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# Second Order Parallel Tensors on Generalized Sasakian Space Forms and Semi Parallel Hypersurfaces in Sasakian Space Forms

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**Abstract.** In this paper, we show that a second order parallel symmetric tensor in a generalized Sasakian space form is proportional to the metric tensor and we deduce that there is no semi parallel hypersurface in a Sasakian space form.

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## 1. Introduction

In 1926, Levy [4] has proved that a second order parallel symmetric non singular tensor in real space forms is proportional to the metric tensor. In 1989, Sharma [8] has proved that a second order parallel tensor in a Kähler space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kählerian metric and the fundamental 2-form. Recently, Das [6] has established the same result for an $\alpha$-Sasakian manifold ($\alpha \in \mathbb{R}_0$). In this paper we generalize this result to generalized Sasakian space form and we prove that there is no semi parallel hypersurface in a Sasakian space form.

## 2. Preliminaries

Let $M$ denote an n-dimensional Riemannian manifold with its metric tensor $g$ and Levi-Civita connection $\widetilde{\nabla}$. Let $\tilde{R}$ denote the Riemann curvature tensor of $M$. If $B$ is a $(0,2)$ tensor which is parallel with respect to $\widetilde{\nabla}$ then we can show that

$$B\left(\tilde{R}(X,Y)Z,W\right)+B\left(Z,\tilde{R}(X,Y)W\right)=0. \tag{1}$$

Das has proved that

**Theorem 2.1.** [6] *On an $\alpha-K$ contact manifold $(\alpha\in R_0)$ a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

The first purpose of this paper is to present a similar result for a generalized Sasakian space form. Let $(M^{2n+1},g)$ be a $2n+1$ dimensional differentiable manifold and let $(\phi,\xi,\eta)$ be tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$ respectively on $M$, such that

$$\eta(\xi)=1 \quad \phi^2=-I+\xi\otimes\eta$$

which implies

$$\eta\circ\phi=0 \qquad \phi(\xi)=0 \qquad rank(\phi)=2n.$$

If $M$ admits a Riemannian metric $g$, such that

$$\begin{aligned} g(\phi X,\phi Y) &= g(X,Y)-\eta(X)\eta(Y) \\ g(X,\xi) &= \eta(X) \end{aligned}$$

then $(\phi,\xi,\eta,g)$ is called an almost contact metric structure on $M$. If moreover

$$\left(\tilde{\nabla}_X\phi\right)Y=g(X,Y)\xi-\eta(Y)X$$

where $\tilde{\nabla}$ denotes the Riemannian connection of $g$, then $(M,\phi,\xi,\eta,g)$ is called a Sasakian manifold [10].

The sectional curvature of the plane section spanned by the unit tangent vector field $X$ orthogonal to $\xi$ and $\phi X$ is called a $\phi$-sectional curvature. If $M$ has a constant $\phi$-sectional curvature $c$, then $M$ is called a Sasakian space form and denoted by $M^{2n+1}(c)$. The Riemannian curvature of a Sasakian space form is given by the following formula

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)X-g(X,Z)Y] \\ &+\frac{c-1}{4}[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X] \\ &+\frac{c-1}{4}[g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi+g(Z,\phi Y)\phi X \\ &-g(Z,\phi X)\phi Y+2g(X,\phi Y)\phi Z]. \end{aligned}$$

**Example 2.2.** [10] We consider $\mathbb{R}^{2n+1}$ with the coordinates $(x^i, y^i, z),\ i=1,\dots,n$ and its usual contact form $\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right)$. The characteristic field $\xi$ is given by $\xi = 2\frac{\partial}{\partial z}$, the tensor field $\phi$ is given by the matrix $\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}$ and the Riemannian metric $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n} (dx^i\,)^2 + (dy^i\,)^2$ is an associated metric for $\eta$. In this case $\mathbb{R}^{2n+1}$ is a Sasakian space form with $\phi$-sectional curvature $c = -3$ denoted by $\mathbb{R}^{2n+1}(-3)$.

Given an almost contact metric $(M, \phi, \xi, \eta, g)$, $M$ is called generalized Sasakian space form if there exist three functions $f_1$, $f_2$ and $f_3$ such that the Riemannian curvature tensor is given by the following formula

$$\begin{aligned} R(X,Y)Z &= f_1[g(Y,Z)X - g(X,Z)Y] + f_2[g(X,\phi Z)\phi Y \\ &\quad - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] + f_3[\eta(X)\eta(Z)Y \\ &\quad - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]. \end{aligned} \tag{2}$$

In such a case, we will write $M(f_1, f_2, f_3)$. This kind of manifold appears as natural generalization of the Sasakian space form by taking:

$$f_1 = \frac{c+3}{4} \text{ and } f_2 = f_3 = \frac{c-1}{4}.$$

The $\phi$-sectional curvature of a generalized Sasakian space form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$ [9].
Let $N^{2n}$ be an immersed hypersurface of $M^{2n+1}(f_1, f_2, f_3)$. We denote the Levi-Civita connection of $M$ by $\widetilde{\nabla}$ and the Levi-Civita connection of $N$ by $\nabla$. Then we have the formulas of Gauss and Weingarten

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X,Y)r \\ \widetilde{\nabla}_X r &= -SX. \end{aligned}$$

$X$ and $Y$ are tangent vector fields, $r$ a unit normal vector normal to $N$ and $h$ the second fundamental form of $N$ and $S$ the shape operator of $N$. Note that $h$ and $S$ are related by $h(X,Y) = g(SX,Y)$. In a hypersurface the $(0,4)$ tensor field $\tilde{R}.h$ is defined by

$$\tilde{R}.h(X,Y,Z,W) = -h\left(\tilde{R}(X,Y)Z, W\right) - h\left(Z, \tilde{R}(X,Y)W\right).$$

In [2] J. Deprez has defined semi parallel immersions which satisfy the condition $\tilde{R}.h = 0$. The authors F. Dillen, J. Fastenakels, S. Haesen, G. Van Der Veken and L. Verstraelen gave a geometrical interpretation of semi parallel submanifolds.

**Proposition 2.3.** [16] *A submanifold $N$ of $M$ is semi parallel if, $\forall p \in M$, the normal vectors $h(u,v)^{*\perp}$ and $h(u^*, v^*)$ coincide for all $u, v \in T_P M$ and for every coordinate parallelogram in $M$, up to second order. Where $u^*$ and $v^*$ are the parallel transport of $u$ and $v$ with respect to $\nabla$ and $h(u,v)^{*\perp}$ is the parallel transport of $h(u,v)$ with respect to the normal connection $\nabla^{\perp}$.*

We have proved in [3] that

**Theorem 2.4.** *There is not a parallel connected hypersurface in a Sasakian space form* $M^{2n+1}(c)$ *with* $n \geq 2$ *and* $c \neq 1$.

The Ricci tensor is given by Kim [13]

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)\, g(X,Y) - (3f_2 + (2n-1) f_3)\, \eta(X)\, \eta(Y).$$

## 3. Main results

**Theorem 3.1.** *In a generalized Sasakian space form* $M(f_1, f_2, f_3)$ *with* $f_1 \neq f_3$, *a second order parallel symmetric tensor* $B$ *is a constant multiple of the associated positive definite metric tensor.*

*Proof.* If $B$ is parallel $\left(\widetilde{\nabla} B = 0\right)$, it follows that

$$B\left(\tilde{R}(X,Y)Z, W\right) + B\left(Z, \tilde{R}(X,Y)W\right) = 0 \tag{3}$$

for $X, Y, Z$ and $W$ vector fields on $M$.

By taking $Y = \xi$ and $Z = W$ and using equation (2), we have

$$(f_1 - f_3)\left(\eta(Z)\, B(X,Z) - g(X,Z)\, B(\xi,Z) + \eta(Z)\, B(Z,X) - g(X,Z)\, B(Z,\xi)\right) = 0$$

since $f_1 \neq f_3$ and $B$ is symmetric we have

$$\eta(Z)\, B(X,Z) = g(X,Z)\, B(Z,\xi)$$

so

$$B(Z,\xi) = \eta(Z)\, B(\xi,\xi)$$

which implies that

$$\eta(Z)\left(B(X,Z) - g(X,Z)\, B(\xi,\xi)\right) = 0.$$

If $\eta(Z) \neq 0$, we have

$$B(X,Z) = g(X,Z)\, B(\xi,\xi). \tag{4}$$

If $\eta(Z) = 0$, so $B(\xi, Z) = 0$ and by substituting $Y = W = \xi$ in equation (4) we get the above equation. □

**Corollary 3.2.** *If the Ricci tensor of a generalized Sasakian space form* $M(f_1, f_2, f_3)$ *with* $f_1 \neq f_3$ *is parallel, then* $M$ *is Einstein.*

We also have

**Theorem 3.3.** *There are no semi parallel hypersurfaces in a Sasakian space form* $M^{2n+1}(c)$ *with* $c \neq 1$ *and* $n \geq 2$.

*Proof.* If $N$ is a semi parallel hypersurface and $h$ the second fundamental form of $N$, we have:

$$-h\left(\tilde{R}(X,Y)Z,W\right)-h\left(Z,\tilde{R}\left(X,Y\right)W\right)=0$$

by using the same argument as in Theorem 3.1 we deduce that

$$h=\lambda g$$

where $\lambda$ is constant. Consequently

$$\tilde{\nabla}h=0$$

which contradicts Theorem 2.3. □

**Corollary 3.4.** *There are no semi parallel hypersurfaces in* $\mathbb{R}^{2n+1}(-3)$ *with* $n\geq 2$.

**Remark 3.5.** Let us consider the $(2n+1)$-dimensional unit sphere, i.e., $S^{2n+1}=\{z\in\mathbb{C}^{n+1}:|z|=1\}$. Any point $z$ of $S^{2n+1}$ can be identified to $(x^1,\ldots,x^n,y^1,\ldots,y^n)\in\mathbb{R}^{2n+2}$. We put $Jz=(-y^1,\ldots,-y^n,x^1,\ldots,x^n)$, where $J$ is the usual complex structure on $\mathbb{C}^{n+1}$. We define the characteristic vector field $\xi$, the 1-form $\eta$ and the $(1,1)$ tensor $\phi$ by

$$\xi=-Jz,\ \eta(X)=g(X,\xi)\ \text{ and } \phi=s\circ J$$

where $g$ is the induced metric of $\mathbb{C}^{n+1}$ on $S^{2n+1}$ and $s$ is the orthogonal projection of $T_x\mathbb{C}^{n+1}$ on $T_xS^{2n+1}$. So, $S^{2n+1}$ is a Sasakian space form with $\phi$-sectional curvature equal to 1.

Now we consider the Clifford hypersurface $M_{p,q}$ defined by

$$M_{p,q}=S^{2p+1}\left(\sqrt{\frac{p}{2n}}\right)\times S^{2q+1}\left(\sqrt{\frac{q}{2n}}\right),\quad p+q=n-1.$$

Then, $M_{p,q}$ is a minimal hypersurface of $S^{2n+1}$ tangent to the structure field $\xi$ of $S^{2n+1}$ and $M_{p,q}$ has a parallel second fundamental form. Therefore the assumption in Theorem 2.4 and Theorem 3.3 on the $\phi$-sectional curvature $c\neq 1$ of the ambient space is essential.

**Remark 3.6.** The condition $n\geq 2$ in Theorem 2.4 and Theorem 3.3 is also essential, there exist parallel surfaces for $n=1$ [14] and [15].

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