Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 50 (2009), No. 2, 369-387.

Conformal Width in Möbius Geometry

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Abstract. In this article we extend the euclidean concept of width to Möbius geometry. For pairs of curves in the plane or in the 2-sphere S^2 which are the two folds of an envelope of circles, the conformal width will be defined as the conformal distance between the osculating circles at corresponding points. We mainly study pairs of curves having constant conformal width. The main results characterize constant conformal width in terms of the geodesic curvature of the family of circles enveloping the pair of curves, seen as a curve in the 3-dimensional de Sitter space, and in terms of the conformal arc-lengths of the two folds of the envelope.

MSC2000: 53A30, 53C40, 53A04
Keywords: curves, conformal width, constant width, conformal 2-sphere, Möbius geometry

1. Introduction

The concept of width in euclidean spaces picks out pairs of points on a geometric object (curves, surfaces etc.) through either parallel tangent spaces or double normals. The distance between these parallel tangent spaces or the length of the

*The second author thanks the University of Burgundy for a month invitation.

double normal segment respectively is called *width*. At least for constant width both concepts coincide.

The width is in the game e.g. at convex bodies, convex bodies of constant width (Gleichdicke) (i.e. convex bodies which can be arbitrarily rotated between two fixed parallel planes without losing contact with either plane) cf. [5], [10], or at parallel curves/surfaces, or rather at Bertrand curves (i.e. pairs of curves with common normal lines, J. Bertrand (1850)) and transnormal curves/submanifolds (i.e. submanifolds which affine normal spaces intersect the submanifold orthogonally at each of their intersection points, cf. [18], [20]).

For the concept of width in spaces of constant curvature see [5], [10], [17], [9].

In this article we develop the width further to Möbius geometry. Mainly things will happen in the conformal sphere S^2.

We consider pairs of regular parameterized curves $X_1(t)$, $X_2(t)$, $t \in I$, related to one another by double normal circles $N(t)$ at corresponding points $X_1(t)$, $X_2(t)$, i.e. $X_1(t), X_2(t) \in N(t)$, $X_1'(t), X_2'(t) \perp N(t)$ ($t \in I$, I interval in $\mathbb{R}$). At corresponding points $X_1(t)$, $X_2(t)$ we take the osculating circles $osc_1(t)$, $osc_2(t)$. We call the "distance" between $osc_1(t)$ and $osc_2(t)$ the *conformal width* $w_c(t)$ (see Definition 1). It is a conformal invariant along the pair of curves under consideration. (For simplicity we suppose all curves etc. to be C^∞-smooth.)

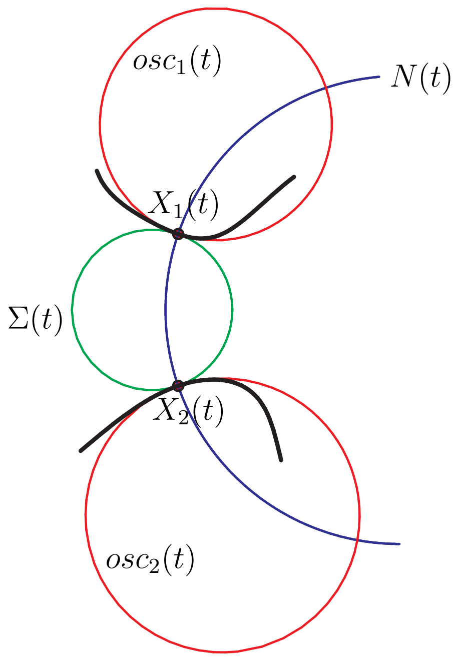

Figure 1. A pair of curves $X_1(t), X_2(t)$ with double normal circle $N(t)$, enveloping circle $\Sigma(t)$ and osculating circles $osc_1(t)$, $osc_2(t)$

The classical "inversive distance" between two circles C_1 and C_2, say in the euclidean plane, is given by

$$w_c(C_1, C_2) = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}\ , \tag{1}$$

where r_1, r_2 are the radii of C_1, C_2 and d is the euclidean distance between the centers of C_1 and C_2, (cf. [6], [1], [8]).

Especially for two intersecting circles, $w_c(C_1, C_2) = \cos\theta$, the cosine of the angle θ between C_1 and C_2.

In the following we mainly investigate pairs of curves related to one another through double normal circles and having constant conformal width.

Example 1. Basic examples are two circles C_1, C_2 ($C_1 \neq C_2$) generating either a pencil of circles in case $C_1 \cap C_2 =$ two distinct points (Steiner pencil), or a pencil of circles with limit points in case $C_1 \cap C_2 = \emptyset$ (Poncelet pencil), or a degenerate pencil of circles in case C_1 tangent C_2 respectively.

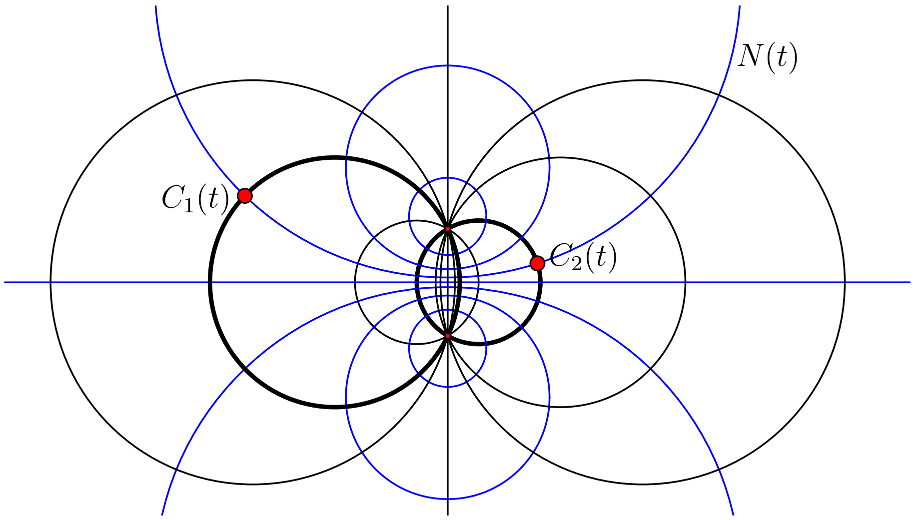

Figure 2. Two circles $C_1(t), C_2(t)$ in a Steiner pencil of circles together with the associated Poncelet pencil of double normal circles $N(t)$

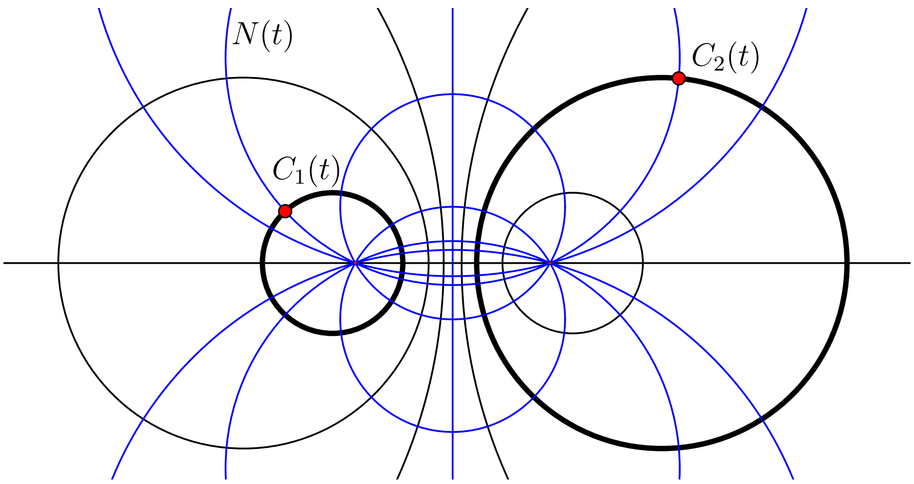

Figure 3. Two circles $C_1(t), C_2(t)$ in a Poncelet pencil of circles together with the associated Steiner pencil of double normal circles $N(t)$

C_1 and C_2 are related to one another through double normal circles building up the associated bundle of circles orthogonal to both C_1 and C_2. These are either Poncelet pencils, or Steiner pencils, or degenerate pencils respectively. Because of $osc_1 = C_1$ and $osc_2 = C_2$, the curves C_1 and C_2 have constant conformal width. Furthermore each pair of distinct circles belonging to the pencil induced by C_1 and C_2 have constant conformal width (see Figure 2 and Figure 3).

In the Lorentz model (see below) these pencils of circles are given by the intersection of a 2-dimensional linear subspace of $\mathbb{R}^4_1$ with the de Sitter sphere Λ^3.

Non-example 2. ("Parallel curves" in the euclidean plane or equivalently pairs of curves with a common involute in the euclidean plane, see Figure 4)

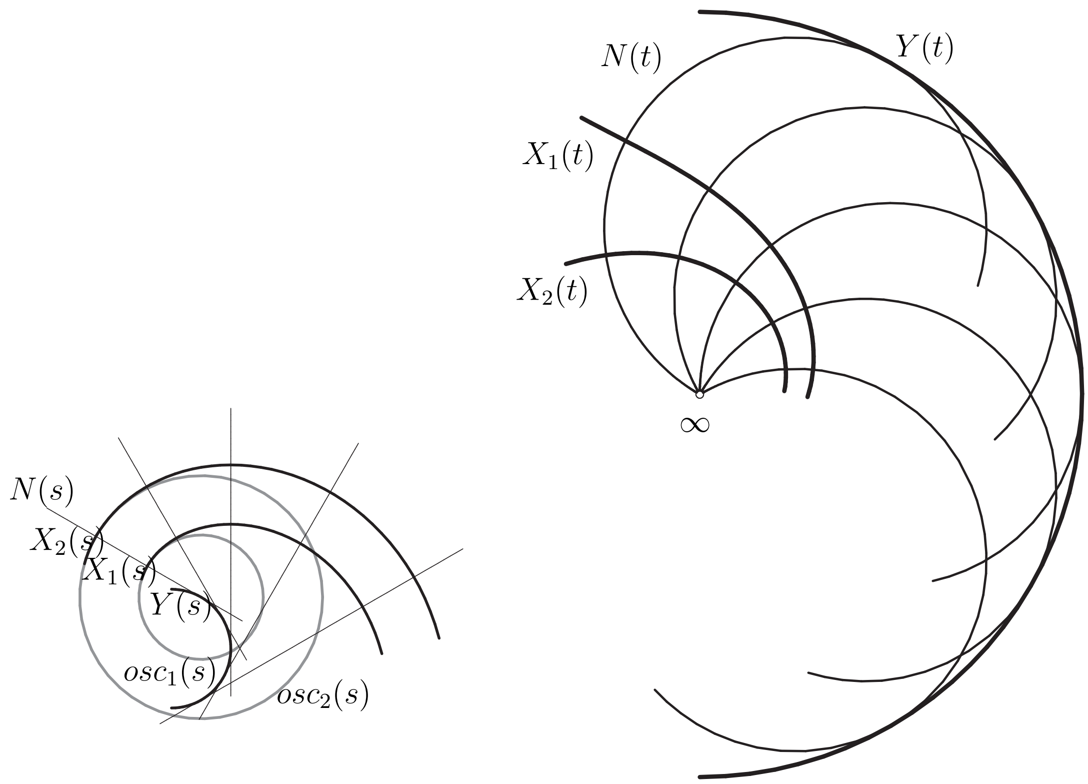

Figure 4. Non-example 2: "Parallel curves" in the euclidean plane

Consider the special situation of pairs of curves $X_1(t)$, $X_2(t)$ related to one another through double normal circles having a point "∞" in common. Through stereographic projection from $S^2 \setminus \{\infty\}$ onto the euclidean plane we see a pair of curves $X_1(t)$, $X_2(t)$ in the plane related to one another through double normal lines. In general $X_1(t)$, $X_2(t)$ are involutes of the envelope $Y(t)$ of the double normal lines. In this picture the osculating circles $osc_1(s)$, $osc_2(s)$ are concentric circles with center $Y(s)$ and radii $s_1 - s$, $s_2 - s$ (s_1, s_2 fixed, $s_1 \neq s_2$, s arc length parameter on Y). Then the inversive distance (1) between $osc_1(s)$ and $osc_2(s)$ is equal to $\frac{(s_1-s)^2+(s_2-s)^2}{2(s_1-s)(s_2-s)}$, which is not constant because $s_1 \neq s_2$. Hence in general constant euclidean width, here $|s_1 - s_2|$, does not imply constant conformal width.

We are considering pairs of curves $X_1(t)$, $X_2(t)$ related to one another through double normal circles $N(t)$. Let $\Sigma(t)$ denote the circle orthogonal to $N(t)$ through $X_1(t)$ and $X_2(t)$. Then equivalently we may consider families of circles $\Sigma(t)$ in S^2 (*one-dimensional canals*) which envelopes are $X_1(t)$ and $X_2(t)$ (see Figure 1).

The considerations here use the Lorentz space model for the Möbius geometry. In detail, say the 2-dimensional case, the model lives in Lorentz space $\mathbb{R}^4_1$ with its Lorentz product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 \ .$$

The projectivization of the light cone $\mathcal{C} = \{x \in \mathbb{R}^4_1 | \langle x, x\rangle = 0\}$ gives the model for the conformal geometry on the sphere S^2 (cf. [11], [7]).

Every vector $x \in \mathcal{C}\backslash\{0\}$ defines a light ray $\mathrm{span}(x) \subset \mathcal{C}$ representing a point $\mathrm{span}(x) = X \in S^2$.

Circles $\Sigma \subset S^2$ are given by intersections of 3-dimensional linear subspaces of mixed type in $\mathbb{R}^4_1$ with the light cone $\mathcal{C}$, respectively equivalently by its orthogonal complement in $\mathbb{R}^4_1$. In this way each point (or more precisely each pair of antipodal points) σ in the de Sitter sphere $\Lambda^3 = \{\sigma \in \mathbb{R}^4_1 | \langle \sigma, \sigma\rangle = 1\}$ represents a circle $\Sigma \subset S^2$ by $\Sigma = (\mathrm{span}(\sigma))^\perp \cap \mathcal{C}$, and vice versa.

The Möbius group acting on S^2 is represented by the projectivization of the Lorentz group acting on $\mathbb{R}^4_1$.

The main results:

First we characterize constant conformal width in terms of the curve σ in the de Sitter sphere $\Lambda^3 \subset \mathbb{R}^4_1$ corresponding to the family of circles $\Sigma(t)$ enveloping X_1 and X_2.

Theorem 1. *Conformal width $w_c(t) = const.$ along $X_1(t)$, $X_2(t)$ is equivalent to geodesic curvature $\kappa_g(t) = const.$ along $\sigma(t)$.*

Secondly we characterize constant conformal width in terms of the conformal arc lengths (see Lemma 4 and [13]) τ_1, τ_2 at related pairs of points.

Theorem 2. *Conformal width $w_c(t) = const.$ along $X_1(t)$, $X_2(t)$ is equivalent to $d\tau_1(t) = d\tau_2(t)$ along $X_1(t)$, $X_2(t)$.*

2. Conformal distances

Let C_1, C_2 be two circles in S^2 represented by two points (more precisely by two pairs of antipodal points) c_1, c_2 in Λ^3. There are various possibilities to define a "conformal distance" between C_1 and C_2:

1. In case of two intersecting circles C_1 and C_2 the angle between C_1 and C_2 defines a conformal invariant "distance" between C_1 and C_2.
2. In case of two disjoint circles C_1 and C_2 they generate a Poncelet pencil with limit points l_1 and l_2. Each circle C of the associated orthogonal Steiner pencil intersects $C_1 \cup C_2$ in four points and contains l_1 and l_2. Take $p_1 \in C_1 \cap C_2$, $p_2 \in C_1 \cap C_2$ in the order l_1, p_1, p_2, l_2 on C, then the absolute

value of the logarithm of the cross-ratio of these four points defines the *modulus* of C_1 and C_2 (cf. [12]). This modulus may serve as conformal invariant "distance" between C_1 and C_2. This because there is a subgroup of the Möbius group acting transitively on the pencil of circles orthogonal to C_1 and C_2 and fixing C_1 and C_2 as circles.

In general:

3. The classical "inversive distance" (1), cf. [6], [1], [8].
4. The circles C_1, C_2 define a pencil of circles. Each circle of the associated orthogonal pencil intersects $C_1 \cup C_2$ in four points. The absolute value of the logarithm of the cross-ratio of these four points is independent of the specific chosen orthogonal circle and defines a conformal invariant "distance" between C_1 and C_2. This because there is a subgroup of the Möbius group acting transitively on the pencil of orthogonal circles to C_1 and C_2, and fixing C_1 and C_2 as circles.
5. Choose a third circle $\tilde{C}$ tangent to both C_1 and C_2. Then in Λ^3 there are two light rays emanating at $\tilde{c}$ and through c_1 and c_2 respectively. Therefore we see in $\Lambda^2 = \text{span}(c_1, c_2, \tilde{c}) \cap \Lambda^3$ a light ray polygon with four vertices $c_1, c_2, \tilde{c}$ and some $\hat{c}$. The 2-dimensional area of the region inside defines a conformal invariant "distance" between C_1 and C_2. This because the subgroup of the Möbius group fixing C_1 and C_2 as circles acts transitively on the set of circles tangent to both C_1 and C_2.
6. The geodesic distance between $\pm c_1$ and $\pm c_2$ in Λ^3 defines a conformal invariant "distance" $w_c(C_1, C_2)$ between C_1 and C_2.
   It is an exercise in differential geometry, left for the reader, to check that this geodesic distance is given by

$$\begin{aligned} \cos(w_c(C_1, C_2)) &= |\langle \pm c_1, \pm c_2 \rangle| \ \text{, or} \\ \cosh(w_c(C_1, C_2)) &= |\langle \pm c_1, \pm c_2 \rangle| \ \text{, or} \\ w_c(C_1, C_2) &= 0 \end{aligned} \tag{2}$$

   in case that $\text{span}(\pm c_1, \pm c_2)$ is space-like ($C_1 \cap C_2 \neq \emptyset$), or is of mixed type ($C_1 \cap C_2 = \emptyset$), or is degenerate ($C_1$ and C_2 are tangent) respectively.

Remark. The cross-ratio of four points on a circle is a well-defined conformal invariant.

Note: Through stereographic projection of S^2 we see four points on a circle in $\mathbb{R}^2$ or equivalently four complex numbers in the complex line $\mathbb{C}^1$ lying on a real circle. But the cross-ratio of four complex numbers in $\mathbb{C}^1$ takes real values if and only if the four points lie on a real circle. And the invariance of the cross-ratio with respect to projective maps on $\mathbb{C}^1$ reflects its conformal invariance on S^2.

Remark. In the limit case of tangent circles C_1 and C_2, there does not exist an ingenious conformal invariant "distance". This because the Möbius group acts transitively on the set of pairs of tangent circles.

Remark. In general these six possibilities lead to different values. But "constant width" has the same meaning with respect to all six possibilities. The sixth possibility is the best adapted one for our purposes. Hence

Definition 1. *Let $osc_1(t), osc_2(t)$ be the curves in Λ^3 representing the osculating circles of a pair of curves $X_1(t), X_2(t), t \in I$, in S^2 related through double normal circles. We call the geodesic distance between $osc_1(t)$ and $osc_2(t)$ in Λ^3 the conformal width $w_c(t)$ along the pair of curves.*

Proposition 1. *Let $X_1(t), X_2(t), t \in I$, be a pair of curves in S^2 related through double normal circles and with constant conformal width. If X_1 is part of a circle C_1, then X_2 is part of a circle C_2.*

Proof. For each $t \in I$, $osc_1(t)$ and $osc_2(t)$ are tangent to $\Sigma(t)$. Therefore the picture in Λ^3 looks as follows: $osc_1(t), osc_2(t), t \in I$, are light-like curves in Λ^3 (cf. [12]), and the corresponding tangent light rays intersect at $\sigma(t)$. Now X_1 is part of a circle, therefore $C_1 = osc_1(t), t \in I$. Hence osc_1 is degenerated to the point $c_1 \in \Lambda^3$ (or more precisely to the pair of antipodal points $\pm c_1$). Because of constant conformal width w_c, $osc_2(t)$ lies in the distance sphere in Λ^3 with center $\pm c_1$ and radius w_c. Taking into account the formulas (2) for the geodesic distance in Λ^3, such a distance sphere is given by the intersection of the parallel hyperplanes $\{x \in \mathbb{R}^4_1 | \ |\langle c_1, x\rangle| = \text{const.}\}$ with Λ^3, that means it is made of antipodal pairs of two-sheet hyperboloids and one-sheet hyperboloids. The first ones are space-like surfaces, in fact four copies of a hyperbolic plane, the second ones are two Λ^2. Therefore in the first case the light-like curve $osc_2(t)$ must degenerate to a point, hence $X_2 = osc_2 =$ circle. In the second case the light-like curve $osc_2(t)$ must be a light ray in Λ^3, but light rays define a degenerate pencil of circles, hence osc_2 must degenerate to a point, hence $X_2 = osc_2 =$ circle. □

3. Conformal width and the σ-curve

The family $\Sigma(t)$ of circles in S^2 enveloping a pair of curves $X_1(t)$, $X_2(t)$ related to one another through double normal circles corresponds to the space-like curve $\sigma(t)$ in the de Sitter sphere Λ^3. We will call the curve $\sigma(t)$ corresponding to $\Sigma(t)$ the *σ-curve.* Let the σ-curve be generic, that is $\dim \operatorname{span}(\sigma(s), \sigma'(s), \sigma''(s)) = 3$, $s \in I$, s arc length parameter. We call such generic curve a *Frenet curve.* We suppose that the geodesic curvature vector $\vec{k}_g(s)$ is nowhere vanishing and nowhere light-like ($\vec{k}_g(s)$ is the component of $\sigma''(s)$ tangent to Λ^3; the length of $\vec{k}_g(s)$ is the geodesic curvature $\kappa_g(s)$ of σ at $\sigma(s)$).

Remark. If $\vec{k}_g(s)$ vanishes identically, then σ is part of a space-like geodesic in Λ^3. In this case the Σ-family is part of a Steiner pencil of circles through two fixed points, and X_1, X_2 degenerate to these two points.

If $\vec{k}_g(s)$ is light-like but nowhere vanishing, then one of the envelopes degenerates to a point.

Curves in Lorentz space can be studied by using moving frames, cf. [21].

Let us define the orthonormal Frenet frame e_0, e_1, e_2, e_3 along the σ-curve as $e_0 = \sigma$, $e_1 = \sigma'$, $\mathrm{span}(e_0, e_1, e_2) = \mathrm{span}(\sigma, \sigma', \sigma'')$. The Frenet formulas write, using matrix representation:

$$\begin{pmatrix} e_0' \\ e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa_g & 0 \\ 0 & -\epsilon_2\kappa_g & 0 & \tau_g \\ 0 & 0 & \tau_g & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{3}$$

with $\epsilon_2 = \langle e_2, e_2\rangle$, geodesic curvature $\kappa_g > 0$ and geodesic torsion τ_g.

Then the Frenet theory yields.

Lemma 1. *The functions $\kappa_g(s) > 0$ and $\pm\tau_g(s)$ determine $\sigma(s)$, respectively $\Sigma(s)$, uniquely up to Lorentz transformations in $\mathbb{R}^4_1$, respectively Möbius transformations in S^2.*

Theorem 1. *Let $X_1(s)$, $X_2(s)$, $s \in I$, be a pair of curves in S^2 related to one another through double normal circles, given as envelopes of a family $\Sigma(s)$ of circles in S^2. Suppose the corresponding space-like σ-curve in the de Sitter sphere Λ^3 to be a Frenet curve. Then conformal width $w_c(s) = const.$ along $X_1(s)$, $X_2(s)$ is equivalent to geodesic curvature $\kappa_g(s) = const.$ along $\sigma(s)$.*

Proof. As the σ-curve is space-like the orthogonal plane to $\mathrm{span}(\sigma, \sigma')$ is of mixed type. Therefore it contains exactly two light rays. The two characteristic points X_1 and X_2 of the envelope of Σ correspond to these two light rays $\mathrm{span}(\sigma, \sigma')^\perp \cap \mathcal{C}$. Alternatively we can see the plane $\mathrm{span}(\sigma, \sigma')^\perp$ as the plane in $T_\sigma\Lambda^3$ orthogonal to σ' (see Figure 5).

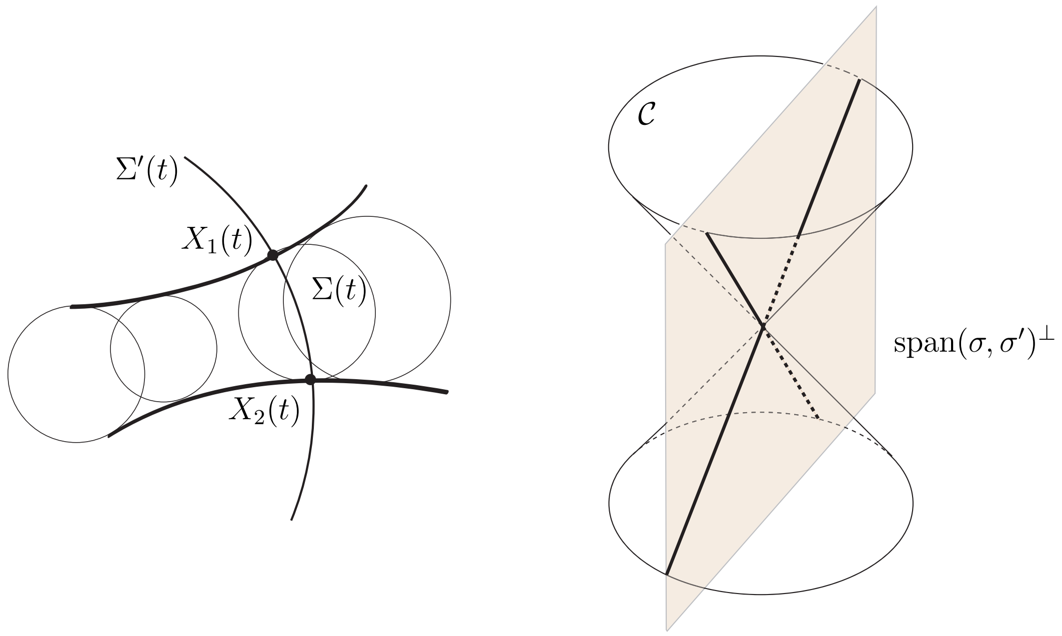

Figure 5. Enveloping circles $\Sigma(t)$ and characteristic points $X_1(t), X_2(t)$

Hence $X_1 = \mathrm{span}(e_2 + e_3)$ and $X_2 = \mathrm{span}(e_2 - e_3)$. The osculating circles of X_1, X_2 are given by

$$osc_1 \perp \mathrm{span}((e_2 + e_3), (e_2 + e_3)', (e_2 + e_3)'') \quad \text{and}$$

$$osc_2 \perp \mathrm{span}((e_2 - e_3), (e_2 - e_3)', (e_2 - e_3)'')$$

(cf. [14], Lemma 3).

Taking into account the Frenet formulas (3), we get

$$osc_1 \perp \mathrm{span}(e_2 + e_3, e_1, e_0 - \kappa_g e_2),$$

$$osc_2 \perp \mathrm{span}(e_2 - e_3, e_1, e_0 - \kappa_g e_2).$$

Hence

$$\begin{aligned} osc_1 &= \epsilon_2 e_0 + \tfrac{1}{\kappa_g}(e_2 + e_3) \quad \text{and} \\ osc_2 &= \epsilon_2 e_0 + \tfrac{1}{\kappa_g}(e_2 - e_3). \end{aligned} \tag{4}$$

Therefore the constant width condition can be expressed as follows

$$\begin{aligned} w_c = const. \quad &\Longleftrightarrow \quad \langle osc_1, osc_2 \rangle = const. \\ &\Longleftrightarrow \quad \frac{\kappa_g^2 + \epsilon_2 - \epsilon_3}{\kappa_g^2} = const. \\ &\Longleftrightarrow \quad \kappa_g = const. \end{aligned}$$

(Note: $\langle e_2, e_2 \rangle = \epsilon_2$, $\langle e_3, e_3 \rangle = \epsilon_3$; e_2, e_3 span a mixed plane, hence $\epsilon_2 - \epsilon_3 = +2$ or -2.) □

Example 3. Take the curves

$$\sigma(t) = (\lambda \cos \omega t, \lambda \sin \omega t, \mu \cosh \nu t, \mu \sinh \nu t) \subset \mathbb{R}^4_1$$

($t \in \mathbb{R}$, $\lambda, \mu, \omega, \nu$ real constants).

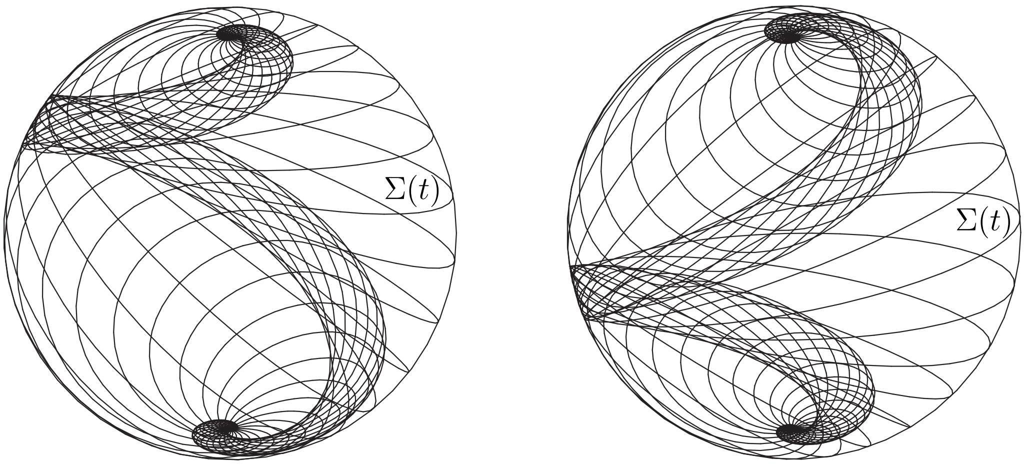

Figure 6. Example 3 ($\lambda = 1/\sqrt{2},\ \mu = 1/\sqrt{2},\ \omega = 2,\ \nu = 1$)

If $\lambda^2 + \mu^2 = 1$, then $\sigma(t) \subset \Lambda^3 \subset \mathbb{R}^4_1$. If $\lambda^2\omega^2 - \mu^2\nu^2 > 0$, then $\sigma'(t)$ space-like.

These curves are orbits of 1-parameter subgroups of the Lorentz group. Hence $\kappa_g = const.$ and $\tau_g = const.$. Moreover the envelopes X_1 and X_2 have each constant conformal curvature. Vice versa each σ-curve with constant geodesic curvature and constant geodesic torsion has the above equation with respect to an appropriate chosen orthonormal base in Lorentz space (see Figure 6).

Example 4. $\kappa_g = \text{const.} > 0$, $\tau_g = 0$ ("one-dimensional Dupin cyclides", see Figure 7, Figure 8 and [15]):

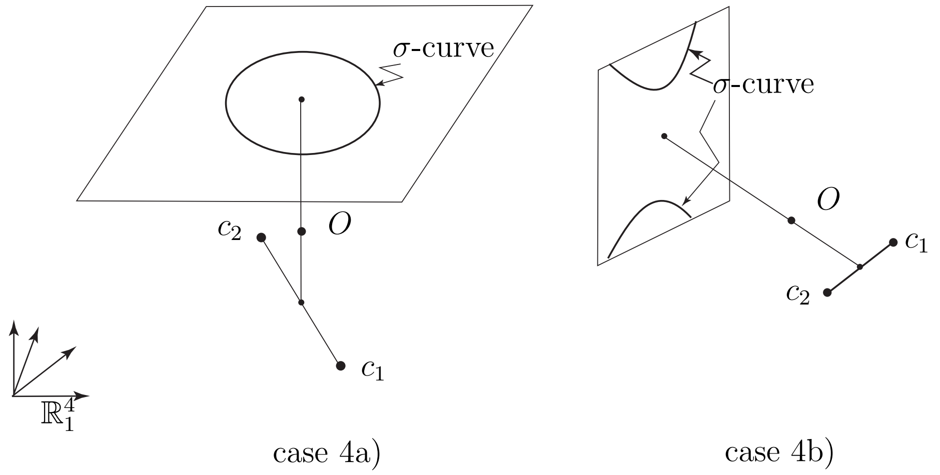

Figure 7. Example 4: One-dimensional Dupin cyclides in Λ^3

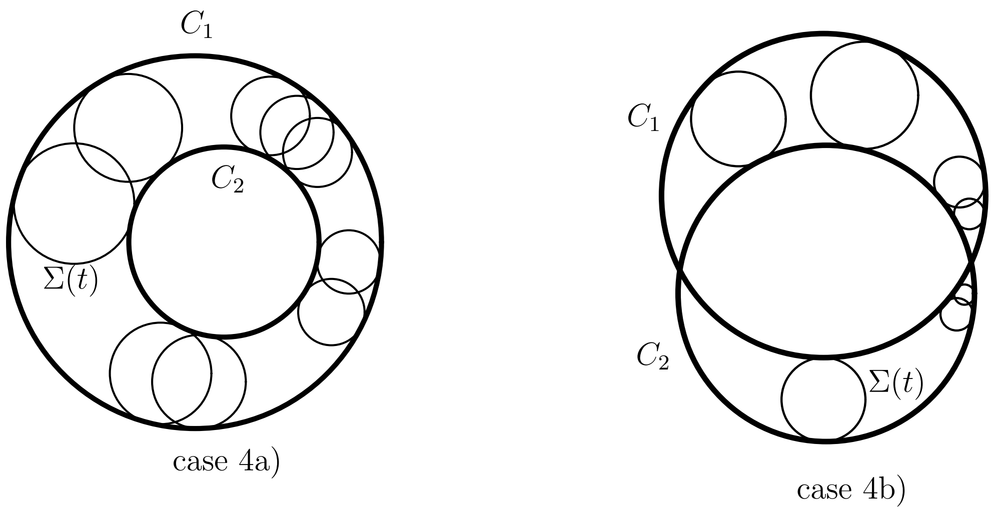

Figure 8. Example 4: One-dimensional Dupin cyclides in the plane

By the Frenet formulas (3) we get $e_3 = \text{const.}$, hence $\text{span}(e_0, e_1, e_2) = \text{const.}$
Case 4a) e_3 time-like: $\text{span}(e_0, e_1, e_2) \cap \Lambda^3 =$ some euclidean 2-sphere S^2. Hence the σ-curve is a circle, given by a plane section of this S^2.

Case 4b) e_3 space-like: $\mathrm{span}(e_0, e_1, e_2) \cap \Lambda^3$ = some de Sitter sphere Λ^2. Hence the σ-curve is a space-like hyperbola given by a plane section of this Λ^2.

The corresponding complete pictures in the Möbius plane are as follows:

In case 4a) $X_1(t)$, $X_2(t)$ are non-intersecting circles C_1, C_2 defining a Poncelet pencil of circles. The double normal circles build up the associated Steiner pencil of circles orthogonal to both C_1 and C_2, $osc_1(t) = C_1$, $osc_2(t) = C_2$.

In case 4b) $X_1(t)$, $X_2(t)$ are intersecting circles C_1, C_2 defining a Steiner pencil of circles. The double normal circles build up the associated Poncelet pencil of circles orthogonal to both C_1 and C_2; $osc_1(t) = C_1$, $osc_2(t) = C_2$.

In both cases there are two families $\Sigma(t)$ of circles enveloping C_1 and C_2. The full picture in $\Lambda^3 \subset \mathbb{R}^4_1$: $C_1 = osc_1(t)$, $C_2 = osc_2(t)$ are two pairs of antipodal points in Λ^3. The light cones of the associated tangent hyperplanes of Λ^3 intersect in two antipodal pairs of plane sections of Λ^3, namely the two possible σ-curves enveloping the circles C_1 and C_2.

Remark. Given a curve $X_1(t)$, $t \in I$, and a constant w_c, then generically and at least locally there exists a curve $X_2(t)$ (in fact many) such that $X_1(t)$, $X_2(t)$ build up a pair of curves related to one another through double normal circles and with constant conformal width w_c. This can be seen as follows: Associated to $X_1(t)$ we have the fiber bundle over $X_1(t)$ which fibers at $X_1(t)$, $t \in I$, are the pencils of circles tangent to X_1 at $X_1(t)$. The sections in this bundle describe exactly the σ-curves enveloping at least X_1 and in general some other curve. Following Theorem 1 we must look for σ-curves with appropriated constant geodesic curvature κ_g. This leads to an ordinary differential equation of second order for the function in t fixing the position inside the 1-dimensional fibers. Generically and at least locally this differential equation can be solved. Then the envelopes of these σ-curves are the affirmed X_1 and X_2.

Proposition 2. *Let $X_1(t)$, $X_2(t)$ and $X_1(t)$, $X_3(t)$, $t \in I$, be different pairs of curves of constant conformal widths and with a common family $N(t)$ of double normal circles. We suppose also that near the points $X_1(t)$, $X_2(t)$, $X_3(t)$ the normal circles $N(t)$ form a foliation. Then X_1, X_2, X_3 are (parts of) circles C_1, C_2, C_3 which are members of a pencil of circles. The common family of double normals belongs to the associated orthogonal pencil of circles.*

Proof. The families $\Sigma(t)$ and $\bar{\Sigma}(t)$ of circles enveloping $X_1(t)$, $X_2(t)$ and $X_1(t)$, $X_3(t)$ have circles $\Sigma(t)$ and $\bar{\Sigma}(t)$ tangent to X_1 at $X_1(t)$, $t \in I$. The three points $X_1(t)$, $X_2(t)$, $X_3(t)$ are distinct and determine the normal circle $N(t)$, hence we have $\Sigma'(t) = \bar{\Sigma}'(t) = N(t)$.

Let $\sigma(s)$ be the σ-curve in Λ^3 of the pair X_1, X_2 with constant geodesic curvature κ_g (cf. Theorem 1), and let s be an arc length parameter on σ.

Then the vector e_2+e_3 is light-like. The light ray $\sigma+\mathrm{span}(e_2+e_3)$ corresponds to the pencil of tangent circles to the curve X_1 at the point under consideration (cf. proof of Theorem 1).

Therefore the $\bar{\sigma}$-curve in Λ^3 of the pair X_1, X_3 can be parametrized by $\bar{\sigma} = \sigma + \rho(e_2 + e_3)$.

Hence, taking into account the Frenet formulas (3) for σ, we get

$$\bar{\sigma}' = \sigma' + \rho'(e_2 + e_3) + \rho(e_2' + e_3') = (1 - \rho\epsilon_2\kappa_g)e_1 + (\rho' + \rho\tau_g)(e_2 + e_3).$$

As $\Sigma' = \bar{\Sigma}' = N$ we have $\mathrm{span}(e_1) = \mathrm{span}(\sigma') = \mathrm{span}(\bar{\sigma}')$. Therefore

$$\rho' + \rho\tau_g = 0. \tag{5}$$

Hence $d\bar{s} = |1 - \rho\epsilon_2\kappa_g|\, ds$, $\bar{s}$ an arc length parameter on $\bar{\sigma}$, then:

$$\frac{d\bar{e}_1}{d\bar{s}} = \frac{de_1}{d\bar{s}} = \frac{de_1}{ds}\frac{ds}{d\bar{s}},$$

and

$$\bar{\kappa}_g = \kappa_g \frac{1}{|1 - \rho\epsilon_2\kappa_g|}. \tag{6}$$

Because κ_g and $\bar{\kappa}_g$ are non zero constants, (6) and (5) yield $\rho = const.$ and $\tau_g = 0$. We leave as an exercise in Frenet theory for the reader to check that the σ-curve and the $\bar{\sigma}$-curve should be contained in sections of Λ^3 by affine planes ("Dupin-like" sections). Therefore, cf. Example 4, X_1, X_2, X_3 are (parts of) circles belonging to a pencil of circles. □

Recall that pairs of curves $X_1, X_2 \subset S^2$ of constant conformal width w_c are characterized through $\kappa_g = const.$ for the σ-curve in Λ^3 (Theorem 1).

Through the Frenet theory curves in Λ^3 are determined up to Lorentz transformations by their geodesic curvature κ_g and geodesic torsion τ_g.

Therefore, starting at the given σ-curve with $\kappa_g = const.$ and geodesic torsion function $\tau_g(s)$, the families $\kappa_g(s,\lambda) = \kappa_g = const.$, $\tau_g(s,\lambda) = (1-\lambda)\tau_g(s)$, $\lambda \in [0,1]$, determine a family $\sigma(s,\lambda)$ of σ-curves with constant geodesic curvature κ_g. For each λ the pair of envelopes $X_1(s,\lambda), X_2(s,\lambda)$, $s \in I$, has constant conformal width w_c. This way we start at a given pair of curves of constant conformal width $(\lambda = 0)$ and we end at Example 4 $(\lambda = 1)$. Hence

Proposition 3. *Given a pair of curves $X_1, X_2 \subset S^2$ of constant conformal width w_c, there exists a homotopy through pairs of curves of constant conformal width w_c in S^2, starting at the given one and ending at a pair of circles (Example 4: one-dimensional Dupin cyclides).*

4. Conformal width and conformal arc length

Now we characterize constant conformal width in terms of the conformal arc length τ. We should suppose that the curves have no vertex, that is never have contact of order three with their osculating circles. We will call *osc-curve* associated to a curve in S^2 the curve in Λ^3 of the osculating circles.

Theorem 2. *Let $X_1(t), X_2(t)$, $t \in I$, be a pair of curves in S^2 without vertices, related to one another through double normal circles. Suppose their osc-curves to be Frenet curves in $\Lambda^3 \subset \mathbb{R}^4_1$. Then:*
conformal width $w_c(t) = const.$ along $X_1(t)$, $X_2(t)$ is equivalent to $d\tau_1(t) = d\tau_2(t)$ along $X_1(t)$, $X_2(t)$.

In order to prove the theorem we first look at the Frenet theory for *osc*-curves in $\Lambda^3 \subset \mathbb{R}^4_1$. We know (see [12]) that the *osc*-curve is light-like, that is have at each point a light-like tangent vector.

Let us now take a Frenet curve $osc(\tau)$, $\tau \in I$, in Λ^3, that means $\dim \mathrm{span}(osc(\tau), \dot{osc}(\tau), \ddot{osc}(\tau)) = 3$, $\tau \in I$, ($\cdot$ denotes the derivation with respect to the parameter τ on the *osc*-curve). Then at each point $\dot{osc}$ is a non vanishing light-like vector. And $\mathrm{span}(\dot{osc})$ defines a curve X in S^2 which is the envelope of the family $osc(\tau)$. The vector $\ddot{osc}$, tangent to the light cone, is space-like. Because the *osc*-curve is a Frenet curve the vector field $\ddot{osc}$ nowhere vanishes. Let τ be the "natural" parameter on the *osc*-curve defined by $\langle \ddot{osc}(\tau), \ddot{osc}(\tau)\rangle = 1$.

We use the Frenet frame $f_0 f_1 f_2 f_3$ in $\mathbb{R}^4_1$ adapted to *osc*: The first vector in $\mathbb{R}^4_1$ is the point $f_0 = osc \in \Lambda^3$. The remaining vectors are $f_1 = \ddot{osc}$, $f_2 = \dddot{osc}/||\dddot{osc}||$ and f_3 orthogonal to $\mathrm{span}(f_0, f_1, f_2)$. As f_0 and f_1 are space-like the plane $\mathrm{span}(f_2, f_3)$ is of mixed type.

Lemma 2. *The light-like vector tangent to the osc-curve satisfies:*

$$\dot{osc} = \dot{f_0} = \lambda(f_2 + f_3).$$

Proof. $\langle f_0, f_0\rangle = 1$, hence $\langle f_0, \dot{f_0}\rangle = 0$. $\langle \dot{f_0}, \dot{f_0}\rangle = 0$, hence $0 = \langle \dot{f_0}, \ddot{f_0}\rangle = \langle \dot{f_0}, f_1\rangle$. Therefore $\dot{f_0} \in \mathrm{span}(f_2, f_3)$. Because $\dot{osc}$ light-like we get $\dot{f_0} = \lambda(f_2 + f_3)$. $\square$

The Frenet formulas write:

$$\begin{pmatrix} \dot{f_0} \\ \dot{f_1} \\ \dot{f_2} \\ \dot{f_3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda & \lambda \\ 0 & 0 & \xi & 0 \\ -\epsilon_2\lambda & -\epsilon_2\xi & 0 & \mu \\ -\epsilon_3\lambda & 0 & \mu & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{7}$$

with $\epsilon_2 = \langle f_2, f_2\rangle$ and $\epsilon_3 = \langle f_3, f_3\rangle$.

Lemma 3. *As the osc-curve is light-like the coefficients of the matrix satisfy two extra relations:*

$$\dot{\lambda} + \lambda\mu = 0,$$
$$-\epsilon_2\lambda\xi = 1.$$

Proof. We have $osc = f_0$, $\ddot{osc} = \ddot{f_0} = f_1$. Taking into account the Frenet formulas (7) we get

$$f_1 = \ddot{f_0} = \dot{\lambda}(f_2 + f_3) + \lambda(\dot{f_2} + \dot{f_3}) =$$
$$= \lambda^2(-\epsilon_2 - \epsilon_3)f_0 - \epsilon_2\lambda\xi f_1 + (\dot{\lambda} + \lambda\mu)(f_2 + f_3). \qquad \square$$

As our *osc*-curve is a Frenet curve the vector field $\dddot{osc}$ is nowhere vanishing. Because $f_1 = \ddot{osc}$ the Frenet formulas (7) imply that ξ is nowhere vanishing. Therefore free choice of a nowhere vanishing ξ implies $\lambda = -\epsilon_2 \frac{1}{\xi}$ and $\mu = -\frac{\dot{\lambda}}{\lambda}$.

Example 5. *osc*-curves with $\xi = \text{const.} \neq 0$ satisfy:

$$\lambda = -\frac{\epsilon_2}{\xi} \quad , \quad \mu = 0.$$

The matrix in the Frenet formulas (7) is constant. As element of the Lie algebra of the Lorentz group it determines a 1-parameter subgroup. The orbit of it through the point $f_0 \in \Lambda^3$ is the corresponding *osc*-curve. Therefore the envelope curve in S^2 has constant conformal curvature.

Lemma 4. *The natural parameter τ on the osc-curve in Λ^3 is equal to the conformal arc length on the point curve $X = \operatorname{span}(\dot{osc})$ in S^2.*

Proof. We use the formula

$$ds_c = \sqrt{\left|\frac{d\kappa_e}{ds_e}(s_e)\right|}\, ds_e, \tag{8}$$

writing the conformal arc length s_c in terms of the euclidean arc length s_e and the euclidean geodesic curvature κ_e (cf. [4], [16], [3]).

We consider a point, arbitrary but fixed, say at $\tau = 0$. And we take the slice $\{x \in \mathbb{R}^4_1 | \langle x, f_l(0)\rangle = -1\} \cap \mathcal{C}$ as euclidean model for S^2, where $l = 2$ or 3 chosen such that $f_l(0)$ is time-like.

Then the envelope of *osc* is

$$X = \operatorname{span}(\dot{osc}) \cap S^2 = \rho\, \dot{osc} = \rho\lambda(f_2 + f_3)$$

with $\rho(0) = 1/\lambda(0)$. Hence

$$\dot{X} = \dot{\rho}\, \dot{osc} + \rho\, \ddot{osc},$$

and, as $\ddot{osc} = f_1$,

$$\dot{X}(0) = \dot{\rho}(0)\lambda(0)(f_2 + f_3)(0) + \frac{1}{\lambda(0)} f_1(0).$$

Because $\dot{X}(0)$ is tangent to S^2 we have $\dot{\rho}(0) = 0$, and therefore

$$ds_e = \frac{1}{|\lambda|}\, d\tau \quad \text{at} \quad \tau = 0. \tag{9}$$

The relation between circles C of radius r in the euclidean model sphere S^2 and the representing point c in Λ^3 is given by the radius function

$$r = \arcsin \frac{1}{\cosh t}, \tag{10}$$

where t is the geodesic distance in Λ^3 between c and the equator $\{x \in \mathbb{R}^4_1 | \langle x, f_l(0)\rangle = 0\} \cap \Lambda^3$. We leave as an exercise in differential geometry for the reader to check that

$$\sinh t = |\langle f_l(0), c\rangle|. \tag{11}$$

The euclidean geodesic curvature of a circle of radius r in S^2 is $\kappa_e = \cot r$. Hence

$$\begin{aligned}
\frac{d\kappa_e}{ds_e} &= -\frac{1}{\sin^2 r}\frac{dr}{ds_e} \\
&= -\frac{1}{\sin^2 r}\frac{1}{\sqrt{1-\frac{1}{\cosh^2 t}}}\frac{d}{ds_e}\left(\frac{1}{\cosh t}\right) \\
&= -\frac{\cosh^3 t}{\sinh t}\frac{d}{ds_e}\left(\frac{1}{\cosh t}\right) \\
&= -\frac{\cosh^3 t}{\sinh t}\,|\lambda|\,\frac{d}{d\tau}\left(\frac{1}{\cosh t}\right) \\
&= |\lambda|\,\cosh t\,\frac{dt}{d\tau}.
\end{aligned}$$

Now, note (11), the geodesic distance of osc from the equator of Λ^3 writes

$$\sinh(t(\tau)) = |\langle f_l(0), osc(\tau)\rangle|.$$

Then through differentiation we get

$$\frac{dt}{d\tau}\cosh(t(\tau)) = |\langle f_l(0), \dot{osc}(\tau)\rangle|.$$

By the Frenet formulas (7) we have $\dot{osc}(\tau) = \dot{f}_0(\tau) = \lambda(\tau)(f_2(\tau) + f_3(\tau))$. At $\tau = 0$ we have $t(0) = 0$ and we get

$$dt/d\tau = |\lambda|.$$

Therefore

$$\frac{d\kappa_e}{ds_e} = \lambda^2 \quad \text{at} \quad \tau = 0. \tag{12}$$

Finally, (8), (9) and (12) yield $ds_c = d\tau$. □

Remark. R. Langevin and J. O'Hara showed that the conformal arc length of an arc can be obtained from a light-like curve in the space of circles, the curve given by the osculating circles (see [13]).

The proof of Theorem 2 now runs as follows:
Let us write the conformal arc length τ (the natural parameter on the osc-curve) using Lemma 4 in terms of the σ-curve. We start from formula (4), i.e.

$$osc_{1,2} = \epsilon_2 e_0 + \frac{1}{\kappa_g}(e_2 \pm e_3).$$

Using the Frenet formulas (3) (the derivative, denoted by $'$, is with respect to the arc length parameter s on the σ-curve) we get

$$osc' = \left(-\frac{\kappa_g'}{\kappa_g^2} \pm \frac{\tau_g}{\kappa_g}\right)(e_2 \pm e_3),$$

and further

$$osc'' = -\epsilon_2 \left(-\frac{\kappa_g'}{\kappa_g} \pm \tau_g \right) e_1 + (\ldots)(e_2 \pm e_3).$$

Because $e_2 \pm e_3$ are light-like and orthogonal to e_1 we have

$$|osc''_{1,2}| = | - \frac{\kappa_g'}{\kappa_g} \pm \tau_g|. \tag{13}$$

The curves X_1 and X_2 are supposed to have no vertices, that means the osculating circles and the curves never have contact of order three. We claim that then τ_g is nowhere vanishing. To this: The osculating circles $osc_{1,2}$ have second order contact with the curves $X_{1,2}$, that means $\langle osc_{1,2}, X_{1,2}\rangle = 0$, $\langle osc_{1,2}, X'_{1,2}\rangle = 0$ and $\langle osc_{1,2}, X''_{1,2}\rangle = 0$ along $X_{1,2}$ (here $X_{1,2}$ a vector representing span$(osc'_{1,2}) = X_{1,2}$). Third order contact between $osc_{1,2}$ and $X_{1,2}$ at a point means in addition $\langle osc_{1,2}, X'''_{1,2}\rangle = 0$ at the point under consideration. Using the second order contact conditions along $X_{1,2}$ and their derivatives we see that the third order condition is equivalent to $\langle X_{1,2}, osc_{1,2}\rangle = 0$, $\langle X_{1,2}, osc'_{1,2}\rangle = 0$, $\langle X_{1,2}, osc''_{1,2}\rangle = 0$ and $\langle X_{1,2}, osc'''_{1,2}\rangle = 0$ at the point under consideration (cf. [14]). In terms of our Frenet frame for the σ-curve we have $X_{1,2} = \mathrm{span}(e_2 \pm e_3)$, $osc_{1,2} = \epsilon_2 e_0 + \frac{1}{\kappa_g}(e_2 \pm e_3)$ (cf. (4)). Taking into account the Frenet formulas (3) for the σ-curve we get that third order contact between $osc_{1,2}$ and $X_{1,2}$ is equivalent to $\tau_g = 0$ at the point under consideration. This proves the claim.

Therefore formula (13) implies

$$|osc''_1| = |osc''_2| \quad \iff \quad \kappa_g' = 0. \tag{14}$$

Finally using (14), $d\tau(s) = |osc''|\, ds$, together with Theorem 1 we get a proof of Theorem 2. □

Remark. In some way Proposition 1 is a limit case of Theorem 2: If X_1 is part of a circle, then its conformal arc length is identically zero. Therefore, in case of constant conformal width, by Theorem 2 the conformal arc length of X_2 must be identically zero, hence X_2 is part of a circle.

For the sake of completeness we give an interpretation of the coefficient ξ in the Frenet matrix (7) of the *osc*-curve as the conformal curvature of the envelope curve X.

Lemma 5. *Let κ_c denote the conformal (or inversive) curvature of the point curve $X = span(\dot{osc})$ in S^2. Then*

$$\kappa_c = -\frac{1}{2}\, \epsilon_2\, \xi^2.$$

Proof. A normalized representation of the point curve X in $\mathbb{R}^4_1$, using again our Frenet frame associated to the *osc*-curve, is given by

$$X = \frac{1}{\xi}(f_2 + f_3)\ ,$$

with $\langle \dot{X}, \dot{X} \rangle = 1$.

The natural parameter on the *osc*-curve is equal to the conformal arc length on X (Lemma 4). Therefore, taking into account the Frenet formulas (7) and [2], [3, §22], [19],

$$\kappa_c = -\frac{1}{2}\langle \ddot{X}, \ddot{X} \rangle = -\frac{1}{2}\,\epsilon_2\,\xi^2.$$

(We do not give here a self-contained proof as the arguments in [2], [3, §22], [19] are lengthy and technical.) □

Proposition 4. *Let $X_1(t), X_2(t), t \in I$, be a pair of curves in S^2 related to one another through double normal circles and with constant conformal width. Suppose the osc-curves of X_1 and X_2 to be Frenet curves in $\Lambda^3 \subset \mathbb{R}^4_1$. Then*

$$|\kappa_{c,1} - \kappa_{c,2}| = 2\left|\frac{\tau_g'}{\tau_g}\right|, \tag{15}$$

where $\kappa_{c,1}, \kappa_{c,2}$ are the conformal curvatures of the curves X_1, X_2 in S^2 (the derivative with respect to the arc length parameter s on the σ-curve is denoted $'$).

Proof. Lemma 5 and the Frenet formulas (7) for the *osc*-curve give

$$\begin{aligned}
\kappa_c &= -\frac{1}{2}\,\epsilon_2\,\xi^2 \\
&= -\frac{1}{2}\,\epsilon_2\,\langle \dot{f}_1, \dot{f}_1 \rangle \\
&= -\frac{1}{2}\,\epsilon_2\,\langle \dddot{osc}, \dddot{osc} \rangle
\end{aligned} \tag{16}$$

(again $\cdot$ is the derivation with respect to the natural parameter τ on the *osc*-curve). Formula (4) gives $osc \in \Lambda^3$ in terms of the σ-curve:

$$osc_{1,2} = \epsilon_2 e_0 + \frac{1}{\kappa_g}(e_2 \pm e_3).$$

We want to compute the third derivative involved in formula (16) using the s parameter on the σ-curve.

We claim:

$$d\tau = \langle osc'', osc'' \rangle^{1/4}\, ds.$$

By definition of the natural parameter on an *osc*-curve we have $1 = \langle \ddot{osc}, \ddot{osc} \rangle$. Changing from τ to s we get

$$\ddot{osc} = \frac{d^2 s}{d\tau^2} osc' + \left(\frac{ds}{d\tau}\right)^2 osc''.$$

Because the *osc*-curves are light-like we have $\langle osc', osc' \rangle = 0$. Hence through differentiation we have $\langle osc', osc'' \rangle = 0$. Therefore

$$1 = \langle \ddot{osc}, \ddot{osc} \rangle = \left(\frac{ds}{d\tau}\right)^4 \langle osc'', osc'' \rangle$$

which proves the claim.

Finally, we put (4) into (16). The computations of the third derivatives $\dddot{osc}_{1,2}$ in terms of σ and derivatives with respect to s are straightforward but ugly and lengthy, so we omit them. They take into account the relation $d/ds = (d\tau/ds)\, d/d\tau$ and the Frenet formulas (3) for the σ-curve. This way we get (15). □

Proposition 4 and Example 3 immediately imply

Proposition 5. *Let $X_1(t), X_2(t), t \in I$, be a pair of curves in S^2 with constant conformal width. Suppose the osc-curves of X_1 and X_2 to be Frenet curves in $\Lambda^3 \subset \mathbb{R}^4_1$. Then*

$$\kappa_{c,1} = \kappa_{c,2} \quad \Longleftrightarrow \quad \tau_g = \text{const.}$$

Moreover $\kappa_{c,1} = \kappa_{c,2} = const.$ in this case.

Remark. We considered pairs of curves related to one another through double normal circles and with constant conformal width. The focus here was on local properties. From the global point of view one-dimensional Dupin cyclides are closed examples of such pairs. They have two components. One-component examples are circles. Circles are closed curves consisting of pairs of points ("antipodal" points) related through double normal circles (in fact in many ways) and with constant conformal width.

Question: Are there other two-component examples of pairs of closed curves of constant conformal width, or other one-component examples of closed curves of constant conformal width respectively?

Acknowledgements. The authors thank Gil Solanes for useful conversations and Gabi Preissler for creating most of the pictures.

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Received November 20, 2007