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On Torsion Free Distributive Modules

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Abstract. Let R be a commutative ring with identity and let M be a torsion free R-module. Several characterizations of distributive modules are investigated. Indeed, among other equivalent conditions, we prove that M is distributive if and only if any primal submodule of M is irreducible, and, if and only if each submodule of M can be represented as an intersection of irreducible isolated components.

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1. Introduction

Let R be a commutative ring with non-zero identity element and let M be a unital R-module. Then M is said to be *distributive* if $(K+L)\cap N = (K\cap N)+(L\cap N)$, for all submodules K, L, N of M.

The notion of distributive modules has been studied and developed as a generalization of Prüfer and Dedekind domains independently by T. M. K. Davison [3] and W. Stephenson [13]. This leads to the considerable research and results on the structure of the modules with distributive lattice of submodules in the last three decades (see for example [1], [4], [5], [6], [14], [15]).

In this paper we will give some new characterizations of distributive modules motivated in large part by the papers [2], [8], [10], [16]. In Section 2 after giving some notation, among other things, we characterize torsion free distributive modules in terms of the irreducibility of certain submodules (Corollary 2.8). Another main result of this section which we need in Section 3 is Proposition 2.9.

In Section 3 we prove the following theorem:

Theorem 1.1. *Let M be a torsion free R-module. The following statements are equivalent.*

(i) *M is a distributive R-module.*

(ii) *Every primal submodule of M is irreducible.*

(iii) *For every maximal ideal $\mathfrak{m}$, the set $W_{\mathfrak{m}}(M)$ is linearly ordered with respect to inclusion.*

(iv) *For every maximal ideal $\mathfrak{m}$ and any $x, y \in M$, $Rx(\mathfrak{m})$ and $Ry(\mathfrak{m})$ are comparable with respect to inclusion.*

(v) *For each submodule N of M and each maximal ideal $\mathfrak{m}$, $E_R(M/N(\mathfrak{m}))$ (the injective hull of $M/N(\mathfrak{m})$) is indecomposable.*

Also we prove in Theorem 3.7 that a torsion free module M is distributive if and only if each submodule of M can be represented as an intersection of irreducible isolated components.

2. Preliminaries

In what follows the notation and terminology is, in general, the same as in [10], [11].

For two subsets X and Y of M the symbol $X :_R Y$ denotes the residual of X by Y which is defined as usual by $X :_R Y = \{r \in R : rY \subseteq X\}$. Let S be a multiplicatively closed subset of R. We set $X(S)$ to denote $\cup_{s \in S}(X :_M s)$. Then $X(S)$ is a subset of M containing X. In particular whenever $\mathfrak{p}$ is a prime ideal of R and $S = R \setminus \mathfrak{p}$, we will denote $X(S)$ by $X(\mathfrak{p})$. Note that $y \in X(\mathfrak{p})$ if and only if $X :_R y \nsubseteq \mathfrak{p}$. We say that the set X is *weakly $\mathfrak{p}$-primal* if $X(\mathfrak{p}) = X$. We will denote the set of all weakly $\mathfrak{p}$-primal subsets of M by $W_{\mathfrak{p}}(M)$. Thus, for each $Y \subseteq M$, $Y(\mathfrak{p})$ is in fact the intersection of all elements of $W_{\mathfrak{p}}(M)$ containing Y.

We note that whenever Y is a submodule of M, then $Y(\mathfrak{p})$ is too, and is called the *isolated $\mathfrak{p}$-component* of Y in the sense of Krull [9]. Then Y is weakly $\mathfrak{p}$-primal if and only if $Z_R(M/Y) \subseteq \mathfrak{p}$, where $Z_R(M/Y)$ denotes the set of all zero divisors of the factor module M/Y. Moreover when M is torsion free (that is $\{x \in M : rx = 0 \text{ for some non-zero } r \in R\} = \{0\}$), then we have $Y(\mathfrak{p}) = Y_{\mathfrak{p}} \cap M$.

Let N be a submodule of M. An element $r \in R$ is said to be *non-prime* (resp. *prime*) *relative to* N if $N \subset N :_M r$ (resp. $N = N :_M r$). Hence $r \in R$ is non-prime relative to N if there exists an element $m \in M \setminus N$ such that $rm \in N$. Clearly the set of all non-prime elements relative to N is $Z_R(M/N)$.

Following [7] we say that N is a *primal submodule* of M if $Z_R(M/N)$ itself forms an ideal $\mathfrak{p}$ called the *adjoint ideal* of N. In this case we say that N is $\mathfrak{p}$-primal. Since the product of two prime element relative to N is again prime relative to N; thus, whenever N is a proper submodule of M, the adjoint ideal is a prime ideal of R which contains $N :_R M$. This fact shows the analogy between the primary submodules and primal submodules. Here we note that N is primal if and only if with two non-prime elements relative to N their difference is a non-prime relative to N.

Finally we recall that a submodule N of M is said to be irreducible if for submodules L_1 and L_2 of M, $N = L_1 \cap L_2$ implies that either $N = L_1$ or $N = L_2$. A submodule Q of M is primary if for all $r \in R$ and $x \in M \setminus Q$, $rx \in Q$ imply that $r^n M \subseteq Q$ for some positive integer n. It is well known that each irreducible submodule of a module satisfying ascending chain condition is a primary submodule. One of the advantages of primal submodules over primary submodules is in the fact that the following propositions hold without any chain condition on M (see [7]).

Proposition 2.1. *Any irreducible submodule of M is a primal submodule.*

Proof. Let N be an irreducible submodule of M, and r, s be two elements of R which are non-prime relative to N. Then we have $N \subset N :_M r$ and $N \subset N :_M s$ which give that $N \subset (N :_M r) \cap (N :_M s) = (N :_M r - s)$. Hence the element $r - s$ is non-prime relative to N and so N is a primal submodule of N. □

Proposition 2.2. *Every submodule of M is the intersection of primal submodules.*

Proof. Since the module M itself, being irreducible, is primal, so the intersection of all primal submodules of M containing N is non-empty. Hence to prove the claim it is enough to show that for each $m \in M \setminus N$ there exists a primal submodule P of M containing N such that $m \in M \setminus P$. Let $\sum = \{L : N \subseteq L \subset M \text{ and } m \in M \setminus L\}$. Then $\sum$ is not empty and by Zorn's lemma it possesses a maximal element with respect to inclusion, say P. We show that P is a primal submodule of M. Let r, s be non-prime elements relative to P. Then there exist $x, y \in M \setminus P$ such that rx, $sy \in P$. Now by the maximality of P we have $m \in P + Rx$ and $m \in P + Ry$. This gives that $rm \in rP + Rrx \subseteq P$ and $sm \in sP + Rsy \subseteq P$. Therefore $(r - s)m \in P$ and so $r - s$ is non-prime relative to P. Consequently P is a primal submodule of M. □

For convenience of the reader we recall the following two well known lemmas about distributive modules.

Lemma 2.3. [3] *M is a distributive R-module if and only if $M_{\mathfrak{p}}$ is a distributive $R_{\mathfrak{p}}$-module for all prime (maximal) ideals $\mathfrak{p}$ of R.*

Recall that the phrase R is *quasi-local* means that R has a unique maximal ideal.

Lemma 2.4. [13, 15] *Let M be an R-module. Then the following two statements are equivalent:*

(i) *M is a distributive R-module.*

(ii) $(Rx :_R y) + (Ry :_R x) = R$, *for all $x, y \in M$.*

Furthermore when R is quasi local, then each of them are equivalent to each of the following:

(iii) *The set of all submodules of M is linearly ordered with respect to inclusion.*

(iv) *The set of all cyclic submodules of M is linearly ordered with respect to inclusion.*

In the sequel we need some auxiliary results on isolated components. We collect them in the following.

Lemma 2.5. *Let N be a $\mathfrak{p}$ primal submodule of M. Then*

(a) $N(\mathfrak{p}) = N$.

(b) *For each prime ideal $\mathfrak{q}$ of R we have $N(\mathfrak{q}) = N$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$.*

(c) *If $N(\mathfrak{q})$ is a $\mathfrak{q}$-primal submodule for some prime ideal $\mathfrak{q}$, then $\mathfrak{q} \subseteq \mathfrak{p}$.*

Proof. (a) Since N is $\mathfrak{p}$-primal it follows that $N(\mathfrak{p}) \subseteq N$. As the opposite inclusion always holds, the result follows.

(b) One can see that $\mathfrak{p} \subseteq \mathfrak{q}$ if and only if $N(\mathfrak{q}) \subseteq N(\mathfrak{p})$, and so the result follows from (a).

(c) Since $N(\mathfrak{q})$ is $\mathfrak{q}$-primal, we have $\mathfrak{q} = \cup_{m \in M \setminus N(\mathfrak{q})} N(\mathfrak{q}) :_R m$. Now let $s \in \mathfrak{q}$. Then there exists $m \in M \setminus N(\mathfrak{q})$ such that $sm \in N(\mathfrak{q})$. Thus there exists $r \in R \setminus \mathfrak{q}$ such that $srm \in N$. However rm is not in N, because otherwise $rm \in N(\mathfrak{q})$, which yields $r \in \mathfrak{q}$. So $s \in Z_R(M/N) = \mathfrak{p}$. □

Proposition 2.6. *The following conditions are equivalent for M.*

(a) *M is a distributive R-module.*

(b) *For each proper submodule N of M and each prime (maximal) ideal $\mathfrak{p}$ of R, $N_\mathfrak{p}$ is an irreducible submodule of $M_\mathfrak{p}$.*

Proof. (a)⇒(b). Let N be a proper submodule of M and let $\mathfrak{p}$ be a prime ideal of R which contains $N :_R M$. Let $N_\mathfrak{p} = K_\mathfrak{p} \cap L_\mathfrak{p}$. By Lemma 2.3, $M_\mathfrak{p}$ is a distributive module over the quasi local ring $R_\mathfrak{p}$. Therefore by Lemma 2.4, either $L_\mathfrak{p} \subseteq K_\mathfrak{p}$ or $K_\mathfrak{p} \subseteq L_\mathfrak{p}$, i.e. either $N_\mathfrak{p} = L_\mathfrak{p}$ or $N_\mathfrak{p} = K_\mathfrak{p}$. Thus $N_\mathfrak{p}$ is irreducible.

(b)⇒(a). Let $\mathfrak{p}$ be a prime ideal of R and let U, V be proper submodules of $M_\mathfrak{p}$. Then by [12, Ex. 9.11], there exist submodules K and L of M such that $U = K_\mathfrak{p}$ and $V = L_\mathfrak{p}$. By assumption $K_\mathfrak{p} \cap L_\mathfrak{p}$ is an irreducible submodule of $M_\mathfrak{p}$, so either $K_\mathfrak{p} \subseteq K_\mathfrak{p} \cap L_\mathfrak{p}$ or $L_\mathfrak{p} \subseteq K_\mathfrak{p} \cap L_\mathfrak{p}$. Consequently either $V \subseteq U$ or $U \subseteq V$ and so the proper submodules of $M_\mathfrak{p}$ are linearly ordered. Now it follows by Lemma 2.4 that M is a distributive R-module. □

Proposition 2.7. *Let N be a submodule of M and let $\mathfrak{p}$ be a prime* (*maximal*) *ideal of R. If $N_\mathfrak{p}$ is an irreducible submodule of $M_\mathfrak{p}$, then $N(\mathfrak{p})$ is an irreducible* submodule *of M. Furthermore if M is torsion free the converse is true.*

Proof. (⇒) Let $N(\mathfrak{p}) = L \cap K$ for some submodules L, K of M. By localizing at $\mathfrak{p}$ and using the fact that $(N(\mathfrak{p}))_\mathfrak{p} = N_\mathfrak{p}$, we have $N_\mathfrak{p} = L_\mathfrak{p} \cap K_\mathfrak{p}$. Hence by assumption either $N_\mathfrak{p} = L_\mathfrak{p}$ or $N_\mathfrak{p} = K_\mathfrak{p}$, which gives that either $N(\mathfrak{p}) = L$ or $N(\mathfrak{p}) = K$.

Furthermore let U, V be submodules of $M_\mathfrak{p}$ such that $N_\mathfrak{p} = U \cap V$. Then by [12, Ex. 9.11], there exist submodules L, K of M such that $U = L_\mathfrak{p}$ and $V = K_\mathfrak{p}$. Hence $N_\mathfrak{p} = L_\mathfrak{p} \cap K_\mathfrak{p}$, which gives that $N(\mathfrak{p}) = N_\mathfrak{p} \cap M = (L_\mathfrak{p} \cap M) \cap (K_\mathfrak{p} \cap M) = L(\mathfrak{p}) \cap K(\mathfrak{p})$. Thus by assumption we have $N(\mathfrak{p}) = L(\mathfrak{p})$ or $N(\mathfrak{p}) = K(\mathfrak{p})$ and so by localizing at $\mathfrak{p}$, $N_\mathfrak{p} = L_\mathfrak{p}$ or $N_\mathfrak{p} = K_\mathfrak{p}$. □

From the above observations we deduce the following characterization for torsion free distributive modules.

Corollary 2.8. *Assume that M is a torsion free R-module. Then the following statements are equivalent.*

(a) *M is a distributive R-module.*

(b) *For any submodule N of M and for each prime (maximal) ideal $\mathfrak{p}$ of R, $N(\mathfrak{p})$ is an irreducible submodule of M.*

The following proposition which is important for its own will be used in the next section.

Proposition 2.9. *Let N be a finitely generated submodule of M and let $\mathfrak{p}$ be a prime ideal of R. Assume that $N_{\mathfrak{p}} \neq 0$. Then $(\mathfrak{p}N)(\mathfrak{p})$ is a $\mathfrak{p}$-primal submodule of M.*

Proof. We show that $Z_R(M/(\mathfrak{p}N)(\mathfrak{p})) = \mathfrak{p}$. First let $r \in Z_R(M/(\mathfrak{p}N)(\mathfrak{p}))$. Then there exists $m \in M \setminus (\mathfrak{p}N)(\mathfrak{p})$ such that $rm \in (\mathfrak{p}N)(\mathfrak{p})$. It follows that $(\mathfrak{p}N :_R m) \nsubseteq \mathfrak{p}$ and so there exists $t \in R \setminus \mathfrak{p}$ such that $rtm \in \mathfrak{p}N$. Hence $rt \in \mathfrak{p}$ and so $r \in \mathfrak{p}$. Consequently $Z_R(M/(\mathfrak{p}N)(\mathfrak{p})) \subseteq \mathfrak{p}$. In order to prove the other inclusion, let $r \in \mathfrak{p}$. Since N is finitely generated and $N_{\mathfrak{p}} \neq 0$, using Nakayama's Lemma we have $(\mathfrak{p}N)_{\mathfrak{p}} \neq N_{\mathfrak{p}}$. This gives that $(\mathfrak{p}N)(\mathfrak{p}) \neq N(\mathfrak{p})$. Since in any case we have $(\mathfrak{p}N)(\mathfrak{p}) \subseteq N(\mathfrak{p})$ and $N(\mathfrak{p}) \subseteq (\mathfrak{p}N)(\mathfrak{p}) :_M r$, so $(\mathfrak{p}N)(\mathfrak{p}) \subset (\mathfrak{p}N)(\mathfrak{p}) :_M r$. This gives that there exists $x \in M \setminus (\mathfrak{p}N)(\mathfrak{p})$ such that $rx \in (\mathfrak{p}N)(\mathfrak{p})$. Hence $r \in Z_R(M/(\mathfrak{p}N)(\mathfrak{p}))$ and the proof is complete. □

In Proposition 2.1 it is proved that any irreducible submodule is a primal submodule. As a well known fact irreducible submodules are primary in Noetherian modules, while a primary submodule need not be irreducible in general. It might be worth while noticing that for torsion free distributive modules the situation is just the opposite. With this in mind, in fact we have:

Theorem 2.10. *If M is a torsion free distributive R-module, then any primary submodule is irreducible.*

Proof. Let Q be a primary submodule of M and let $\sqrt{Q :_R M} = \mathfrak{p}$. Let $Q = K \cap L$ for some submodule L of M. Localizing in $\mathfrak{p}$ we have $Q_{\mathfrak{p}} = K_{\mathfrak{p}} \cap L_{\mathfrak{p}}$. In view of Lemma 2.3 and Lemma 2.4 the set of all submodules of $M_{\mathfrak{p}}$ is linearly ordered, so that either $Q_{\mathfrak{p}} = K_{\mathfrak{p}}$ or $Q_{\mathfrak{p}} = L_{\mathfrak{p}}$. Suppose $Q_{\mathfrak{p}} = K_{\mathfrak{p}}$. We have $K_{\mathfrak{p}} \cap M = K(\mathfrak{p})$. Since Q is $\mathfrak{p}$-primary one can easily see that $Q_{\mathfrak{p}} \cap M = Q$. So $K \subseteq K(\mathfrak{p}) = Q$. Because the converse of this inclusion is obvious, we have $Q = K$ and the result follows. □

3. Main results

In this section we will give several characterizations of distributive modules in terms of the primal submodules and the comparability of the isolated components of cyclic submodules. In particular we prove that the R-module M is distributive if and only if any proper submodule of M can be represented as an intersection of irreducible isolated component. We will prove all of these in Theorems 3.6 and 3.7.

Let N be a proper submodule of M. Following [16, p. 72], we define a prime ideal $\mathfrak{p}$ of R to be a *Krull associated prime* of M/N if for every element $t \in \mathfrak{p}$, there exists $x \in M$ such that $t \in N :_R x \subseteq \mathfrak{p}$. We denote by $\mathrm{K}(M/N)$ (resp. by $\mathrm{MaxK}(M/N)$) the set of all Krull associated primes of M/N (resp. the set of all maximal members of $\mathrm{K}(M/N)$). A prime ideal $\mathfrak{p}$ is called *weak Bourbaki associated prime* of M/N if it is minimal prime divisor of $N :_R x$ for some $x \in M \setminus N$. We will denote the set of all weak Bourbaki associated primes of M/N by $\mathrm{wB}(M/N)$. It is known that (see for example [11, Lemma 2.15]), the set $\mathrm{wB}(M/N)$ is non empty. The fact that the set $\mathrm{K}(M/N)$ is not empty is given in the following.

Lemma 3.1. *Let N be a proper submodule of M. Then* $\mathrm{wB}(M/N) \subseteq \mathrm{K}(M/N)$. *A prime ideal $\mathfrak{q}$ of R is a Krull associated prime of M/N if and only if $\mathfrak{q}$ is a set-theoretic union of weak Bourbaki associated primes of M/N.*

Proof. Let $\mathfrak{p} \in \mathrm{wB}(M/N)$. Then there exists $x \in M \setminus N$ such that $\mathfrak{p}$ is a minimal prime divisor of $N :_R x$. Now in view of [9, p. 737], $(N :_R x)(\mathfrak{p})$ is $\mathfrak{p}$-primary ideal. Let $s \in \mathfrak{p}$. There exists a least positive integer n such that $s^n \in (N :_R x)(\mathfrak{p})$, i.e., there exists $t \in R \setminus \mathfrak{p}$ such that $s^n tx \in N$. From this we have $s \in N :_R s^{n-1}tx$. To prove $\mathfrak{p} \in \mathrm{K}(M/N)$, it suffices to show that $N :_R s^{n-1}tx \subseteq \mathfrak{p}$. If this is not the case, pick $r \in N :_R s^{n-1}tx$ with $r \in R \setminus \mathfrak{p}$. This gives that $s^{n-1} \in (N :_R x)(\mathfrak{p})$, which is impossible because of the minimality of n. Hence $N :_R s^{n-1}tx \subseteq \mathfrak{p}$ and the claim is proved. It follows that a prime ideal $\mathfrak{q}$ of R is a Krull associated prime of M/N if it is a set-theoretic union of weak-Bourbaki associated primes of M/N. The converse is clear, since $r \in N :_R y \subseteq \mathfrak{q}$ implies that r is contained in every minimal prime of $N :_R y$. □

Lemma 3.2. *Let N be a proper submodule of M. Then*

$$Z_R(M/N) = \bigcup_{\mathfrak{p} \in \mathrm{wB}(M/N)} \mathfrak{p} = \bigcup_{\mathfrak{p} \in \mathrm{MaxK}(M/N)} \mathfrak{p}.$$

Proof. We note that $\cup_{\mathfrak{p} \in \mathrm{MaxK}(M/N)} \mathfrak{p} = \cup_{\mathfrak{p} \in \mathrm{K}(M/N)} \mathfrak{p}$. Also from the previous lemma we have $\mathfrak{q} \in \mathrm{K}(M/N)$ if and only if $\mathfrak{q}$ is a union of elements of $\mathrm{wB}(M/N)$. This in turn, gives that $\cup_{\mathfrak{p} \in \mathrm{K}(M/N)} \mathfrak{p} = \cup_{\mathfrak{p} \in \mathrm{wB}(M/N)} \mathfrak{p}$. Hence the second equality is clear. To verify the first, let $r \in Z_R(M/N)$. Then there exists $x \in M \setminus N$ such that $rx \in N$. Let $\mathfrak{p} \in \mathrm{wB}(M/N)$ be a minimal prime divisor of $N :_R x$. Then $r \in N :_R x \subseteq \mathfrak{p}$.

In order to prove the opposite inclusion, let $\mathfrak{p} \in \mathrm{wB}(M/N)$ and $r \in \mathfrak{p}$. There exists $x \in M \setminus N$ such that $\mathfrak{p}$ is a minimal prime divisor of $N :_R x$ and so $N(\mathfrak{p}) :_R x = (N :_R x)(\mathfrak{p})$ is a $\mathfrak{p}$-primary ideal by [9, p. 737]. This gives that

$r^n \in N(\mathfrak{p}) :_R x$ for some positive integer n. Thus $sr^n x \in N$ for some $s \in R \setminus \mathfrak{p}$. Now since $N :_R x \subseteq \mathfrak{p}$ and $s \in R \setminus \mathfrak{p}$, so sx is not an element of N. Consequently r^n and so r is an element of $Z_R(M/N)$. $\square$

Corollary 3.3. *Let N be a primal submodule of M with adjoint prime ideal $\mathfrak{p}$. Then $\mathfrak{p} \in K(M/N)$.*

Proof. By the previous lemma $\mathfrak{p} = Z_R(M/N) = \cup_{\mathfrak{p}' \in \mathrm{wB}(M/N)} \mathfrak{p}'$ is the set union of elements of $\mathrm{wB}(M/N)$. Hence $\mathfrak{p} \in \mathrm{K}(M/N)$ by Lemma 3.1. $\square$

Remark 3.4. It should be noted that for each proper submodule N of M, $N = \cap_{\mathfrak{p} \in \mathrm{Spec}(R)} N(\mathfrak{p})$; that is N is the intersection of its isolated components. This components in general do not need to be primal. However if we focus our attention on the isolated components which belongs to the elements $\mathrm{Max}\,\mathrm{K}(M/N)$, then we have a representation of N such that each component is isolated component and primal. In fact $N = \cap_{\mathfrak{p} \in \mathrm{Max}\,\mathrm{K}(M/N)} N(\mathfrak{p})$. (To see this let $x \in M \setminus N$. If $\mathfrak{q}$ is a minimal prime divisor of $N :_R x$, then there exists $\mathfrak{p} \in \mathrm{Max}\,\mathrm{K}(M/N)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Hence $N :_R x \subseteq \mathfrak{p}$ and it follows that x is not an element of $N(\mathfrak{p})$. Consequently x is not in $\cap_{\mathfrak{p} \in \mathrm{Max}\,\mathrm{K}(M/N)} N(\mathfrak{p})$ and we conclude the claim).

Now we state and prove the following theorem which is of significance and will be used in the proof of the main theorem of this paper.

Theorem 3.5. *Let N be a proper submodule of M. Then we have $N = \bigcap_{\mathfrak{p} \in \mathrm{Max}\,\mathrm{K}(M/N)} N(\mathfrak{p})$, where the isolated components $N(\mathfrak{p})$ are primal submodules with distinct and incomparable adjoint primes $\mathfrak{p}$.*

Proof. By the above remark it suffices to prove that if $\mathfrak{p} \in \mathrm{K}(M/N)$, then the isolated $\mathfrak{p}$-component $N(\mathfrak{p})$ of N is $\mathfrak{p}$-primal submodule; i.e. $Z_R(M/N(\mathfrak{p})) = \mathfrak{p}$. It is clear that the elements of $R \setminus \mathfrak{p}$ are prime relative to $N(\mathfrak{p})$ and hence $Z_R(M/N(\mathfrak{p})) \subseteq \mathfrak{p}$. To prove the other inclusion let $r \in \mathfrak{p}$. Since $\mathfrak{p} \in \mathrm{K}(M/N)$, there exists $x \in M \setminus N(\mathfrak{p})$ such that $r \in N(\mathfrak{p}) :_R x \subseteq \mathfrak{p}$. Therefore we have $N(\mathfrak{p}) \subset N(\mathfrak{p}) :_M r$ and so $r \in Z_R(M/N(\mathfrak{p}))$. $\square$

Now we state and prove the main theorem of this section.

Theorem 3.6. *Let M be a torsion free R-module. The following statements are equivalent.*

(i) *M is a distributive R-module.*

(ii) *Every primal submodule of M is irreducible.*

(iii) *For every maximal ideal $\mathfrak{m}$, the set $W_{\mathfrak{m}}(M)$ is linearly ordered with respect to inclusion.*

(iv) *For every maximal ideal $\mathfrak{m}$ and any $x, y \in M$, $Rx(\mathfrak{m})$ and $Ry(\mathfrak{m})$ are comparable with respect to inclusion.*

(v) *For each submodule N of M and each maximal ideal $\mathfrak{m}$, $E_R(M/N(\mathfrak{m}))$ (the injective hull of $M/N(\mathfrak{m})$) is indecomposable.*

Proof. (i)⇒(ii). Let N be a $\mathfrak{p}$-primal submodule of M. Then by Corollary 2.8, $N(\mathfrak{p})$ is irreducible. Now it follows by Lemma 2.5(a) that N is irreducible.

(ii)⇒(i). By virtue of Lemma 2.3, it is enough to show that for each maximal ideal $\mathfrak{m}$ of R, $M_\mathfrak{m}$ is a distributive module over the quasi local ring $R_\mathfrak{m}$. To do this, by Lemma 2.4, it suffices to prove that for any $x, y \in M$, either $< x/1 >\subseteq < y/1 >$ or $< y/1 >\subseteq < x/1 >$. To this end, Let $N =< x, y >$ and $N_\mathfrak{m} \neq 0$. (If $N_\mathfrak{m} = 0$, then $x/1 = y/1 = 0$ and there is nothing to prove.) Then by Lemma 2.9, $(\mathfrak{m}N)(\mathfrak{m})$ is an $\mathfrak{m}$-primal and so by our assumption an irreducible submodule of M. Hence by Proposition 2.7, $(\mathfrak{m}N)_\mathfrak{m}$ is an irreducible submodule of $M_\mathfrak{m}$. This gives that $N_\mathfrak{m}/(\mathfrak{m}N)_\mathfrak{m}$ as a finite dimensional vector space over the field $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ is of dimension one, and therefore is isomorphic to $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. So we have either $N_\mathfrak{m} =< x/1 > +(\mathfrak{m}N)_\mathfrak{m}$ or $N_\mathfrak{m} =< y/1 > +(\mathfrak{m}N)_\mathfrak{m}$. Hence by the Nakayama's Lemma $N_\mathfrak{m} =< x/1 >$ or $N_\mathfrak{m} =< y/1 >$, which gives that either $< x/1 >\subseteq< y/1 >$ or $< y/1 >\subseteq< x/1 >$ and the result follows.

(iii)⇒(iv) is clear.

(iv)⇒(i). Assume that (i) does not hold. Then, by Lemma 2.4, there exist $x, y \in M$ and a maximal ideal $\mathfrak{m}$ of R such that $(Ry :_R x) + (Rx :_R y) \subseteq \mathfrak{m}$. It follows that $y \in Ry(\mathfrak{m}) \setminus Rx(\mathfrak{m})$ and $x \in Rx(\mathfrak{m}) \setminus Ry(\mathfrak{m})$, contracting to (iv).

(i)⇒(iii). Suppose the contrary; i.e., there exist a maximal ideal $\mathfrak{m}$ and $L, K \in W_\mathfrak{m}(M)$, such that $L \nsubseteq K$ and $K \nsubseteq L$. Let $x \in L \setminus K$ and $y \in K \setminus L$. Our assumption together with Lemma 2.4, give that there exists $r \in R$ such that $rx \in K$ and $(1-r)y \in L$. Since $\mathfrak{m}$ is a maximal ideal, at least one of the elements r, $1 - r$ is not contained in $\mathfrak{m}$. If the first possibility is true, it follows that $x \in K(\mathfrak{m}) = K$, a contradiction. With the second possibility we come to the contradiction $y \in L(\mathfrak{m}) = L$.

(i)⇔(v). We note that a submodule K of M is irreducible if and only if $E_R(M/K)$ is indecomposable. Thus it follows from Corollary 2.8. □

The result of the previous theorem gives that for each submodule N of a distributive module M, the representation of N as an intersection of isolated components given in Theorem 3.5 is a decomposition of N into irreducible isolated components. In the next theorem we show that for the torsion free R-modules this condition is in fact sufficient for M to be distributive.

Theorem 3.7. *Let M be a torsion free R-module. The following statements are equivalent.*

(i) *M is a distributive R-module.*

(ii) *For each proper submodule N of M, $N = \bigcap_{\mathfrak{p}\in \mathrm{MaxK}(M/N)} N(\mathfrak{p})$ is an irreducible decomposition of N.*

Proof. (i)⇒(ii) follows from Theorems 3.5 and 3.6.

(ii)⇒(i). By Lemma 2.3 it is enough to show that for each maximal ideal $\mathfrak{m}$ of R the cyclic submodules of $M_\mathfrak{m}$ are totally ordered. To this end, let $x, y \in M$. Set $N =< x, y >$. Then by assumption $\mathfrak{m}N = \bigcap_{\mathfrak{p}\in \mathrm{MaxK}(M/\mathfrak{m}N)}(\mathfrak{m}N)(\mathfrak{p})$, is an irreducible decomposition of $\mathfrak{m}N$. We claim that $\mathfrak{m} \in \mathrm{MaxK}(M/\mathfrak{m}N)$. If this is not the case, then $(\mathfrak{m}N)(\mathfrak{p}) = N(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{MaxK}(M/\mathfrak{m}N)$. This gives that $\mathfrak{m}N = \bigcap_{\mathfrak{p}\in\mathrm{MaxK}(M/\mathfrak{m}N)} N(\mathfrak{p}) \supseteq N$, which is impossible, since N is finitely generated. Hence $\mathfrak{m} \in \mathrm{MaxK}(M/\mathfrak{m}N)$ and $(\mathfrak{m}N)(\mathfrak{m})$ is an irreducible submodule of M. Thus by Proposition 2.7, $(\mathfrak{m}N)_{\mathfrak{m}}$ is an irreducible submodule of $M_{\mathfrak{m}}$. The result now follows by the same argument as in the proof of the Theorem 3.6 part (ii)⇒(i). □

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