id_A \otimes \psi)$$

$$(\Delta_C \otimes id_A) \circ \psi = (id_C \otimes \psi) \circ (\psi \otimes id_C) \circ (id_A \otimes \Delta_C)$$
$$(\epsilon_C \otimes id_A) \circ \psi = id_A \otimes \epsilon_C$$
$$\psi \circ (id_C \otimes \iota) = \iota \otimes id_C.$$

These relations are respectively equivalent to

$$(|(aa')_\alpha|/|c|)(c^\alpha \otimes (aa')_\alpha) = (|a'_\alpha|/|c|)(|a_\beta|/|c^\alpha|)(c^{\alpha\beta} \otimes a_\beta a'_\alpha)$$
$$(|a_\alpha|/|c|)(\Delta_C(c^\alpha) \otimes a_\alpha) = (|a_\alpha|/|c_1|)(|a_{\alpha\beta}|/|c_2|)({c_1}^\alpha \otimes {c_2}^\beta \otimes a_{\alpha\beta})$$
$$(|a_\alpha|/|c|)(\epsilon_C(c^\alpha)a_\alpha) = a\epsilon_C(c)$$
$$(|1_\alpha|/|c|)(c^\alpha \otimes 1_\alpha) = c \otimes 1.$$

The map $\psi$ is called a graded entwining map, and $A$ and $C$ are said to be graded entwined by $\psi$. By [5, 32.1], $\mathcal{C} = C \otimes A$ is a graded $A$-coring with $A$-multiplications $a'(c \otimes a)a'' = \psi(a' \otimes c)aa''$, coproduct

$$\Delta_{\mathcal{C}} : C \otimes A \to C \otimes A \otimes_A C \otimes A \cong C \otimes C \otimes A; \quad c \otimes a \mapsto \Delta_C(c) \otimes a$$

and counit $\epsilon_{\mathcal{C}}(c \otimes a) = \epsilon_C(c)a$.

Let $M$ be a graded left $A$-module. Then $C \otimes M$ becomes a graded left $A$-module if we set $a(c \otimes m) = (|a_\alpha|/|c)|)c^\alpha \otimes (a_\alpha m)$. We say that a vector space $M$ is a graded left-left $(A, C, \psi)$-entwined module if $M$ is a graded left $A$-module and a graded left $C$-comodule such that $\rho_{M,C}$ is a graded left $A$-linear map; i.e.,

$$\rho_{M,C}(am) = (|a_\alpha|/|(m_{-1})|)(m_{-1})^\alpha \otimes (a_\alpha m_0).$$

We denote by ${}^{gr-C}_{gr-A}\mathcal{M}(\psi)$ the category of graded left-left $(A, C, \psi)$-entwined modules: its morphisms are the graded left $A$-linear maps and the graded left $C$-colinear maps. We can show that ${}^{gr-C}_{gr-A}\mathcal{M}(\psi)$ is isomorphic to ${}^{gr-\mathcal{C}}\mathcal{M}$.

• Graded right-right entwined modules.

A graded right-right entwining structure over $k$ is a triple $(A, C, \psi)$ with a graded $k$-linear map $C \otimes A \to A \otimes C$; $c \otimes a \mapsto (|c|/|a_\alpha|)a_\alpha \otimes c^\alpha$ satisfying the following conditions [5, 32.1]:

$$\psi \circ (id_C \otimes \mu) = (\mu \otimes id_C) \circ (id_A \otimes \psi) \circ (\psi \otimes id_A)$$
$$(id_A \otimes \Delta_C) \circ \psi = (\psi \otimes id_C) \circ (id_C \otimes \psi) \circ (\Delta_C \otimes id_A)$$
$$(id_A \otimes \epsilon_C) \circ \psi = \epsilon_C \otimes id_A$$
$$\psi \circ (id_C \otimes \iota) = \iota \otimes id_C.$$

These relations are respectively equivalent to

$$(|c|/|(aa')_\alpha|)((aa')_\alpha \otimes c^\alpha) = (|c|/|a_\alpha|)(|c^\alpha|/|a'_\beta|)(a_\alpha a'_\beta \otimes c^{\alpha\beta})$$
$$(|c|/|a_\alpha|)(a_\alpha \otimes \Delta_C(c^\alpha)) = (|(c_2)|/|a_\alpha|)(|(c_1)|/|a_{\alpha\beta}|)(a_{\alpha\beta} \otimes {c_1}^\beta \otimes {c_2}^\alpha)$$
$$(|c|/|a_\alpha|)(a_\alpha\epsilon_C(c^\alpha)) = \epsilon_C(c)a$$

$$(|c|/|1_\alpha|)(1_\alpha \otimes c^\alpha) = 1 \otimes c.$$

By [5, 32.1], $\mathcal{C} = A \otimes C$ is a graded $A$-coring with $A$-multiplications $a'(a \otimes c)a'' = a'a\psi(c \otimes a'')$, coproduct

$$\Delta_{\mathcal{C}} : A \otimes C \to A \otimes C \otimes_A A \otimes C \cong A \otimes C \otimes C; \quad a \otimes c \mapsto a \otimes \Delta_C(c)$$

and counit $\epsilon_{\mathcal{C}}(a \otimes c) = a\epsilon_C(c)$.

Let $M$ be a graded right $A$-module. Then $M \otimes C$ becomes a graded right $A$-module if we set $(m \otimes c)a = (|c|/|a_\alpha|)(ma_\alpha) \otimes c^\alpha$. A vector space $M$ is a graded right-right $(A, C, \psi)$-entwined module if $M$ is a graded right $A$-module and a graded right $C$-comodule via such that $\rho_{M,C}$ is a graded right $A$-linear map; i.e.,

$$\rho_{M,C}(ma) = (|(m_1)|/|a_\alpha|)(m_0 a_\alpha) \otimes (m_1)^\alpha.$$

We denote by $\mathcal{M}(\psi)^{gr-C}_{gr-A}$ the category of graded right-right $(A, C, \psi)$-entwined modules: its morphisms are the graded right $A$-linear maps and the graded right $C$-colinear maps. We can show that this category is isomorphic to $\mathcal{M}^{gr-\mathcal{C}}$.

• Graded left-right entwined modules.

A graded left-right entwining structure over $k$ is a triple $(A, C, \psi)$ with a graded $k$-linear map $A \otimes C \to A \otimes C$; $a \otimes c \mapsto a_\alpha \otimes c^\alpha$ satisfying the following conditions [5, 32.1]:

$$\psi \circ (\mu \otimes id_C) = (\mu \otimes id_C) \circ (id_A \otimes \tau^{-1}) \circ \circ(\psi \otimes id_A) \circ (id_A \otimes \tau) \circ (id_A \otimes \psi)$$

$$(id_A \otimes \Delta_C) \circ \psi = (\tau^{-1} \otimes id_C) \circ (id_C \otimes \psi) \circ (\tau \otimes id_C) \circ (\psi \otimes id_C) \circ (id_A \otimes \Delta_C)$$

$$(id_A \otimes \epsilon_C) \circ \psi = id_A \otimes \epsilon_C$$

$$\psi \circ (\iota \otimes id_C) = \iota \otimes id_C.$$

These relations are respectively equivalent to

$$(aa')_\alpha \otimes c^\alpha = (|a'_\alpha|/|c^\alpha|)(|c^{\alpha\beta}|/|a'_\alpha|)(a_\beta a'_\alpha \otimes c^{\alpha\beta})$$

$$a_\alpha \otimes \Delta_C(c^\alpha) = (|a_\alpha|/|c_1{}^\alpha|)(|c_1{}^\alpha|/|a_{\alpha\beta}|)(a_{\alpha\beta} \otimes c_1{}^\alpha \otimes c_2{}^\beta)$$

$$a_\alpha \epsilon_C(c^\alpha) = a\epsilon_C(c)$$

$$1_\alpha \otimes c^\alpha = 1 \otimes c.$$

Let $M$ be a graded left $A$-module. Then $M \otimes C$ becomes a graded left $A$-module if we set $a(m \otimes c) = (|m|/|c|)(|c^\alpha|/|m|)(a_\alpha m \otimes c^\alpha)$. A vector space $M$ is a graded left-right $(A, C, \psi)$-entwined module if $M$ is a graded left $A$-module and a graded right $C$-comodule such that $\rho_{M,C}$ is a graded left $A$-linear map; i.e.,

$$\rho_{M,C}(am) = (|m_0|/|m_1|)(|(m_1)^\alpha|/|m_0|)(a_\alpha m_0 \otimes (m_1)^\alpha.$$

We denote by ${}_{gr-A}\mathcal{M}(\psi)^{gr-C}$ the category of graded left-right $(A, C, \psi)$-entwined modules: its morphisms are the graded left $A$-linear maps and the graded right $C$-colinear maps.

• Graded right-left entwined modules.

A graded right-left entwining structure over $k$ is a triple $(A, C, \psi)$ with a graded $k$-linear map $C \otimes A \to C \otimes A$; $c \otimes a \mapsto c^{\alpha} \otimes a_{\alpha}$ satisfying the following conditions [5, 32.1]:

$$\psi \circ (id_C \otimes \mu) = (id_C \otimes \mu) \circ (\tau \otimes id_A) \circ (id_A \otimes \psi) \circ (\tau^{-1} \otimes id_A) \circ (\psi \otimes id_A)$$

$$(\Delta_C \otimes id_A) \circ \psi = (id_C \otimes \tau) \circ (\psi \otimes id_C) \circ (id_C \otimes \tau^{-1}) \circ (id_C \otimes \psi) \circ (\Delta_C \otimes id_A)$$

$$(\epsilon_C \otimes id_A) \circ \psi = \epsilon_C \otimes id_A$$

$$\psi \circ (id_C \otimes \iota) = id_C \otimes \iota$$

where $\tau : C \otimes A \to A \otimes C$; $c \otimes a \mapsto (|c|/|a|)a \otimes c$. These relations are respectively equivalent to

$$c^{\alpha} \otimes (aa')_{\alpha} = (|c^{\alpha}|/|a_{\alpha}|)(|a_{\alpha}|/|c^{\alpha\beta}|)(c^{\alpha\beta} \otimes a_{\alpha}a'_{\beta})$$

$$\Delta_C(c^{\alpha}) \otimes a_{\alpha} = (|{c_2}^{\alpha}|/|a_{\alpha}|)(|a_{\alpha\beta}|/|{c_2}^{\alpha}|)({c_1}^{\beta} \otimes {c_2}^{\alpha} \otimes a_{\alpha\beta})$$

$$\epsilon_C(c^{\alpha})a_{\alpha} = \epsilon_C(c)a$$

$$c_{\alpha} \otimes 1^{\alpha} = c \otimes 1.$$

Let $M$ be a graded right $A$-module. Then $C \otimes M$ becomes a graded right $A$-module if we set $(c \otimes m)a = (|c|/|m|)(|m|/|c^{\alpha}|)(c^{\alpha} \otimes ma_{\alpha})$. A vector space $M$ is a graded right-left $(A, C, \psi)$-entwined module if $M$ is a graded right $A$-module and a graded left $C$-comodule such that $\rho_{M,C}$ is a graded right $A$-linear map; i.e.,

$$\rho_{M,C}(ma) = (|m_{-1}|/|m_0|)(|m_0|/|(m_{-1})^{\alpha}|)(m_{-1})^{\alpha} \otimes (m_0 a_{\alpha}).$$

We denote by ${}^{gr-C}\mathcal{M}(\psi)_{gr-A}$ the category of graded right-left $(A, C, \psi)$-entwined modules: its morphisms are the graded right $A$-linear maps and the graded left $C$-colinear maps.

### 3.3.1. Graded Doi-Hopf modules

In this section, $H$ is a Hopf colour algebra with a bijective antipode $S_H$, $A$ is a colour algebra and $C$ is a graded coalgebra.

We say that $A$ is a graded left $H$-comodule algebra if it is a graded left $H$-comodule via $\rho_{A,H}(a) = a_{[-1]} \otimes a_{[0]}$ such that $\rho_{A,H}(aa') = (|a_{[0]}|/|a'_{[-1]}|)(a_{[-1]}a'_{[-1]}) \otimes a_{[0]} \otimes a'_{[0]}$ and $\rho_{A,H}(1_A) = 1_H \otimes 1_A$. This is equivalent to say that the multiplication and the unit are graded left $H$-colinear, where the left $H$-coaction on $A \otimes A$ is defined by $(a \otimes a')_{[-1]} \otimes (a \otimes a')_{[0]} = (|a_{[0]}|/|a'_{[-1]}|)(a_{[-1]}a'_{[-1]}) \otimes a_{[0]} \otimes a'_{[0]}$.

We say that $A$ is a graded right $H$-comodule algebra if it is a graded right $H$-comodule via $\rho_{A,H}(a) = a_{[0]} \otimes a_{[1]}$ such that $\rho_{A,H}(aa') = (|a_{[1]}|/|a'_{[0]}|)a_{[0]} \otimes a'_{[0]} \otimes (a_{[1]}a'_{[1]})$ and $\rho_{A,H}(1_A) = 1_A \otimes 1_H$. This is equivalent to say that the multiplication and the unit are graded right $H$-colinear, where the right $H$-coaction on $A \otimes A$ is defined by $(a \otimes a')_{[0]} \otimes (a \otimes a')_{[1]} = (|a_{[1]}|/|a'_{[0]}|)a_{[0]} \otimes a'_{[0]} \otimes (a_{[1]}a'_{[1]})$.

We say that $C$ is a graded left $H$-module coalgebra if $C$ is a graded left $H$-module such that $\Delta_C(h \rightharpoonup c) = (|h_2|/|c_1|)(h_1 \rightharpoonup c_1) \otimes (h_2 \rightharpoonup c_2)$ and $\epsilon_C(h \rightharpoonup c) = \epsilon_H(h)\epsilon_C(c)$. This is equivalent to say that $\Delta_C$ and $\epsilon_C$ are graded left $H$-linear, where the left $H$-action on $C \otimes C$ is defined by

$$h \rightharpoonup (c \otimes c') = (|h_2|/|c_1|)(h_1 \rightharpoonup c_1) \otimes (h_2 \rightharpoonup c_2).$$

We say that $C$ is a graded right $H$-module coalgebra if $C$ is a graded right $H$-module such that $\Delta_C(c \leftharpoonup h) = (|c_2|/|h_1|)(c_1 \leftharpoonup h_1) \otimes (c_2 \leftharpoonup h_2)$ and $\epsilon_C(c \leftharpoonup h) = \epsilon_H(h)\epsilon_C(c)$. This is equivalent to say that $\Delta_C$ and $\epsilon_C$ are graded right $H$-linear, where the right $H$-action on $C \otimes C$ is defined by

$$(c \otimes c') \leftharpoonup h = (|c_2|/|h_1|)(c_1 \leftharpoonup h_1) \otimes (c_2 \leftharpoonup h_2).$$

• Graded left-left Doi-Hopf modules.

Let $A$ be a graded left $H$-comodule algebra and $C$ a graded left $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded left-left Doi-Hopf datum.

The category ${}^{gr-C}_{gr-A}\mathcal{M}(H)$ of graded left-left Doi-Hopf modules is the category whose objects are the graded left $A$-modules and the graded left $C$-comodules $M$ such that $\rho_{M,C}(am) = (|a_{[0]}|/|m_{-1}|)(a_{[-1]} \rightharpoonup m_{-1}) \otimes (a_{[0]}m_0)$. The morphisms of this category are the graded left $A$-linear maps and the graded left $C$-colinear maps. Any graded left-left Doi-Hopf datum $(H, A, C)$ gives rise to a graded left-left entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(a \otimes c) = (|a_{[0]}|/|c|)(a_{[-1]} \rightharpoonup c) \otimes a_{[0]}$. The corresponding category of graded left-left entwined modules coincides with the category ${}^{gr-C}_{gr-A}\mathcal{M}(H)$.

• Graded right-right Doi-Hopf modules.

Let $A$ be a graded right $H$-comodule algebra and $C$ a graded right $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded right-right Doi-Hopf datum.

The category $\mathcal{M}(H)^{gr-C}_{gr-A}$ of graded right-right Doi-Hopf modules is the category whose objects are the graded right $A$-modules and the graded right $C$-comodules $M$ such that $\rho_{M,C}(ma) = (|m_1|/|a_{[0]}|)(m_0 a_{[0]}) \otimes (m_1 \leftharpoonup a_{[1]})$. The morphisms of this category are the graded right $A$-linear maps and the graded right $C$-colinear maps. Any graded right-right Doi-Hopf datum $(H, A, C)$ gives rise to a graded right-right entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(c \otimes a) = (|c|/|a_{[0]}|)a_{[0]} \otimes (c \leftharpoonup a_{[1]})$. The corresponding category of graded right-right entwined modules coincides with $\mathcal{M}(H)^{gr-C}_{gr-A}$.

• Graded left-right Doi-Hopf modules.

Let $A$ be a graded right $H$-comodule algebra and $C$ a graded left $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded left-right Doi-Hopf datum.

The category ${}_{gr-A}\mathcal{M}(H)^{gr-C}$ of graded left-right Doi-Hopf modules is the category whose objects are the graded left $A$-modules and the graded right $C$-comodules $M$ such that $\rho_{M,C}(am) = (|a_{[1]}|/|m_0|)(a_{[0]}m_0) \otimes (a_{[1]} \rightharpoonup m_1)$. The morphisms of this category are the graded left $A$-linear maps and the graded right $C$-colinear maps. Any graded left-right Doi-Hopf datum $(H, A, C)$ gives rise to a graded left-right entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(a \otimes c) = a_{[0]} \otimes (a_{[1]} \rightharpoonup c)$. The corresponding category of graded left-right entwined modules coincides with ${}_{gr-A}\mathcal{M}(H)^{gr-C}$.

• Graded right-left Doi-Hopf modules.

Let $A$ be a graded left $H$-comodule algebra and $C$ a graded right $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded right-left Doi-Hopf datum.

The category ${}^{gr-C}\mathcal{M}(H)_{gr-A}$ of graded right-left Doi-Hopf modules is the category whose objects are the graded right $A$-modules and the graded left $C$-comodules $M$ such that $\rho_{M,C}(ma) = (|m_0|/|a_{[-1]}|)(m_{-1} \leftharpoonup a_{[-1]}) \otimes (m_0 a_{[0]})$. The morphisms of this category are the graded right $A$-linear maps and the graded left $C$-colinear maps. Any graded right-left Doi-Hopf datum $(H, A, C)$ gives rise to a graded right-left entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(c \otimes a) = (c \leftharpoonup a_{[-1]}) \otimes a_{[0]}$. The corresponding category of graded right-left entwined modules coincides with ${}^{gr-C}\mathcal{M}(H)_{gr-A}$.

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Received January 15, 2009