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Degree of Triangle Centers and a Generalization of the Euler Line

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Abstract. We introduce a concept "degree of triangle centers", and give a formula expressing the degree of triangle centers on generalized Euler lines. This generalizes the well known 2 : 1 point configuration on the Euler line. We also introduce a natural family of triangle centers based on the Ceva conjugate and the isotomic conjugate. This family contains many famous triangle centers, and we conjecture that the degree of triangle centers in this family always takes the form $(-2)^k$ for some $k \in \mathbf{Z}$.

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Introduction

In this paper we present a new method to study triangle centers in a systematic way. Concerning triangle centers, there already exist tremendous amount of studies and data, among others Kimberling's excellent book and homepage [32], [36], and also various related problems from elementary geometry are discussed in the surveys and books [4], [7], [9], [12], [23], [26], [41], [50], [51], [52]. In this paper we introduce a concept "degree of triangle centers", and by using it, we clarify the mutual relation of centers on generalized Euler lines (Proposition 1, Theorem 2). Here the term "generalized Euler line" means a line connecting the centroid G and a given triangle center P, and on this line an infinite number of centers lie in a fixed order, which are successively constructed from the initial center P

(for precise definition, see Section 3). This generalizes the well known 2 : 1 point configuration on the Euler line, concerning centroid, orthocenter, circumcenter, nine-point center, etc. In addition we exhibit a new class of triangle centers $\mathcal{P}_C$ based on the Ceva conjugate and the isotomic conjugate, which contains many famous centers and possesses an intimate relationship to the concept "degree of triangle centers".

Now we state the contents of this paper. In Section 1 we define the concept "triangle centers" in terms of barycentric coordinate. In this paper we only treat triangle centers whose barycentric coordinates $f(a, b, c)$ are expressed as a quotient of polynomials of edge lengths a, b, c. In Section 2 we give a definition of degree of triangle centers $d(f)$. We remark that for most famous centers the value $d(f)$ takes the form $(-2)^k$ ($k \in \mathbf{Z}$). In Section 3 we generalize the notion of the Euler line by using barycentric coordinates, and state the relation between centers on generalized Euler lines and the sequence of infinite homothetic triangles constructed successively by taking the medians of edges (Proposition 1). This result is more or less well known for many situations. For example we already know that the circumcenter of ΔABC is equal to the orthocenter of the medial triangle of ΔABC, which is also equal to the nine-point center of the anticomplementary triangle of ΔABC (see Figure 3).

In Section 4 we give a formula on the degree $d(f)$ of centers on generalized Euler lines (Theorem 2), which is the principal result of the present paper. By this formula, in case $\deg f \not\equiv 0 \pmod 3$, we can read the relative position of centers on the generalized Euler line from the value $d(f)$. On the other hand, in case $\deg f \equiv 0 \pmod 3$, the degree $d(f)$ gives an invariant of the generalized Euler line. In Section 5 we give three other invariants of generalized Euler lines. In Section 6 we introduce a new class of triangle centers $\mathcal{P}_C$ based on two conjugates: the Ceva conjugate and the isotomic conjugate. This family contains many famous triangle centers, including centers on the generalized Euler line, and is quite naturally adapted to the concept degree $d(f)$. But unfortunately its explicit form is not determined yet. In the final section Section 7, we give several conjectures on the family $\mathcal{P}_C$. Among others we conjecture that for centers $f(a, b, c)$ in the family $\mathcal{P}_C$ we have $d(f) = (-2)^k$ (Conjecture 2). The number -2 appears in several places in elementary geometry, and we believe that this conjecture gives one basepoint of our approach to the new understanding of elementary geometry, after it is settled affirmatively.

1. Triangle centers

We denote by Δ the set of triangles in the plane $\mathbf{R}^2$. Remark that vertices and edges of triangles are unnamed at this stage. A triangle center is a map $\varphi : \Delta \longrightarrow \mathbf{R}^2$, satisfying the following condition: The map φ commutes with the action of homothety g, i.e., the following diagram is commutative:

$$\begin{array}{ccc} \Delta & \xrightarrow{\varphi} & \mathbf{R}^2 \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ \Delta & \xrightarrow[\varphi]{} & \mathbf{R}^2 \end{array}$$

Here, $g : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is a map defined by

$$g\begin{pmatrix} x \\ y \end{pmatrix} = kA\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix},$$

where $k > 0$, $A \in O(2)$, $\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbf{R}^2$, and $\tilde{g} : \Delta \longrightarrow \Delta$ is a map naturally induced from g. This condition means that triangle centers do not depend on the choice of a coordinate of $\mathbf{R}^2$, and also not on scaling.

Note that in some special cases we must restrict the domain of φ to a subset of Δ. For example to define the Feuerbach point, we must exclude equilateral triangles from Δ.

For later use we put names to vertices and edges of triangles, and reformulate the above definition of triangle centers. Let ΔABC be a triangle in the plane with vertices A, B, C, and we denote by a, b, c the edge lengths of ΔABC, i.e., $a = BC$, $b = CA$, $c = AB$. Then any point P of the plane can be uniquely expressed as $P = xA + yB + zC$ $(x + y + z = 1)$. The triple (x, y, z) is called the barycentric coordinate of P.

Since the shape of a triangle is determined by its edge lengths a, b, c and the position of a triangle center depends only on the shape of the triangle, we know that the barycentric coordinate (x, y, z) of a triangle center $\varphi(\Delta ABC)$ is expressed as functions of a, b, c, i.e., $\varphi(\Delta ABC) = f(a, b, c)A + g(a, b, c)B + h(a, b, c)C$.

Next, since the position of a triangle center $\varphi(\Delta ABC)$ does not depend on the naming of vertices A, B, C (for example, $\varphi(\Delta ABC) = \varphi(\Delta ACB)$ etc.), we obtain the equalities such as

$$f(a, b, c)A + g(a, b, c)B + h(a, b, c)C = f(a, c, b)A + g(a, c, b)C + h(a, c, b)B.$$

From these conditions it is easy to see that the map $\varphi(\Delta ABC) = f(a, b, c)A + g(a, b, c)B + h(a, b, c)C$ gives a triangle center if and only if $g(a, b, c) = f(b, c, a)$, $h(a, b, c) = f(c, a, b)$ and

$$\begin{aligned} &f(a, b, c) + f(b, c, a) + f(c, a, b) = 1, \\ &f(a, b, c) = f(a, c, b), \\ &f(ka, kb, kc) = f(a, b, c) \ \forall k > 0. \end{aligned}$$

Hence we may say that a triangle center is uniquely determined by a function $f(a, b, c)$ satisfying the above three conditions. (See also [2], [28], [31], [33] concerning this formulation.) For example, we have $f(a, b, c) = a/(a + b + c)$ for the incenter X_1, $f(a,b,c) = 1/3$ for the centroid X_2 and $f(a,b,c) = a^2(-a^2+b^2+c^2)/(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$ for the circumcenter X_3. Here the symbol X_k means the k-th triangle center listed in [29], [32], [36].

Clearly, the essential part of a function $f(a,b,c)$ is its numerator, since we can recover the denominator of $f(a,b,c)$ by a cyclic sum of its numerator. Two numerators $f_1(a,b,c)$ and $f_2(a,b,c)$ define the same function $f(a,b,c)$ if and only if $f_2(a,b,c) = f_1(a,b,c)\psi(a,b,c)$ for some symmetric function $\psi(a,b,c)$.

In this paper, for the reason which will be clarified in Section 2, we only treat triangle centers $f(a,b,c)$ whose numerators (and also denominators) are expressed as real homogeneous polynomials of a, b, c. (So the Fermat point, Napoleon point, etc., are beyond our scope in this paper.)

We reformulate the above setting as follows: Let $\mathcal{H}$ be the set of all real homogeneous polynomials $f(a,b,c)$ satisfying

$$f(a,b,c) = f(a,c,b), \qquad f(a,b,c) + f(b,c,a) + f(c,a,b) \neq 0.$$

We say $f(a,b,c)$ and $f'(a,b,c) \in \mathcal{H}$ are equivalent if $f'(a,b,c) = f(a,b,c)\psi(a,b,c)$ for some symmetric function $\psi(a,b,c)$, and denote it by $f(a,b,c) \sim f'(a,b,c)$. Then the quotient set $\mathcal{P} = \mathcal{H}/\sim$ constitutes a natural subclass of triangle centers. In the following we often express the representative class of $f(a,b,c)$ by the same symbol. Note that the position P of a triangle center defined by $f(a,b,c) \in \mathcal{P}$ is given by

$$P = \frac{f(a,b,c)A + f(b,c,a)B + f(c,a,b)C}{f(a,b,c) + f(b,c,a) + f(c,a,b)},$$

and abuse of notations, we often express this point as $f(a,b,c)$, if there is no confusion.

The above formulation on triangle centers seems to be a quite natural one. But actually it contains many redundant points. For example, a point P on the Euler line with $GP : PO = 1 : 100$ does not perhaps possess any geometrical significance. And so some more restricted class of triangle centers should be considered. We will introduce such a class in Section 6 as one attempt. (For another important formulation of triangle centers, see [43], [44], [45].)

2. Degree of triangle centers

Now we define the degree $d(f)$ of triangle centers. We assume that the polynomial $f(a,b,c) \in \mathcal{P}$ corresponding to a triangle center does not possess a symmetric polynomial of a, b, c as its factor. This is always possible, because we may divide $f(a,b,c)$ by a symmetric polynomial of a, b, c in case $f(a,b,c)$ has such a factor. Then after this modification, the polynomial $f(a,b,c)$ is uniquely determined up to a non-zero constant.

Under this preparation, we define a map $d : \mathcal{P} \longrightarrow \mathbf{R} \cup \{\infty\}$ by

$$d(f) = \begin{cases} \dfrac{f(1,\omega,\omega^2)}{f(1,1,1)} & f(1,1,1) \neq 0, \\ \infty & f(1,1,1) = 0, \end{cases}$$

where $\omega, \omega^2 = (-1 \pm \sqrt{3}i)/2$. Note that the value $d(f)$ is well defined since $f(a,b,c)$ is symmetric with respect to b, c and $f(a,b,c)$ is uniquely determined up to a non-zero constant, as we explained above. Also note that $d(f)$ takes a real value since $\omega + \omega^2 = -1$ and $\omega \cdot \omega^2 = 1$. The equality $f(1,1,1) = 0$ holds if and only if the triangle center corresponding to $f(a,b,c)$ is undefined for equilateral triangles (such as the Feuerbach point).

Since b and c represent the edge lengths of a triangle, they must be real numbers, and so the value $f(1,\omega,\omega^2)$ itself does not possess any geometric meaning. But, as we shall explain later, we can attach a nice geometric meaning to the value $d(f)$, which actually expresses a "degree of triangle centers" in a sense.

The assumption that a polynomial $f(a,b,c)$ has no symmetric factor is indispensable in order to obtain the definite value. In fact if we use the expression $f(a,b,c) = (a+b)(b+c)(c+a)(-a+b+c)$ instead of $f(a,b,c) = -a+b+c$, we have $d(f) = 1/4$, though the actual value is $d(f) = -2$.

Interestingly, as the following examples show, the value $d(f)$ takes the form $(-2)^k \quad (k \in \mathbf{Z})$ in many (or rather, for most) cases.

Example 1.

X_1 (I, Incenter) : $f = a, \quad d(f) = 1,$

X_2 (G, Centroid) : $f = 1, \quad d(f) = 1,$

X_3 (O, Circumcenter) : $f = a^2(-a^2+b^2+c^2), \quad d(f) = -2,$

X_4 (H, Orthocenter) : $f = (a^2-b^2+c^2)(a^2+b^2-c^2), \quad d(f) = 4,$

X_5 (N, Nine-point center) : $f = a^2(b^2+c^2) - (b^2-c^2)^2, \quad d(f) = 1,$

X_8 (Na, Nagel point) : $f = -a+b+c, \quad d(f) = -2,$

X_{10} (S, Spieker center) : $f = b+c, \quad d(f) = -\frac{1}{2},$

X_{11} (F, Feuerbach point) : $f = (b-c)^2(-a+b+c), \quad d(f) = \infty,$

X_{944} (Ho, Hofstadter trapezoid point) : $f = -3a^4 + 2a^3(b+c) + 2a^2(b-c)^2 - 2a(b-c)^2(b+c) + (b^2-c^2)^2, \quad d(f) = -20.$

3. A generalization of the Euler line

The Euler line is a fundamental line of a triangle, which passes through many famous triangle centers such as centroid, circumcenter, orthocenter, nine-point center, de Longchamps point, etc. Also the Nagel line passes through the centers such as centroid, incenter, Nagel point, Spieker center, etc. In this section, by using $f(a,b,c) \in \mathcal{P}$, we generalize these lines and centers lying on it in a unified way (Proposition 1). This result is a preparation for the next section.

First, for a given $f(a,b,c) \in \mathcal{P}$ we define a new center $f_n \in \mathcal{P}$ ($n \in \mathbf{Z}$) by

$$f_n(a,b,c) = 2^n(f_a + f_b + f_c) + (-1)^n(2f_a - f_b - f_c),$$

where $f_a = f(a,b,c)$, $f_b = f(b,c,a)$ and $f_c = f(c,a,b)$. Clearly this definition does not depend of the choice of representatives $f(a,b,c)$ in $\mathcal{P}$. We can easily see that

$$f_0 = f, \qquad (f_m)_n = f_{m+n}, \qquad (m,n \in \mathbf{Z}).$$

(Strictly speaking, we have $f_0 = 3f$, and so we should write it as $f_0 \sim f$. But in the following, we write $f_0 = f$ in such a situation, if there is no danger of confusion.) In particular, two correspondences $f \longmapsto f_1$ and $f \longmapsto f_{-1}$ give mutually the inverse map to each other since $(f_1)_{-1} = (f_{-1})_1 = f_0 = f$.

Example 2. In case $f(a,b,c) = a^2(-a^2+b^2+c^2)$, then we have $f_1(a,b,c) = a^2(b^2+c^2)-(b^2-c^2)^2$ and $f_{-1}(a,b,c) = (a^2-b^2+c^2)(a^2+b^2-c^2)$. Note that f corresponds to the circumcenter and f_1, f_{-1} correspond to the nine-point center and the orthocenter, respectively.

The center $f_n(a,b,c) \in \mathcal{P}$ constructed above possesses the following geometric meaning. First we consider an infinite series of triangles $\Delta_n(ABC)$ $(n \in \mathbf{Z})$, whose vertices are given by

$$\begin{aligned} A_n &= \tfrac{1}{3}\{A+B+C+\left(-\tfrac{1}{2}\right)^n(2A-B-C)\}, \\ B_n &= \tfrac{1}{3}\{A+B+C+\left(-\tfrac{1}{2}\right)^n(2B-C-A)\}, \\ C_n &= \tfrac{1}{3}\{A+B+C+\left(-\tfrac{1}{2}\right)^n(2C-A-B)\}. \end{aligned}$$

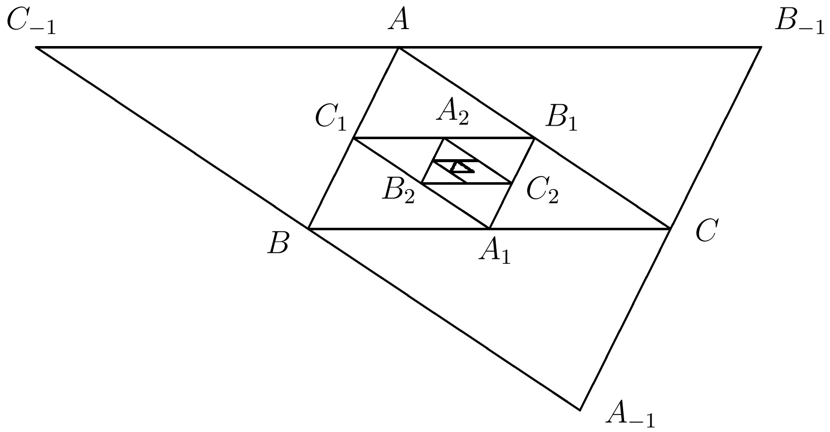

Figure 1

Clearly, the triangle $\Delta_0(ABC)$ is the initial triangle ΔABC itself. It is easy to see that $\Delta_1(ABC)$ is the medial triangle of $\Delta_0(ABC)$, and $\Delta_2(ABC)$ is the medial triangle of $\Delta_1(ABC)$, etc. Also $\Delta_{-1}(ABC)$ is the anticomplementary triangle of $\Delta_0(ABC)$, and $\Delta_{-2}(ABC)$ is the anticomplementary triangle of $\Delta_{-1}(ABC)$, etc. In general, the triangle $\Delta_{n+1}(ABC)$ is the medial triangle of $\Delta_n(ABC)$, and conversely, $\Delta_n(ABC)$ is the anticomplementary triangle of $\Delta_{n+1}(ABC)$. Further, the property $\Delta_m(\Delta_n(ABC)) = \Delta_{m+n}(ABC)$ holds. We call $\Delta_n(ABC)$ the n-th iterated triangle of ΔABC. Note that the centroid of $\Delta_n(ABC)$ coincides with the centroid of ΔABC for any $n \in \mathbf{Z}$, and the triangle $\Delta_n(ABC)$ converges to the centroid as $n \to \infty$. Then we have the following proposition.

Proposition 1. *Let $f(a,b,c) \in \mathcal{P}$ and $n \in \mathbf{Z}$.*

(1) *The triangle center of ΔABC determined by the polynomial $f_n(a,b,c) \in \mathcal{P}$ coincides with the triangle center of the n-th iterated triangle $\Delta_n(ABC)$ which is determined by the initial $f(a,b,c)$. Or more generally, for any $k \in \mathbf{Z}$, this point is equal to the center of $\Delta_k(ABC)$ determined by the polynomial $f_{n-k}(a,b,c)$.*

(2) *The infinite centers of ΔABC determined by $f_n(a,b,c)$ ($n \in \mathbf{Z}$) together with the centroid G of ΔABC are collinear. For any $n \in \mathbf{Z}$ the centroid G is situated between two points $f_n(a,b,c)$, $f_{n+1}(a,b,c)$, and the ratio $f_n(a,b,c)\,G : G f_{n+1}(a,b,c)$ is always $2:1$, i.e., $\overrightarrow{Gf_n} = -2\,\overrightarrow{Gf_{n+1}}$. Consequently, $\lim_{n\to\infty} f_n(a,b,c) = G$.*

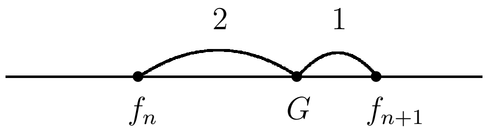

Figure 2

The above fact (1) implies that in spite of their appearances, the centers determined by the family of polynomials $\{f_n(a,b,c)\}_{n\in\mathbf{Z}}$ define essentially "one" geometric concept in the family of iterated triangles $\{\Delta_n(ABC)\}_{n\in\mathbf{Z}}$.

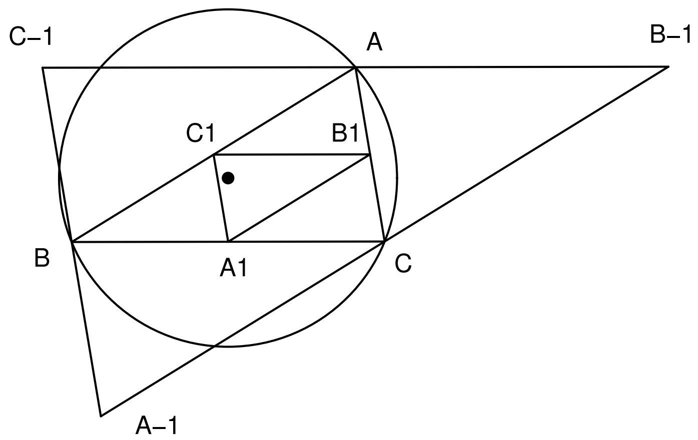

Figure 3

For example, from Figure 3, we can easily see that the circumcenter of ΔABC, the orthocenter of the medial triangle, and the nine-point center of the anticomplementary triangle give the same point in the plane. This is the special case of the above proposition. We put $n = 1$ and $f(a,b,c) = (a^2-b^2+c^2)(a^2+b^2-c^2)$, which corresponds to the orthocenter of ΔABC. Then we have $f_1(a,b,c) = a^2(-a^2+b^2+c^2)$ and $f_2(a,b,c) = a^2(b^2+c^2)-(b^2-c^2)^2$, which correspond to the circumcenter and the nine-point center of ΔABC, respectively. And hence, from

Proposition 1 (1), three points determined by f_1 in $\Delta_0(ABC)$, f_0 in $\Delta_1(ABC)$ and f_2 in $\Delta_{-1}(ABC)$ coincide, which is nothing but the above statement.

Moreover, we may add the de Longchamps point to this family of centers, since the de Longchamps point is by definition the orthocenter of the anticomplementary triangle. Actually it corresponds to the polynomial $f_{-1}(a,b,c) = 3a^4 - 2a^2(b^2 + c^2) - (b^2 - c^2)^2$, and is the circumcenter of $\Delta_{-2}(ABC)$. We can also add the point X_{140} to this family since it corresponds to the polynomial $f_3(a,b,c) = 2a^4 - 3a^2(b^2 + c^2) + (b^2 - c^2)^2$.

Therefore, the orthocenter, the nine-point center, the de Longchamps point, etc., are the circumcenters of $\Delta_n(ABC)$ for some $n \in \mathbf{Z}$, and so we may say that they indicate essentially one concept in the set of triangle centers. Or in other words, we can draw the Euler line by only plotting circumcenters of triangles $\Delta_n(ABC)$. This fact is already stated in many publications.

Similarly, we consider the case $f(a,b,c) = a$ which determines the incenter. Then we have $f_1(a,b,c) = b+c$, $f_{-1}(a,b,c) = -a+b+c$, and these linear polynomials correspond to the Spieker center X_{10} and the Nagel point X_8, respectively. From the above proposition, we know that the incenter of ΔABC coincides with the Spieker center of the anticomplementary triangle, and also coincides with the Nagel point of the medial triangle. Or in other words, the Nagel point of ΔABC is the incenter of the anticomplementary triangle, and the Spieker center of ΔABC is the incenter of the medial triangle, which is nothing but the definition of the Spieker center. These three points are collinear, including the centroid G. The ratios of distance of these points keep the value 2 : 1, and this line is called the Nagel line. The Nagel line also contains the points $f_2(a,b,c) = 2a+b+c$, $f_{-2}(a,b,c) = -3a+b+c$, which are X_{1125} and X_{145}, respectively in Kimberling's list. In this case the Nagel line is obtained by only drawing incenters of $\Delta_n(ABC)$, which is also a well known fact.

Starting from the Feuerbach point X_{11} corresponding to the polynomial $f = (b-c)^2(-a+b+c)$, we know that the points X_{149} (f_{-2}), X_{100} ($f_{-1} = a(a-b)(a-c)$), X_{11} (f_0), X_{3035} (f_1) and $X_2 = G$ are collinear (see [32], [36]).

In this way, the above proposition shows that there are an infinite number of lines that possess similar property as the Euler line. In the following we call such lines the generalized Euler lines determined by $f(a,b,c)$ (or $f_n(a,b,c)$).

Perhaps the contents of Proposition 1 are more or less well known, and the proof is not so difficult. For example, see references [2], [6], [8], [14], [17], [18], [19], [22],[46], [47], [48], [61], [62], [63], etc. But as far as the author knows, a unified treatment as stated in Proposition 1 cannot be found anywhere, and so we give here its complete proof.

Proof of Proposition 1. First, we state one remark. In considering the center of $\Delta_n(ABC)$ corresponding to the polynomial f, we must substitute in f the edge lengths of the triangle $\Delta_n(ABC)$, instead of a, b, c. But since the edge lengths of $\Delta_n(ABC)$ are given by $B_nC_n = a/2^n$, etc., and since f is a homogeneous polynomial, we may use the initial edge lengths a, b, c to calculate the position of the center.

(1) We show that the center of $\Delta_k(ABC)$ determined by $f_{n-k}(a,b,c)$ is independent of k. This center is represented by

$$\frac{f_{n-k}(a,b,c)A_k + f_{n-k}(b,c,a)B_k + f_{n-k}(c,a,b)C_k}{f_{n-k}(a,b,c) + f_{n-k}(b,c,a) + f_{n-k}(c,a,b)}.$$

We substitute

$$\begin{aligned} &f_{n-k}(a,b,c) = 2^{n-k}(f_a + f_b + f_c) + (-1)^{n-k}(2f_a - f_b - f_c), \\ &A_k = \tfrac{1}{3}\{A + B + C + \left(-\tfrac{1}{2}\right)^k (2A - B - C)\}, \end{aligned}$$

etc., to the above point. Here $f_a = f(a,b,c)$, $f_b = f(b,c,a)$, $f_c = f(c,a,b)$, as we stated before. Then after some calculations, it follows that the coefficient of A is equal to

$$\frac{2^n(f_a + f_b + f_c) + (-1)^n(2f_a - f_b - f_c)}{3 \cdot 2^n(f_a + f_b + f_c)} = \frac{f_n(a,b,c)}{f_n(a,b,c) + f_n(b,c,a) + f_n(c,a,b)}.$$

And thus the integer k disappears. This point corresponds to the center of ΔABC determined by $f_n(a,b,c)$.

(2) The point determined by $f_n(a,b,c)$ is given by

$$\frac{f_n(a,b,c)A + f_n(b,c,a)B + f_n(c,a,b)C}{f_n(a,b,c) + f_n(b,c,a) + f_n(c,a,b)}$$

and hence we have

$$\overrightarrow{Gf_n} = \frac{f_n(a,b,c)A + f_n(b,c,a)B + f_n(c,a,b)C}{f_n(a,b,c) + f_n(b,c,a) + f_n(c,a,b)} - \frac{1}{3}(A + B + C).$$

Substituting $f_n(a,b,c) = 2^n(f_a + f_b + f_c) + (-1)^n(2f_a - f_b - f_c)$, etc., into this equality, we have finally

$$\overrightarrow{Gf_n} = \left(-\frac{1}{2}\right)^n \frac{(2f_a - f_b - f_c)A + (2f_b - f_c - f_a)B + (2f_c - f_a - f_b)C}{3(f_a + f_b + f_c)}.$$

All results follow immediately from this equality. q.e.d.

As we can easily see, the polynomiality assumption on $f(a,b,c)$ is actually unnecessary to prove this proposition. The Euler line itself possesses several interesting properties related to other triangle centers. (See for example [49], [54], [55], [56].) It is natural to ask to what extent these similar properties hold for generalized Euler lines.

4. Main theorem

In Proposition 1 we showed that the center of ΔABC defined by the polynomial $f_n(a,b,c)$ coincides with the center of the n-th iterated triangle $\Delta_n(ABC)$ defined by the initial $f(a,b,c)$. And so we may say that the absolute value $|n|$ in

$f_n(a,b,c)$ expresses a sort of complexity of the center, since the geometric situation of $f_n(a,b,c)$ becomes more complicated as the value $|n|$ becomes large.

The following theorem shows that the degree $d(f_n)$ of a center $f_n(a,b,c)$, which we introduced in §2, essentially determines the value n in case $\deg f \not\equiv 0 \,(\mathrm{mod}\, 3)$, where $\deg f$ means the usual degree of the polynomial f. This implies that a complexity of $f(a,b,c)$ can be directly read from $d(f)$ in this case.

Theorem 2. *Assume that $f(a,b,c) \in \mathcal{P}$ does not possess a symmetric factor. Then $f_n(a,b,c)$ also does not possess a symmetric factor for any $n \in \mathbf{Z}$, and the following equality holds in case $f(1,1,1) \neq 0$:*

$$d(f_n) = \begin{cases} \left(-\frac{1}{2}\right)^n d(f) & \deg f \not\equiv 0 \,(\mathrm{mod}\, 3), \\ d(f) & \deg f \equiv 0 \,(\mathrm{mod}\, 3). \end{cases}$$

Thus, if $\deg f \not\equiv 0 (\mathrm{mod} 3)$ and $f(1,1,1) \neq 0$, we can read the value n from the data $d(f_n)/d(f)$. This number n indicates the relative position of f_n in the generalized Euler line. If n is sufficiently large, its position is close to the centroid G, and conversely if n is sufficiently small, it is situated far from G on the generalized Euler line. For example, the center X_{60} defined by $f(a,b,c) = a^2(a+b)^2(a+c)^2(-a+b+c)$ is situated near G since $d(f) = (-1/2)^3$.

Example 3. On the Euler line, according to the distance from the centroid G, the nine-point center, the circumcenter, the orthocenter, and the de Longchamps point lie in this order. The degree of these centers is 1, -2, $(-2)^2$, $(-2)^3$, respectively.

Similarly, on the Nagel line, X_{1125}, the Spieker center, the incenter, the Nagel point, and X_{145} lie in this order, and the degree of these centers is given by $(-1/2)^2$, $-1/2$, 1, -2 and $(-2)^2$, respectively.

We remark that two different classes $\{f_n(a,b,c)\}_{n\in\mathbf{Z}}$ and $\{f'_n(a,b,c)\}_{n\in\mathbf{Z}}$ may define geometrically the same line in the plane. For example, the Exeter point X_{22} also lies on the Euler line. But it is defined by the sextic polynomial $a^2(-a^4+b^4+c^4)$, and clearly does not belong to the standard family of centers on the Euler line, which is generated by $f(a,b,c) = a^2(-a^2+b^2+c^2)$. Similarly, the Schiffler point X_{21} also lies on the Euler line, and this point corresponds to the quartic polynomial $a(-a+b+c)(a+b)(a+c)$. But it is easy to see that this point also does not belong to the standard family.

Proof of Theorem 2. In general, if f' is divided by a symmetric polynomial, then $f'_{-n} = 2^{-n}(f'_a + f'_b + f'_c) + (-1)^{-n}(2f'_a - f'_b - f'_c)$ is also divided by the same symmetric polynomial, where $f'_a = f'(a,b,c)$, etc. Now assume that f does not possess a symmetric factor, but f_n admits a symmetric factor. Then by the above reason $(f_n)_{-n} = f_0 = f$ possess a symmetric factor, which is a contradiction. Hence f_n does not possess a symmetric factor.

Now we prove the main part. We express $f(a,b,c)$ as $f(a,b,c) = \sum_{i,j,k} f_{ijk} a^i (b+c)^j (bc)^k$, where $\deg f = i+j+2k$. Then, since

$$f_n(a,b,c) = 2^n(f_a + f_b + f_c) + (-1)^n(2f_a - f_b - f_c),$$

we have $f_n(1,1,1) = 3 \cdot 2^n f(1,1,1)$. In particular, $f(1,1,1) \neq 0$ if and only if $f_n(1,1,1) \neq 0$. On the other hand, we have

$$\begin{aligned} f_n(1,\omega,\omega^2) = 2^n\{f(1,\omega,\omega^2) + f(\omega,\omega^2,1) + f(\omega^2,1,\omega)\} \\ + (-1)^n\{2f(1,\omega,\omega^2) - f(\omega,\omega^2,1) - f(\omega^2,1,\omega)\}. \end{aligned}$$

Here we have

$$\begin{aligned} f(1,\omega,\omega^2) &= \sum f_{ijk}(-1)^j, \\ f(\omega,\omega^2,1) &= \sum f_{ijk}\omega^i(-\omega)^j\omega^{2k} = \sum f_{ijk}(-1)^j\omega^{i+j+2k}, \\ f(\omega^2,1,\omega) &= \sum f_{ijk}\omega^{2i}(-\omega^2)^j\omega^k = \sum f_{ijk}(-1)^j\omega^{2i+2j+k} \\ &= \sum f_{ijk}(-1)^j\omega^{2(i+j+2k)}. \end{aligned}$$

Hence, by putting $i+j+2k = p$, we have

$$\begin{aligned} f_n(1,\omega,\omega^2) = 2^n \left(\sum f_{ijk}(-1)^j\right)(1+\omega^p+\omega^{2p}) \\ + (-1)^n \left(\sum f_{ijk}(-1)^j\right)(2-\omega^p-\omega^{2p}). \end{aligned}$$

If $p \equiv 0 \,(\mathrm{mod}\, 3)$, we have $f_n(1,\omega,\omega^2) = 3 \cdot 2^n \sum f_{ijk}(-1)^j = 3 \cdot 2^n f(1,\omega,\omega^2)$, which implies

$$d(f_n) = \frac{f_n(1,\omega,\omega^2)}{f_n(1,1,1)} = \frac{f(1,\omega,\omega^2)}{f(1,1,1)} = d(f).$$

If $p \not\equiv 0 \,(\mathrm{mod}\, 3)$, we have $f_n(1,\omega,\omega^2) = 3(-1)^n \sum f_{ijk}(-1)^j = 3(-1)^n f(1,\omega,\omega^2)$. Hence

$$d(f_n) = \frac{f_n(1,\omega,\omega^2)}{f_n(1,1,1)} = \left(-\frac{1}{2}\right)^n \frac{f(1,\omega,\omega^2)}{f(1,1,1)} = \left(-\frac{1}{2}\right)^n d(f).$$

This completes the proof. q.e.d.

5. Invariants of generalized Euler lines

From Theorem 2 we may say that in case $\deg f \equiv 0 \,(\mathrm{mod}\, 3)$, the degree $d(f_n)$ gives an invariant of the generalized Euler line, since the value $d(f_n)$ does not depend on n. In this section we introduce other types of invariants naturally associated with generalized Euler lines.

We first consider the following quantity $q(f)$, assuming that $f(a,b,c) \in \mathcal{P}$ does not possess a symmetric factor:

$$q(f) = \begin{cases} \dfrac{f(1,\alpha,\beta)}{f(1,1,1)} & f(1,1,1) \neq 0, \\ \infty & f(1,1,1) = 0, \end{cases}$$

where α, $\beta = (-1 \pm \sqrt{3})/2$. Clearly $q(f)$ is well defined. Then we have the following proposition.

Proposition 3. *Assume* $\deg f = 2$ *and* f *does not possess a symmetric factor. Then the value* $q(f)$ *gives an invariant of the generalized Euler line determined by* f*.*

Proof. By putting $f(a,b,c) = pa^2 + qa(b+c) + r(b^2+c^2) + sbc$, we have easily $f_n(1,\alpha,\beta) = 3\cdot 2^n(p-q+2r-s/2) = 3\cdot 2^n f(1,\alpha,\beta)$ and $f_n(1,1,1) = 3\cdot 2^n(p+2q+2r+s) = 3\cdot 2^n f(1,1,1)$, which imply $f_n(1,\alpha,\beta)/f_n(1,1,1) = f(1,\alpha,\beta)/f(1,1,1)$ for any $n \in \mathbf{Z}$. Here, we use the relation $\alpha+\beta = -1$, $\alpha\beta = -1/2$, $\alpha^2+\beta^2 = 2$ to check this fact. q.e.d.

Curiously, for many triangle centers, the value of this invariant $q(f)$ also takes the form $(-2)^k$, as in the case of $d(f)$. But in the case of general degree ($\deg f \neq 2$), the value $q(f)$ does not give an invariant of the generalized Euler line. For example we have $q(f_n) = (-1/2)^n$ for $f_n = 2^n(a+b+c) + (-1)^n(2a-b-c)$.

Next, considering the direction of the generalized Euler line, we can construct another invariant of the generalized Euler line.

In the proof of Proposition 1, we showed that the vector

$$\frac{(2f_a - f_b - f_c)A + (2f_b - f_c - f_a)B + (2f_c - f_a - f_b)C}{3(f_a + f_b + f_c)}$$

is parallel to the generalized Euler line determined by the family $\{f_n(a,b,c)\}_{n\in\mathbf{Z}}$. Essentially this direction is determined by the coefficient of A

$$\frac{2f_a - f_b - f_c}{3(f_a + f_b + f_c)},$$

since remaining two coefficients can be obtained by cyclic permutation of a, b, c. But the above coefficient depends on n, i.e., if we use f_n instead of f, then n appears in the coefficient. Slightly modifying this quantity, we obtain another invariant as follows.

Proposition 4. *Assume* $f(a,b,c),\ f'(a,b,c) \neq 1$.

(1) *Two points* $f(a,b,c)$ *and* $f'(a,b,c)$ *lie on the same generalized Euler line if and only if*

$$2{f'}_a - {f'}_b - {f'}_c = (2f_a - f_b - f_c)\psi(a,b,c)$$

for some symmetric function $\psi(a,b,c)$.

(2) *Assume* $\deg f \not\equiv 0 \pmod 3$, $f(1,\omega,\omega^2) \neq 0$, *and* f *does not possess a symmetric factor. Then* $f_n(1,\omega,\omega^2) \neq 0$, *and the polynomial*

$$\frac{2(f_n)_a - (f_n)_b - (f_n)_c}{f_n(1,\omega,\omega^2)}$$

does not depend on n. *Hence the polynomial* $(2f_a - f_b - f_c)/f(1,\omega,\omega^2)$ *gives an invariant of the generalized Euler line determined by the family* $\{f_n(a,b,c)\}_{n\in\mathbf{Z}}$.

Proof. The statement (1) is clear from the explanation stated above. We show (2). The property $f_n(1,\omega,\omega^2) \neq 0$ follows immediately from the equality $f_n(1,\omega,\omega^2) = 3(-1)^n f(1,\omega,\omega^2)$, as we explained in the proof of Theorem 2 in case $\deg f \not\equiv 0 \pmod 3$. By the definition of $f_n(a,b,c)$, we immediately obtain the equality $2(f_n)_a - (f_n)_b - (f_n)_c = 3(-1)^n(2f_a - f_b - f_c)$. Hence the statement (2) follows. q.e.d.

Remark. The above invariant is related to the direction of the generalized Euler line as follows. We further assume $f(1,1,1) \neq 0$ in addition to the assumption in Proposition 4 (2). Then we have $f_n(1,1,1) \neq 0$, as explained in the proof of Theorem 2, and so the degree $d(f_n)$ has a non-zero finite value. Since $\deg f \not\equiv 0 \pmod 3$, we have $d(f_{n+1}) = -\frac{1}{2}d(f_n)$ from Theorem 2. Then, since $\overrightarrow{Gf_{n+1}} = -\frac{1}{2}\overrightarrow{Gf_n}$ (Proposition 1 (2)), we see that the vector $\overrightarrow{Gf_n}/d(f_n)$ does not depend on n, and it is equal to

$$\frac{\{2(f_n)_a - (f_n)_b - (f_n)_c\}A + 1\{2(f_n)_b - (f_n)_c - (f_n)_a\}B + \{2(f_n)_c - (f_n)_a - (f_n)_b\}C}{3d(f_n)\{(f_n)_a + (f_n)_b + (f_n)_c\}}.$$

The coefficient of A is equal to

$$\begin{aligned} \frac{2(f_n)_a - (f_n)_b - (f_n)_c}{3d(f_n)\{(f_n)_a + (f_n)_b + (f_n)_c\}} &= \frac{2(f_n)_a - (f_n)_b - (f_n)_c}{f_n(1,\omega,\omega^2)} \cdot \frac{f_n(1,1,1)}{3\{(f_n)_a + (f_n)_b + (f_n)_c\}} \\ &= \frac{2(f_n)_a - (f_n)_b - (f_n)_c}{f_n(1,\omega,\omega^2)} \cdot \frac{f(1,1,1)}{3(f_a + f_b + f_c)}, \end{aligned}$$

because $f_n(1,1,1) = 3 \cdot 2^n f(1,1,1)$ and $(f_n)_a + (f_n)_b + (f_n)_c = 3 \cdot 2^n(f_a + f_b + f_c)$. Since $f(1,1,1)/3(f_a + f_b + f_c)$ does not depend on n, we know once again that $\{2(f_n)_a - (f_n)_b - (f_n)_c\}/f_n(1,\omega,\omega^2)$ gives an invariant of the generalized Euler line.

In this remark, we showed that the symmetric polynomial

$$\frac{(f_n)_a + (f_n)_b + (f_n)_c}{f_n(1,1,1)}$$

is also an invariant of the generalized Euler line. In the following, assuming that $f(a,b,c) \in \mathcal{P}$ does not possess a symmetric factor, we put

$$e(f) = \begin{cases} \dfrac{2f_a - f_b - f_c}{f(1,\omega,\omega^2)} & f(1,\omega,\omega^2) \neq 0, \\ \infty & f(1,\omega,\omega^2) = 0, \end{cases}$$

$$s(f) = \begin{cases} \dfrac{f_a + f_b + f_c}{f(1,1,1)} & f(1,1,1) \neq 0, \\ \infty & f(1,1,1) = 0. \end{cases}$$

These two quantities are also well defined. As seen above, the direction of the generalized Euler line is essentially determined by the quotient of two polynomials $e(f)/s(f)$. Remark that $e(f)$ is not an invariant of the generalized Euler line in case $\deg f \equiv 0 \pmod 3$, as we see in the next example, though for the invariant $s(f)$ such an assumption is unnecessary.

From Proposition 4 (1), we know that two centers determined by $f(a,b,c)$ and $f'(a,b,c)$ lie on the same generalized Euler line if and only if $e(f)$ and $e(f')$ are equal up to a symmetric function (under the assumption $f(1,\omega,\omega^2)$, $f'(1,\omega,\omega^2) \neq 0$).

Example 4. For the centers on the Euler line determined by the circumcenter, the orthocenter, the nine-point center, we have the common expression

$$\begin{aligned} e(f) &= \tfrac{1}{2}\{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2\}, \\ s(f) &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c). \end{aligned}$$

For the Schiffler point X_{21}, which also lies on the Euler line, the invariant $s(f)$ takes a little different form:

$$\begin{aligned} e(f) &= \tfrac{1}{2}\{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2\}, \\ s(f) &= \tfrac{1}{4}(a+b+c)\{(-a+b+c)(a-b+c)(a+b-c) + 3abc\}. \end{aligned}$$

Remark that X_{21} is not contained in the above family of centers on the Euler line.

For other points on the Euler line such as X_{22}, X_{23}, X_{24}, we have

$$X_{22} \; : \; \begin{cases} e(f) = \frac{1}{2}(a^2+b^2+c^2)\{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2\}, \\ s(f) = (-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2) + 2a^2b^2c^2, \end{cases}$$

$$X_{23} \; : \; \begin{cases} e(f) = \frac{1}{3}(a^2+b^2+c^2)\{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2\}, \\ s(f) = \infty, \end{cases}$$

$$X_{24} \; : \; \begin{cases} e(f) = -\frac{1}{8}\{(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2) + 4a^2b^2c^2\} \\ \qquad\qquad \times\{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2\}, \\ s(f) = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\ \qquad\qquad \times\{2a^2b^2c^2 - (-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)\}. \end{cases}$$

As for the centers X_{22} and X_{23}, corresponding to the sextic polynomials $f(a,b,c) = a^2(-a^4+b^4+c^4)$ and $f(a,b,c) = a^2(-a^4+b^4+c^4-b^2c^2)$, we have

$$\begin{aligned} e(f_n) &= \tfrac{1}{2}(-\tfrac{1}{2})^n(a^2+b^2+c^2)\{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2\}, \\ e(f_n) &= \tfrac{1}{3}(-\tfrac{1}{2})^n(a^2+b^2+c^2)\{2a^4 - a^2(b^2+c^2) - (b^2-c^2)^2\}, \end{aligned}$$

respectively, and these values clearly depend on n. The values $e(f)$ for these centers imply that X_{21}, X_{22}, X_{23}, X_{24} all lie geometrically on the same Euler line, though they do lie on "algebraically different" generalized Euler lines.

As for the Nagel line, we have

$$e(f_n) = 2a - b - c, \qquad s(f_n) = a + b + c$$

for $f_n(a,b,c) = 2^n(a+b+c) + (-1)^n(2a-b-c)$, and so the polynomial $2a-b-c$ gives a fundamental invariant of this line.

Finally, we remark that $f(a,b,c)$ has a natural decomposition into three quantities $d(f)$, $e(f)$, $s(f)$ as follows. Assume $f(1,1,1)$, $f(1,\omega,\omega^2) \neq 0$. Then we have

$$\begin{aligned} d(f)e(f) + s(f) &= \frac{f(1,\omega,\omega^2)}{f(1,1,1)} \cdot \frac{2f_a - f_b - f_c}{f(1,\omega,\omega^2)} + \frac{f_a + f_b + f_c}{f(1,1,1)} \\ &= \frac{3}{f(1,1,1)} f(a,b,c) = f(a,b,c), \end{aligned}$$

since $3/f(1,1,1)$ is a non-zero constant. This equality means that a triangle center in $\mathcal{P}$ is uniquely determined by these three quantities (or invariants).

6. A new family of centers

In this section we introduce a new family of triangle centers $\mathcal{P}_C$ that is based on two conjugates: The isotomic conjugate and the Ceva conjugate. This family $\mathcal{P}_C$ is a subset of $\mathcal{P}$ that was introduced in Section 1. The family of centers $\mathcal{P}$ is a natural one, on which we can introduce the concept degree $d(f)$. But most centers in $\mathcal{P}$ do not possess a geometric meaning. For example, on the Euler line, the point which divides the centroid and the circumcenter with ratio $1 : x$ seems to have no geometric meaning for most $x \in \mathbf{R}$. (Perhaps, geometrically important centers appear discretely, though the term "geometrically important" has no precise definition here.) So it is desirable to restrict centers to a subclass of $\mathcal{P}$, all of which possess some definite geometric meaning. The class $\mathcal{P}_C$ which we introduce here contains many famous centers. It also contains the family $\{f_n(a,b,c)\}_{n\in\mathbf{Z}}$ on the generalized Euler line if $f(a,b,c) \in \mathcal{P}_C$.

Definition. *$\mathcal{P}_C$ is a minimum subset of $\mathcal{P}$ satisfying the following conditions:*

(i) $1,\ a \in \mathcal{P}_C$,

(ii) $f,\ f' \in \mathcal{P}_C \Longrightarrow f_b{f'}_c + {f'}_b f_c \in \mathcal{P}_C$,

(iii) $f,\ f' \in \mathcal{P}_C \Longrightarrow {f'}_a(-f_a{f'}_a + f_b{f'}_b + f_c{f'}_c) \in \mathcal{P}_C$.

Here, ${f'}_b$ means $f'(b,c,a)$ etc., as before. Note that the conditions (ii), (iii) do not depend on the choice of representatives of f and f'. The subclass $\mathcal{P}_C$ actually exists. In fact it is the intersection of all subsets of $\mathcal{P}$ satisfying the above conditions (i), (ii), (iii). But unfortunately, its explicit form is not clear at present (see Section 7).

Before stating several properties of $\mathcal{P}_C$, we must give some explanation on the definition of $\mathcal{P}_C$. The condition (i) clearly means that the centroid G and the incenter I are contained in $\mathcal{P}_C$. To explain the remaining two conditions (ii) and (iii), we review the concept "Ceva conjugate".

Let P and Q be two centers corresponding to $f(a,b,c)$, $f'(a,b,c)$, respectively. Then the P-Ceva conjugate of Q, which we denote by $C_P(Q)$, is a point defined by

$$f'_a\left(-\frac{f'_a}{f_a}+\frac{f'_b}{f_b}+\frac{f'_c}{f_c}\right).$$

We remark that if P, $Q \in \mathcal{P}$, then $C_P(Q) \in \mathcal{P}$, because the above expression is equivalent to

$$f'_a(-f'_a f_b f_c + f_a f'_b f_c + f_a f_b f'_c) \in \mathcal{P}$$

up to a symmetric function. The geometric meaning and many examples of $C_P(Q)$ are exhibited in Kimberling's book [32]. By computation, we can easily check that the property $C_P(C_P(Q)) = Q$ holds for any P, Q, i.e., if we put $C_P(Q) = R$, then we have $C_P(R) = Q$.

In this situation, from two points Q and R, we can determine the point P as follows. We put

$$f''_a = f'_a\left(-\frac{f'_a}{f_a}+\frac{f'_b}{f_b}+\frac{f'_c}{f_c}\right),$$

which corresponds to the point R. Then we have

$$f''_b = f'_b\left(\frac{f'_a}{f_a}-\frac{f'_b}{f_b}+\frac{f'_c}{f_c}\right), \qquad f''_c = f'_c\left(\frac{f'_a}{f_a}+\frac{f'_b}{f_b}-\frac{f'_c}{f_c}\right).$$

From these equalities we have

$$\frac{2f'_a}{f_a} = \frac{f''_b}{f'_b}+\frac{f''_c}{f'_c} = \frac{f'_b f''_c + f''_b f'_c}{f'_b f'_c},$$

and thus we have

$$f_a = \frac{1}{f'_b f''_c + f''_b f'_c}.$$

(Note that $2f'_a f'_b f'_c$ is a symmetric function, and so we may cut it.) The point P is usually called the Ceva point of Q and R.

Here we assume that a set of centers is closed under the following three geometric operations

(a) the isotomic conjugate,

(b) the Ceva conjugate,

(c) the operation $(f, f') \longmapsto \dfrac{1}{f_b f'_c + f'_b f_c}$.

Then this set satisfies the conditions (ii) and (iii).

In fact, the condition (a) means that if f is contained in this set of centers, then $1/f = f_b f_c$ is also contained in the set. (Remind that in terms of barycentric coordinates, the isotomic conjugate of f is given by $1/f$.) Then from the condition (c), this set satisfies the condition (ii). Similarly, from the conditions (b) and (a), the condition (iii) follows.

Conversely, assume that a set of centers satisfies the conditions (ii) and (iii). Then it satisfies (a), (b), (c). First, we put $f' = f$ in (ii). Then we know that $2f_b f_c = 1/f$ is contained in this set, which implies the condition (a). Next, from the conditions (iii) and (a), we have (b). The condition (c) immediately follows from (ii) and (a).

Thus we see that the conditions (ii) and (iii) in the definition of $\mathcal{P}_C$ have quite natural geometric meaning based on two conjugates. Of course, by only imposing the conditions (ii), (iii), there exists a trivial set $\{1\}$, consisting only one center (the centroid). Thus we add the condition (i) to exclude this case.

Now we state some properties of $\mathcal{P}_C$ that can be obtained directly from the definition.

Proposition 5. *Let* $f(a,b,c) \in \mathcal{P}_C$. *Then:*

(1) $f^n \in \mathcal{P}_C$ *for any* $n \in \mathbf{Z}$. (*We consider* $f^{-1} = f_b f_c$, $f^{-2} = (f_b f_c)^2$, *etc.*)

(2) $(f_a)^n(f_b + f_c) \in \mathcal{P}_C$ *for any* $n \in \mathbf{Z}$.

(3) $f_a({f_b}^2 + {f_c}^2) \in \mathcal{P}_C$.

(4) $f_n \in \mathcal{P}_C$ *for any* $n \in \mathbf{Z}$.

(5) $f_m f_n \in \mathcal{P}_C$ *for any* m, $n \in \mathbf{Z}$.

Proof. By putting $f'(a,b,c) = 1$ in (ii) and (iii), we know that $f_1(= f_b + f_c)$, $f_{-1}(= -f_a + f_b + f_c) \in \mathcal{P}_C$ if $f \in \mathcal{P}_C$. Hence, using the property $(f_m)_n = f_{m+n}$, we have inductively $f_n \in \mathcal{P}_C$, which shows (4).

We prove f^n, $(f_b f_c)^n(f_b + f_c) \in \mathcal{P}_C$ for integer $n \geq 0$ by induction. The case $n = 0$ is clear. We assume f^n, $(f_b f_c)^n(f_b + f_c) \in \mathcal{P}_C$ for some integer $n \geq 0$. Then by putting $f = (f_b f_c)^n(f_b + f_c)$, $f' = f^n$ in (iii), we obtain $f^{n+1} \in \mathcal{P}_C$. Next, by putting $f = f^{n+1}$, $f' = (f_b f_c)^n(f_b + f_c)$ in (iii), we have $(f_b f_c)^{n+1}(f_b + f_c) \in \mathcal{P}_C$. Hence by induction, we obtain f^n, $(f_b f_c)^n(f_b + f_c) \in \mathcal{P}_C$ for integer $n \geq 0$. The latter means $(f_a)^{-n}(f_b + f_c) \in \mathcal{P}_C$.

We already showed that $1/f = f_b f_c \in \mathcal{P}_C$ if $f \in \mathcal{P}_C$. Hence, (1) immediately follows from this fact.

To prove the remaining part of (2), we show $(f_a)^n(f_b + f_c) \in \mathcal{P}_C$ for integer $n \geq 0$. Assume $(f_a)^n(f_b + f_c) \in \mathcal{P}_C$ for some integer $n \geq 0$. Then by putting $f = (f_a)^{-n}$, $f' = (f_a)^n(f_b + f_c)$ in (iii), we have $(f_a)^{n+1}(f_b + f_c) \in \mathcal{P}_C$, which completes the proof of (2).

Next, in (ii), we put $f' = f_b f_c$. Then we have $f_b {f'}_c + {f'}_b f_c = f_a({f_b}^2 + {f_c}^2) \in \mathcal{P}_C$, and hence (3) follows.

Finally we prove (5). From (1) and (4) we have ${f_n}^2 \in \mathcal{P}_C$. Next, by putting $f = 1$ and $f' = f_{n+1}$ in (iii), the property $f_n f_{n+1} \in \mathcal{P}_C$ holds for any integer n.

Hence, to complete the proof of (5), we have only to show the following property: If $f_n f_{n+k} \in \mathcal{P}_C$ for any $n \in \mathbf{Z}$, then the polynomial $f_n f_{n+k+2}$ is also contained in $\mathcal{P}_C$ for any $n \in \mathbf{Z}$. Assume $f_n f_{n+k} \in \mathcal{P}_C$. Then, putting $f = f_n$ and $f' = f_n f_{n+k}$ in (ii), we have $(f_n)_b(f_n f_{n+k})_c + (f_n f_{n+k})_b(f_n)_c = (f_n)_b(f_n)_c(f_{n+k+1})_a \in \mathcal{P}_C$, after some calculations. Then taking the inverse, we have $(f_n)_a(f_{n+k+1})_b(f_{n+k+1})_c \in \mathcal{P}_C$. Finally we put $f = (f_n)_a(f_{n+k+1})_b(f_{n+k+1})_c$ and $f' = f_{n+k+1}$ in (iii). Then we have ${(f_{n+k+1})_a}^2(f_{n+k+1})_b(f_{n+k+1})_c\ \{-(f_n)_a + (f_n)_b + (f_n)_c\} = f_{n-1}f_{n+k+1} \in \mathcal{P}_C$, which implies $f_n f_{n+k+2} \in \mathcal{P}_C$ for any integer n. q.e.d.

Starting from 1 and a in $\mathcal{P}_C$, we can successively construct famous centers as follows: First, from Proposition 5 (1), we know $a^2 \in \mathcal{P}_C$, which corresponds to the Lemoine point X_6. Then, since the circumcenter X_3 is the X_2-Ceva conjugate of X_6, we know that X_3 is contained in $\mathcal{P}_C$. From this fact it follows that the centers on the Euler line such as the orthocenter X_4, the nine-point center X_5, the de Longchamps point X_{20}, etc., are also contained in $\mathcal{P}_C$.

Similarly, since the incenter X_1 is contained in $\mathcal{P}_C$, the centers on the Nagel line such as the Nagel point X_8, the Spieker center X_{10}, etc., are contained in $\mathcal{P}_C$. The Gergonne point X_7 is the isotomic conjugate of X_8, and the Mittenpunkt X_9 is the X_2-Ceva conjugate of X_1, and so these points are also contained in $\mathcal{P}_C$.

By developing such procedures, we know that among Kimberling's list X_k ($k = 1 \sim 400$) in [32], at least the following 167 points are contained in $\mathcal{P}_C$:

$k =$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 19, 20, 21, 22, 25, 27, 28, 29, 31, 32, 33, 34, 37, 38, 39, 40, 41, 42, 43, 46, 47, 48, 51, 52, 53, 54, 55, 56, 57, 58, 63, 65, 66, 69, 71, 72, 73, 75, 76, 77, 78, 81, 82, 83, 84, 85, 86, 87, 92, 95, 97, 140, 141, 142, 144, 145, 154, 155, 157, 158, 159, 160, 165, 169, 170, 184, 189, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 233, 251, 253, 255, 261, 264, 269, 270, 271, 273, 274, 275, 276, 278, 279, 280, 281, 282, 283, 284, 286, 288, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 318, 321, 322, 324, 326, 329, 330, 331, 332, 333, 341, 342, 343, 345, 346, 347, 348, 349, 393, 394.

We may say that the family $\mathcal{P}_C$ constitutes a network of triangle centers, and this network is constructed from the initial centers G and I, by applying only two geometric operations (ii) and (iii).

On the other hand, unfortunately, we have no effective method to show $X_k \notin \mathcal{P}_C$ for a given center X_k at present. For example, we cannot determine whether X_{24}, X_{26}, X_{35}, X_{49}, ... are contained in $\mathcal{P}_C$ or not, and we conjecture certainly that the Feuerbach point X_{11} is not contained in $\mathcal{P}_C$ (see Conjecture 2 in Section 7). Finally we remark that there were several attempts to grasp a meaninful set of triangle centers by Kimberling [30], [34], [35], etc., and there may further exist other type of natural set of centers deserve to study.

7. Conjectures on $\mathcal{P}_C$

Explicit determination of the set $\mathcal{P}_C$ is an important but difficult problem. The essential difficulty comes from a reduction by a symmetric factor, which suddenly appears during the operations (ii) and (iii). (See the example after Conjecture 2.) In this section we state several conjectures concerning $\mathcal{P}_C$.

Conjecture 1. *The set $\mathcal{P}_C$ is closed under the multiplication, i.e., the following property holds*:

$$f,\ f' \in \mathcal{P}_C \Longrightarrow ff' \in \mathcal{P}_C.$$

For example, we conjecture $X_{60} \in \mathcal{P}_C$, since it corresponds to the polynomial $f(a,b,c) = a^2(a+b)^2(a+c)^2(-a+b+c)$ and both X_8 ($f = -a+b+c$), X_{81} ($f = a(a+b)(a+c)$) are contained in $\mathcal{P}_C$. At present we cannot find a counterexample to Conjecture 1.

The following is the most important conjecture on $\mathcal{P}_C$. This gives one necessary condition for a point $f(a,b,c) \in \mathcal{P}$ to belong to $\mathcal{P}_C$.

Conjecture 2. *For any $f(a,b,c) \in \mathcal{P}_C$ the degree $d(f)$ takes the form $(-2)^k$ ($k \in \mathbf{Z}$).*

As above, we have no counterexample to this conjecture at present. For all centers $X_k \in \mathcal{P}_C$ ($k \leq 400$) listed at the end of Section 6, this property holds. The following "incomplete proof" partially supports our Conjecture 2.

"*Incomplete*" *proof of Conjecture* 2. Let f, $f' \in \mathcal{P}_C$, and we put $f'' = f_b f'_c + f'_b f_c$, i.e., $f''(a,b,c) = f(b,c,a)f'(c,a,b) + f'(b,c,a)f(c,a,b)$. Then we have $f''(1,1,1) = 2f(1,1,1)f'(1,1,1)$, and

$$f''(1,\omega,\omega^2) = f(\omega,\omega^2,1)f'(\omega^2,1,\omega) + f'(\omega,\omega^2,1)f(\omega^2,1,\omega).$$

Now assume that $\deg f = p$ and $\deg f' = q$. Then we have $f(\omega,\omega^2,1) = f(\omega,\omega^2,\omega^3) = \omega^p f(1,\omega,\omega^2)$, and similarly $f(\omega^2,1,\omega) = \omega^{2p} f(1,\omega,\omega^2)$. Hence we have

$$f''(1,\omega,\omega^2) = (\omega^{p+2q} + \omega^{2p+q}) f(1,\omega,\omega^2) f'(1,\omega,\omega^2).$$

Thus, if the polynomial f'' does not possesses a symmetric factor, we have $d(f'') = \frac{1}{2}(\omega^{p+2q} + \omega^{2p+q}) d(f) d(f')$. We can easily see that

$$\tfrac{1}{2}(\omega^{p+2q} + \omega^{2p+q}) = \begin{cases} 1 & p \equiv q \pmod 3, \\ -\frac{1}{2} & p \not\equiv q \pmod 3. \end{cases}$$

Hence if both $d(f)$, $d(f')$ are of the form $(-2)^k$, and in addition if f'' does not possess a symmetric factor, then $d(f'')$ is also the power of -2.

In case $f'' = f'_a(-f_a f'_a + f_b f'_b + f_c f'_c)$ we can similarly calculate its degree under the same assumption:

$$d(f'') = \begin{cases} d(f)d(f')^2 & p+q \equiv 0 \pmod 3, \\ -2d(f)d(f')^2 & p+q \not\equiv 0 \pmod 3. \end{cases}$$

Hence under the assumption that f'' never possess a symmetric factor under the operations (ii) and (iii), we have "proved" that $d(f) = (-2)^k$ for any $f \in \mathcal{P}_C$. "q.e.d".

But of course, the above assumption does not hold in general. For example we consider the following case:

$$f = (b+c)(-a+b+c),$$
$$f' = (b+c)(-a+b+c)\{a^3(b+c) - 2a^2bc - a(b+c)(b^2-3bc+c^2) - bc(b-c)^2\}.$$

Then we have

$$\begin{aligned} f'' &= f_b f'_c + f'_b f_c \\ &= 2a^2(a+b)(b+c)(c+a)(-a+b+c)(a-b+c)(a+b-c) \\ &= a^2. \end{aligned}$$

As this example shows, it is hard to judge whether f'' admits a symmetric factor or not by only seeing f and f', and in case it admits, it is also hard to find its symmetric factor without explicit calculations. This is the principal difficulty in the explicit determination of $\mathcal{P}_C$.

But if we can succeed to show Conjecture 2, we can easily see that some centers does not belong to $\mathcal{P}_C$. For example, for the point X_{45} corresponding to the polynomial $f(a,b,c) = a(-a+2b+2c)$, we have $d(f) = -1$, and so we "know" that $X_{45} \notin \mathcal{P}_C$. Similarly concerning the Feuerbach point X_{11}, we can "show" $X_{11} \notin \mathcal{P}_C$ by applying Conjecture 2, since $d(f) = \infty$ in this case.

The converse to Conjecture 2 certainly does not hold. For example, for the quadratic polynomial $f(a,b,c) = a^2+b^2+c^2-(-2)^n(a^2+b^2+c^2-3bc)$, we have $d(f) = (-2)^n$. But we conjecture that $f(a,b,c) \notin \mathcal{P}_C$ for $n \neq 0, 1$, as we will explain later (Conjecture 4). It seems that some more conditions are necessary to characterize the set $\mathcal{P}_C$ in addition to Conjecture 2. As known from the above "incomplete proof", the degree $d(f)$ roughly gives the number of operations (ii), (iii) to obtain the center f from the initial one, unless symmetric factors appear during these operations. Hence we may say again that the degree $d(f)$ indicates a sort of complexity of the center f.

Next, we state two conjectures concerning polynomials in $\mathcal{P}_C$ with lower degree. For linear polynomials, we have the following conjecture.

Conjecture 3. *Linear polynomials in $\mathcal{P}_C$ are exhausted by*

$$2^n(a+b+c) + (-1)^n(2a-b-c) \qquad (n \in \mathbf{Z}).$$

Note that these centers are lying on the Nagel line and we already know that they are contained in $\mathcal{P}_C$. We can easily see that this conjecture is an easy consequence of Conjecture 2. If Conjecture 3 holds, we can uniquely specify the linear polynomials in $\mathcal{P}_C$ by one invariant $d(f)$.

Concerning quadratic polynomials in $\mathcal{P}_C$, we have the following conjecture.

Conjecture 4. *Quadratic polynomials in $\mathcal{P}_C$ are exhausted by the following two families, depending on three parameters l, m, $n \in \mathbf{Z}$:*

$$\begin{aligned}
Q_1: \quad & (a+b+c)^2 - (-\tfrac{1}{2})^{m+n-1}(a^2+b^2+c^2-ab-bc-ca) \\
& \qquad + \{(-\tfrac{1}{2})^{l+m} + (-\tfrac{1}{2})^{l+n}\}(a+b+c)(2a-b-c) \\
& \qquad\qquad + (-\tfrac{1}{2})^{l+m+n}\{2a^2-2a(b+c)-b^2+4bc-c^2\}, \\
Q_2: \quad & (a+b+c)^2 - (-\tfrac{1}{2})^{m+n}(a^2+b^2+c^2-ab-bc-ca) \\
& \qquad + \{(-\tfrac{1}{2})^{l+m+1} + (-\tfrac{1}{2})^{l+n+1}\}(a+b+c)(2a-b-c) \\
& \qquad\qquad + (-\tfrac{1}{2})^{l+m+n}\{2a^2-2a(b+c)-b^2+4bc-c^2\}.
\end{aligned}$$

These two families are actually contained in $\mathcal{P}_C$. In fact, by putting $f(a,b,c) = a$, the first family Q_1 coincides with the polynomial $(f_m f_n)_l$. (See Proposition 5 (5), (4).) Under the same notations, the second family Q_2 is given by $\{(f_m)_b(f_n)_c + (f_n)_b(f_m)_c\}_l$.

These two families Q_1 and Q_2 have quite similar expressions. But they actually constitute different families. For example, the polynomial a^2 is contained in Q_1, but not in Q_2. Conversely, the polynomial a^2+bc is contained in Q_2, but not in Q_1. (The intersection $Q_1 \cap Q_2$ is non-empty, because bc is contained in both families.) We also note that these polynomials can not be divided by the symmetric linear polynomial $a+b+c$, and hence these polynomials are essentially quadratic.

Now we calculate the invariants of these quadratic polynomials. First we have $d(f) = (-1/2)^{l+m+n}$ for both families. The exponent $l+m+n$ is almost equal to the number of operations (ii), (iii) to obtain polynomials in Q_1, Q_2 from the initial $f = a$.

For the polynomial in Q_1, we have $q(f) = (-1/2)^{m+n}$ and

$$e(f) = \tfrac{1}{3}\left[2a^2-2a(b+c)-b^2+4bc-c^2+\{(-2)^m+(-2)^n\}(a+b+c)(2a-b-c)\right],$$

and for the polynomial in Q_2, we have $q(f) = (-1/2)^{m+n+1}$ and

$$e(f) = \tfrac{1}{3}\left[2a^2-2a(b+c)-b^2+4bc-c^2+\{(-2)^{m-1}+(-2)^{n-1}\}(a+b+c)(2a-b-c)\right].$$

The invariant $s(f)$ has a common expression in terms of $q(f)$:

$$s(f) = \tfrac{1}{3}\{(a+b+c)^2 + 2q(f)(a^2+b^2+c^2-ab-bc-ca)\}.$$

Triangle centers defined by linear polynomials $2^n(a+b+c)+(-1)^n(2a-b-c)$ all lie on the Nagel line, as we examined above. As for the quadratic centers $Q_1 \cup Q_2$, they constitute a curious figure like a milky way, possibly contains several lines. We here plot them for small l, m, n as a reference (Figure 4).

For higher degrees it seems difficult to characterize the centers in $\mathcal{P}_C$. We conjecture that polynomials in $\mathcal{P}_C$ with degree n contain $2n-1$ free parameters as in the case of $n = 1, 2$. (See [24] for other interesting point configuration consisting of triangle centers.) The points in $\mathcal{P}_C$ are situated discretely in the plane, and we may say that three invariants $d(f)$, $e(f)$ and $s(f)$ give a “discrete coordinate” of the points in $\mathcal{P}_C$. Conjecture 2 implies that among these invariants $d(f)$ must

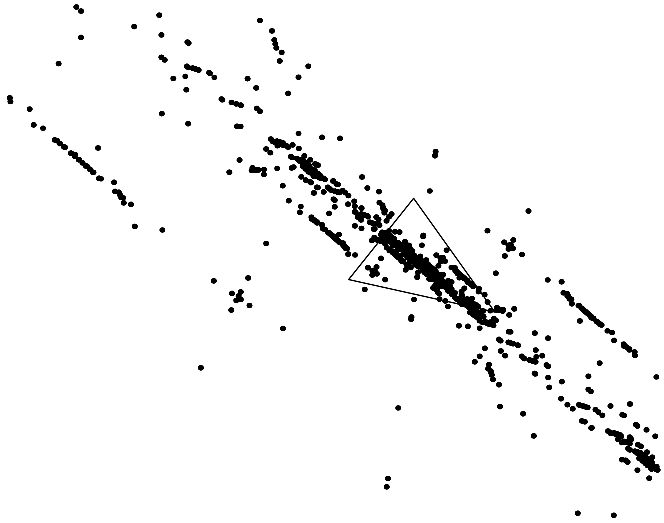

Figure 4

take a special form. Perhaps there are another restrictions on polynomials $e(f)$ and $s(f)$, and it is an important problem to characterize the set of these polynomials. We conjecture that $e(f)$ and $s(f)$ admit some decompositions consisting of finer invariants. And many geometric facts can be verified in terms of these finer invariants. For example, we suppose that three points in $\mathcal{P}_C$ are collinear if and only if these finer invariants satisfy some simple algebraic relations, as in Proposition 4 (1) for generalized Euler lines. It is our final problem to give an atlas of triangle centers in terms of several invariants, by which we can understand many geometric facts in a simple unified manner.

Finally, we state one more conjecture concerning isogonal conjugate. In defining the set $\mathcal{P}_C$, we used the barycentric coordinate. But if we use trilinear coordinate instead of barycentric coordinate, we obtain another class of triangle centers $\mathcal{P}'_C$. (We consider $f(a,b,c)$ as a trilinear coordinate of the triangle center, and construct $\mathcal{P}'_C$ completely by the same procedure as $\mathcal{P}_C$.) Geometrically we can say that the set $\mathcal{P}'_C$ is closed under the Ceva conjugate and the isogonal conjugate, since $1/f(a,b,c)$ means the isogonal conjugate of $f(a,b,c)$ in trilinear coordinate. But this set may coincide with $\mathcal{P}_C$ itself.

Conjecture 5. $\mathcal{P}_C = \mathcal{P}'_C$.

In barycentric coordinate the isogonal conjugate of $f(a,b,c)$ is given by $a^2/f(a,b,c)$. Hence if Conjecture 1 holds, then Conjecture 5 follows immediately, since $a^2 \in \mathcal{P}_C$.

In this paper we considered a sequence of infinite triangles $\Delta_n(ABC)$. But there are many other sequences such as orthic triangles, as recently discussed in [16]. (For other examples, see [1], [3], [5], [10], [11], [13], [15], [20], [21], [25], [27], [38], [39], [40], [42], [53], [57], [58], [59], [60], [64], [65], etc.) (See the list of publications in References at the end of this paper.) It seems an interesting

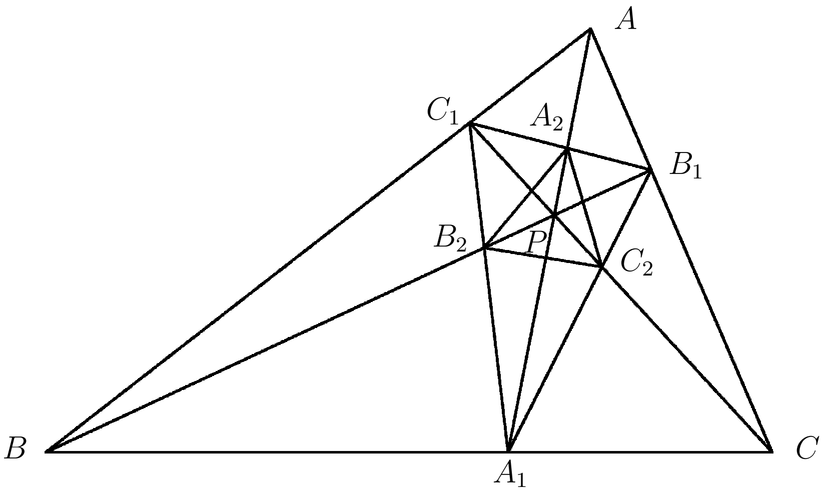

Figure 5

problem to investigate a trajectory of a fixed triangle center for each sequence of triangles, keeping in mind the results in Proposition 1.

As one example, we consider a sequence of infinite Cevian triangles determined by a fixed triangle center P (see Figure 5). In case P is the centroid of ΔABC, this sequence is nothing but $\Delta_n(ABC)$ we discussed above. For a general triangle center P, we can show that the trajectory of the centroid of this sequence of triangles lies on a cubic curve, not on a line. If P is the orthocenter H of ΔABC, then this cubic curve possesses a cusp singularity at H, and three asymptotic lines of this cubic curve form a triangle which is homothetic to the original ΔABC. Many interesting properties seem to hold in this setting. As one important problem, we ask whether a similar result as in Theorem 2 holds or not in this sequence of Cevian triangles, or more generally for other sequences of central triangles listed in [32], [37].

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