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Cardinality Estimates for Piecewise Congruences of Convex Polygons

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Abstract. Two convex polygons $P, P' \subseteq \mathbb{R}^2$ are congruent by dissection with respect to a given group G of transformations of $\mathbb{R}^2$ if both can be dissected into the same finite number k of polygonal pieces $Q_1, \dots, Q_k$ and $Q'_1, \dots, Q'_k$ such that corresponding pieces Q_i, Q'_i are congruent with respect to G, $1 \le i \le k$. In that case $\deg_G(P, P')$ denotes the smallest k with the above property.

For the group Isom^+ of proper Euclidean isometries we prove two general upper estimates for $\deg_{\mathrm{Isom}^+}(P, P')$, the first one in terms of the numbers of vertices and the diameters of P, P', the second one depending moreover on the radii of inscribed circles. A particular result concerns regular polygons P, P'.

For the groups Sim^+ and Sim of proper and general similarities we give upper bounds for $\deg_{\mathrm{Sim}^+}(P, P')$ and $\deg_{\mathrm{Sim}}(P, P')$ in terms of the numbers of vertices.

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1. Introduction

Given a group G of affine transformations of the Euclidean plane $\mathbb{R}^2$, two polygons $P, P' \subseteq \mathbb{R}^2$ are called *congruent by dissection* (or *equidissectable*) *with respect to* G if there exist a number $k \in \{1, 2, \ldots\}$ and dissections of P into polygons $Q_1, \ldots, Q_k$ and of P' into polygons $Q'_1, \ldots, Q'_k$ such that, for every $i \in \{1, \ldots, k\}$, Q_i and Q'_i are congruent with respect to G. Here a *polygon* is meant to be a union of finitely many triangles. We say that P is *dissected* into $Q_1, \ldots, Q_k$ if $P = Q_1 \cup \cdots \cup Q_k$ and the interiors of distinct pieces Q_i, Q_j, $i \neq j$, are disjoint. If P and P' are equidissectable, the minimal number k admitting dissections with the above property is called the *degree* of the congruence by dissection of P and P'. This optimal number of pieces is denoted by $\deg_G(P, P')$.

Of course, if P and P' are congruent by dissection with respect to a subgroup H of G, then P and P' are equidissectable with respect to G, too, and

$$\deg_G(P, P') \leq \deg_H(P, P').$$

The present paper is mainly devoted to degree estimates for congruences by dissection of convex polygons with respect to the group Isom of Euclidean isometries and the subgroup Isom^+ of proper isometries.

The classical Wallace-Bolyai-Gerwien theorem says that any two polygons of the same area are equidissectable with respect to Isom (see [4] and [10, Chapter 3] for historical remarks). The group containing all translations and all central reflections is known to be the smallest subgroup of Isom satisfying the above property (see [5, 1]). However, the question for the degree of congruences by dissection is rather open. The following theorem by Hertel seems to give the first upper estimate for $\deg_{\mathrm{Isom}}(P, P')$ concerning general polygons of equal area.

Theorem 1. ([6], Satz 2) *Let P_m and P'_n be an m-gon and an n-gon of the same area whose diameters are d and d', respectively. Suppose that there exist dissections of P_m into $m-2$ triangles $T_1, \ldots, T_{m-2}$ and of P'_n into $n-2$ triangles $T'_1, \ldots, T'_{n-2}$ and define*

$$c = \min\{\mathrm{diam}(T_1), \ldots, \mathrm{diam}(T_{m-2}), \mathrm{diam}(T'_1), \ldots, \mathrm{diam}(T'_{n-2})\}.$$

Then

$$\deg_{\mathrm{Isom}}(P_m, P'_n) \leq 4(m-2)(n-2)\left(\tfrac{\max\{d,d'\}}{c} + 2\right)^2.$$

Our first goal in Section 2 is an upper estimate for $\deg_{\mathrm{Isom}}(P_m, P'_n)$ for arbitrary convex P_m and P'_n only depending on m, n, d, and d'. We shall see that Theorem 1 gives a bound of that kind depending on m and n like a polynomial of degree 4 and of quadratic behaviour in $\max\left\{\frac{d}{d'}, \frac{d'}{d}\right\}$. Then we establish a stronger estimate for for $\deg_{\mathrm{Isom}^+}(P_m, P'_n)$ quadratic in m, n and linear in $\max\left\{\frac{d}{d'}, \frac{d'}{d}\right\}$ (see Theorem 2). An important technical step to this main result concerns the piecewise congruence of triangles (see Lemma 3).

The estimate of Theorem 2 can be improved if P_m and P'_n are known to contain inscribed circles of sufficiently large radii (see Theorem 3). In particular,

$\deg_{\mathrm{Isom}^+}(P_m^r, P_n^r) \leq 7(m+n-1)$ for regular polygons of the same area with m and n vertices, respectively (see Theorem 4). This improves the bound

$$\deg_{\mathrm{Isom}}(P_m^r, P_n^r) \leq (2m+4)(n+1) \quad \text{for} \quad 3 \leq m < n \tag{1}$$

given by Doyen and Landuyt without proof (see [2]).

Nevertheless, our estimates do not use to be sharp in particular situations. For example, isometric congruences by dissection of regular P_m^r and P_n^r with very small m, n, say $m, n \leq 12$, are known to require much less than $7(m+n-1)$ pieces. We refer to Theobald's frequently updated web page [9].

Let us point out that the estimates for $\deg_{\mathrm{Isom}^+}(P_m, P_n')$ from Theorems 2, 3, and 4 can be realized by dissections into convex pieces.

In the last section we shall summarize new degree estimates for congruences by dissection with respect to similarities. The paper is closed with a remark about the group of translations.

We use the following notations: Given three points $x_1, x_2, x_3 \in \mathbb{R}^2$, the symbols $l(x_1, x_2)$, x_1x_2, $|x_1x_2| \big(= d(x_1, x_2)\big)$, $\angle x_1x_2x_3$, and $|\angle x_1x_2x_3|$ stand for the straight line passing through x_1, x_2, the line segment between x_1, x_2, the length of that segment (which is the Euclidean distance of x_1, x_2), the angle determined by x_1, x_2, x_3, and the size of that angle, respectively. $\mathrm{int}(A)$, $\mathrm{bd}(A)$, $\mathrm{conv}(A)$, $\mathrm{diam}(A) = \sup\{d(x_1, x_2) : x_1, x_2 \in A\}$, and $\lambda(A)$ denote the interior, the boundary, the convex hull, the diameter, and the area measure, respectively, of a (measurable) subset $A \subseteq \mathbb{R}^2$. $\lfloor \xi \rfloor$ and $\lceil \xi \rceil$ are the largest lower and the smallest upper integer bound of $\xi \in \mathbb{R}$.

2. Congruence by dissection with respect to isometries

2.1. An estimate in terms of vertex numbers and diameters based on Theorem 1

Corollary. *Let P_m and P_n' be convex polygons of the same area whose numbers of vertices are m and n and whose diameters are d and d', respectively. Then*

$$\deg_{\mathrm{Isom}}(P_m, P_n') \leq 4(m-2)(n-2)\left(\max\{d, d'\}\max\left\{\tfrac{\lfloor \frac{m}{2} \rfloor}{d}, \tfrac{\lfloor \frac{n}{2} \rfloor}{d'}\right\} + 2\right)^2.$$

In particular

$$\deg_{\mathrm{Isom}}(P_m, P_n') < mn(m+n)^2\left(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\right)^2.$$

For proving this corollary, we need a lower bound for the value c from Theorem 1.

Lemma 1. *Every convex m-gon P_m admits a dissection into $m-2$ triangles $T_1, \ldots, T_{m-2}$ such that*

$$\min\{\mathrm{diam}(T_1), \ldots, \mathrm{diam}(T_{m-2})\} \geq \tfrac{1}{\lfloor \frac{m}{2} \rfloor}\mathrm{diam}(P_m).$$

One can obtain a strict inequality if $m \neq 3$.

Proof. We proceed by induction. The case $m = 3$ is trivial.

In the case $m = 4$ let P_4 have the vertices x_1, x_2, x_3, x_4. Let $d_1 = d(x_1, x_3)$ and $d_2 = d(x_2, x_4)$ be the lengths of the diagonals, say $d_1 \geq d_2$. The convexity of P_4 and the triangle inequality yield

$$\operatorname{diam}(P_4) = \max\nolimits_{1 \leq i < j \leq 4} d(x_i, x_j) < d_1 + d_2.$$

Cutting P_4 along x_1x_3 gives a dissection into two triangles each having a diameter of at least

$$d(x_1, x_3) = d_1 \geq \tfrac{d_1+d_2}{2} > \tfrac{\operatorname{diam}(P_4)}{2} = \tfrac{1}{\lfloor \frac{4}{2} \rfloor} \operatorname{diam}(P_4).$$

Now let $m \geq 5$. Let $x_1, \ldots, x_m$ be the vertices of P_m in their order along the boundary of P_m.

Case 1. $\operatorname{diam}(P_m)$ is the length of a diagonal, say of x_1x_k, $k \in \{3, \ldots, m-1\}$.

Then x_1x_k dissects P_m into a k-gon $P_{k,1} = \operatorname{conv}\{x_1, \ldots, x_k\}$ and an $(m-k+2)$-gon $P_{m-k+2,2} = \operatorname{conv}\{x_k, x_{k+1}, \ldots, x_m, x_1\}$, both with diameter $d(x_1, x_k) = \operatorname{diam}(P_m)$. Application of the induction hypothesis to $P_{k,1}$ and $P_{m-k+2,2}$ yields a dissection of P_m with the required properties.

Case 2. $\operatorname{diam}(P_m)$ is the length of an edge of P_m, say of x_1x_m.

Case 2.1. $d(x_1, x_3) > \frac{1}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m)$.

We cut P_m along x_1x_3 into the triangle $T = \triangle x_1x_2x_3$ and the $(m-1)$-gon $P_{m-1} = \operatorname{conv}\{x_3, x_4, \ldots, x_m, x_1\}$ of diameter $d(x_1, x_m) = \operatorname{diam}(P_m)$. Then $\operatorname{diam}(T) \geq d(x_1, x_3) > \frac{1}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m)$. Dissection of P_{m-1} according to the induction hypothesis gives $m-3$ additional triangles of sufficiently large diameters.

Case 2.2. $d(x_1, x_3) \leq \frac{1}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m)$.

Then

$$d(x_3, x_m) > d(x_1, x_m) - d(x_1, x_3) \geq \big(1 - \tfrac{1}{\lfloor \frac{m}{2} \rfloor}\big) \operatorname{diam}(P_m) = \tfrac{\lfloor \frac{m-2}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m).$$

We split P_m along x_3x_m into the quadrilateral $P_{4,1} = \operatorname{conv}\{x_1, x_2, x_3, x_m\}$ with $\operatorname{diam}(P_{4,1}) = d(x_1, x_m) = \operatorname{diam}(P_m)$ and into $P_{m-2,2} = \operatorname{conv}\{x_3, \ldots, x_m\}$ with $\operatorname{diam}(P_{m-2,2}) \geq d(x_3, x_m) > \frac{\lfloor \frac{m-2}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m)$. The induction hypothesis gives dissections of $P_{4,1}$ and of $P_{m-2,2}$ into two and into $m-4$ triangles, respectively. The two pieces T of $P_{4,1}$ satisfy

$$\operatorname{diam}(T) > \tfrac{1}{2} \operatorname{diam}(P_{4,1}) = \tfrac{1}{2} \operatorname{diam}(P_m) \geq \tfrac{1}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m).$$

The $m-2$ pieces U of $P_{m-2,2}$ have diameters

$$\operatorname{diam}(U) \geq \tfrac{1}{\lfloor \frac{m-2}{2} \rfloor} \operatorname{diam}(P_{m-2,2}) > \tfrac{1}{\lfloor \frac{m-2}{2} \rfloor} \tfrac{\lfloor \frac{m-2}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m) = \tfrac{1}{\lfloor \frac{m}{2} \rfloor} \operatorname{diam}(P_m).$$

This completes the proof. □

Proof of the Corollary. We apply Theorem 1 to P_m and P'_n. According to Lemma 1 there exist dissections of P_m and P'_n such that the value c from the theorem satisfies

$$c \geq \min\big\{\tfrac{d}{\lfloor \frac{m}{2} \rfloor}, \tfrac{d'}{\lfloor \frac{n}{2} \rfloor}\big\}, \quad \text{that is} \quad \tfrac{1}{c} \leq \max\big\{\tfrac{\lfloor \frac{m}{2} \rfloor}{d}, \tfrac{\lfloor \frac{n}{2} \rfloor}{d'}\big\}. \tag{2}$$

Combining this with the statement of the theorem we obtain the first estimate of the corollary. The second one is proved by

$$\begin{aligned}
\deg_{\mathrm{Isom}}(P_m, P_n') &\leq 4(m-2)(n-2)\big(\max\{d,d'\}\max\big\{\tfrac{\frac{m}{2}}{d},\tfrac{\frac{n}{2}}{d'}\big\}+2\big)^2\\
&\leq 4(m-2)(n-2)\big(\max\big\{\tfrac{d}{d'},\tfrac{d'}{d}\big\}\max\big\{\tfrac{m}{2},\tfrac{n}{2}\big\}+2\big)^2\\
&= (m-2)(n-2)\big(\max\big\{\tfrac{d}{d'},\tfrac{d'}{d}\big\}\max\{m,n\}+4\big)^2\\
&\leq (m-2)(n-2)\big(\max\{m,n\}+4\big)^2\big(\max\big\{\tfrac{d}{d'},\tfrac{d'}{d}\big\}\big)^2\\
&\leq (m-2)(n-2)\big((m+n-3)+4\big)^2\big(\max\big\{\tfrac{d}{d'},\tfrac{d'}{d}\big\}\big)^2\\
&= (m-2)(m+n+1)(n-2)(m+n+1)\big(\max\big\{\tfrac{d}{d'},\tfrac{d'}{d}\big\}\big)^2\\
&< m(m+n)n(m+n)\big(\max\big\{\tfrac{d}{d'},\tfrac{d'}{d}\big\}\big)^2. \quad \square
\end{aligned}$$

We close this subsection by a claim showing that Lemma 1 is sharp. In this sense the estimate (2) is best possible. This justifies the formulation of the corollary.

Lemma 2. *For every $m \in \{3,4,5,\dots\}$ and every $\varepsilon > 0$, there exists a convex m-gon P_m such that every dissection of P_m into $m-2$ triangles $T_1,\dots,T_{m-2}$ satisfies*

$$\min\{\mathrm{diam}(T_1),\dots,\mathrm{diam}(T_{m-2})\} < \big(\tfrac{1}{\lfloor \frac{m}{2}\rfloor}+\varepsilon\big)\,\mathrm{diam}(P_m).$$

Sketch of the proof. We assume $m \geq 4$, because the case $m = 3$ is trivial. Let $\delta > 0$ be small. We define $P_m = \mathrm{conv}\{x_1,\dots,x_m\}$ where

$$x_i = \begin{cases} \big(\cos\big((i-1)\delta+\delta^2\big), \sin\big((i-1)\delta+\delta^2\big)\big) & \text{if } i \text{ is odd},\\ \big(\cos(i\delta), \sin(i\delta)\big) & \text{if } i \text{ is even}.\end{cases}$$

Then $\mathrm{diam}(P_m) = d(x_1, x_m)$ (if δ is sufficiently small).

Suppose that P_m is dissected into $m-2$ triangles $T_1,\dots,T_{m-2}$. It can be shown by Euler's formula that all vertices of the triangles are vertices of P_m, too. Any such triangulation contains two different triangles each of them sharing two edges with the boundary of P_m. Only one of them can contain the long edge x_1x_m. Hence at least one of them is of the form $T = \triangle x_ix_{i+1}x_{i+2}$ with $1 \leq i \leq m-2$. Consequently, $\mathrm{diam}(T) = d(x_i, x_{i+2}) = d(x_1, x_3) = d(x_2, x_4)$. We obtain

$$\tfrac{\mathrm{diam}(T)}{\mathrm{diam}(P_m)} = \tfrac{d(x_1,x_3)}{d(x_1,x_m)} \quad \text{and} \quad \lim_{\delta\downarrow 0}\tfrac{d(x_1,x_3)}{d(x_1,x_m)} = \lim_{\delta\downarrow 0}\tfrac{2\delta}{\lfloor\frac{m}{2}\rfloor 2\delta} = \tfrac{1}{\lfloor\frac{m}{2}\rfloor}.$$

Thus P_m satisfies the claim of Lemma 2 if δ is sufficiently small. $\square$

2.2. Congruence by dissection of triangles

Lemma 3. *Let T and T' be triangles of the same area having the diameters d and d', respectively. Then*

$$\deg_{\mathrm{Isom}^+}(T, T') \leq 4\left\lceil \tfrac{1}{2}\max\big\{\tfrac{d}{d'},\tfrac{d'}{d}\big\}\right\rceil + 3.$$

Proof. We use the method of crossposing triangle strips (see [3, Chapter 12]).

Let $T = \triangle x_0x_1\hat{x}_0$ with $d = d(x_0, x_1)$. c denotes the centre of $x_0\hat{x}_0$. Let τ be the translation mapping x_0 onto x_1 and let σ be the central reflection with respect to c. We define $x_i = \tau^i(x_0)$, $T_i = \tau^i(T)$, $\hat{x}_i = \sigma(x_i)$, and $\hat{T}_i = \sigma(T_i)$ for $i \in \mathbb{Z}$. Then the triangles $T_i, \hat{T}_i$, $i \in \mathbb{Z}$, form an infinite dissection of a strip Σ bounded by $l = l(x_0, x_1)$ and $\hat{l} = l(\hat{x}_0, \hat{x}_1)$ (see Figure 1). The sizes of the inner angles

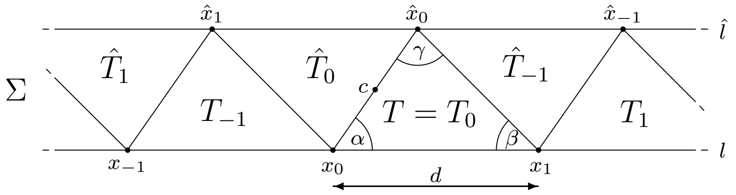

Figure 1. The triangle strip Σ

of T at $x_0, x_1, \hat{x}_0$ are denoted by α, β, γ, respectively. Since $d(x_0, x_1) = d$ is the diameter of T, the width of Σ is at most $\frac{\sqrt{3}}{2}\,d$. Based on T', we introduce a strip Σ' and respective terms c', τ', σ', x_i', T_i', $\hat{x}_i'$, $\hat{T}_i'$, l', $\hat{l}'$, α', β', γ' analogously. We assume $T = \triangle x_0x_1\hat{x}_0$ and $T' = \triangle x_0'x_1'\hat{x}_0'$ to be oriented in the same way.

Without loss of generality, $d \le d'$. Then $\alpha' \le \alpha$ or $\beta' \le \beta$, because T and T' have the same area. Again without loss of generality, $\alpha' \le \alpha$.

We suppose that $c' = c$ (which can be obtained by translating T'). Finally, we assume that the intersection $\Sigma \cap \Sigma'$ is a parallelogram P, whose vertices p_1, p_2, p_3, p_4 represent the intersections $l \cap l'$, $l \cap \hat{l}'$, $\hat{l} \cap \hat{l}'$, $\hat{l} \cap l'$, respectively, such that $d(p_2, p_3) = d'$ and $\delta = |\angle p_1p_2p_3| \le \frac{\pi}{3}$ (see Figure 2). In fact, this situation can be obtained by suitably rotating Σ' around c, because the width of Σ does not exceed $\frac{\sqrt{3}}{2}d \le \frac{\sqrt{3}}{2}d'$.

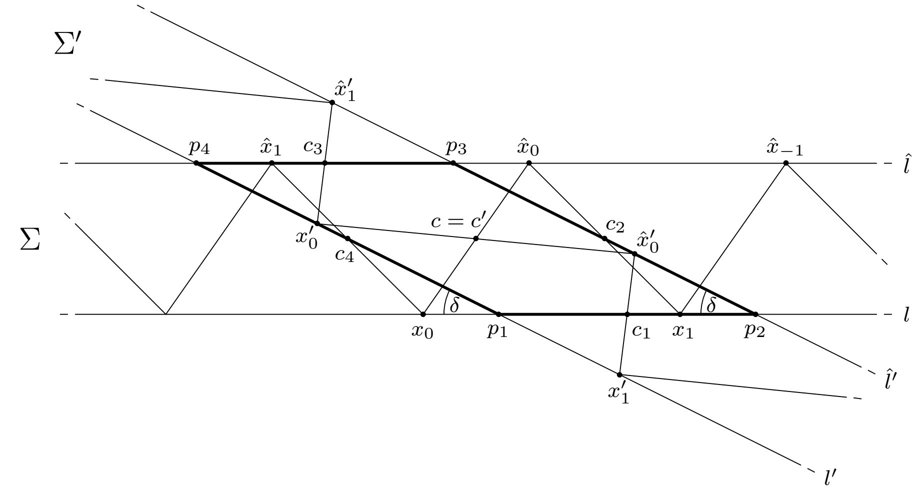

Figure 2. Crossposing the strips Σ and Σ'

The midpoints c_1 and c_3 of p_1p_2 and p_3p_4 satisfy $d(c, c_1) = d(c, c_3) = \frac{d'}{2}$ and hence agree with the centres of $x_1'\hat{x}_0'$ and $x_0'\hat{x}_1'$, respectively. The area of P is twice that of T' and so twice that of T, too. This shows that $d(p_1, p_2) = d$ and the midpoints c_2 and c_4 of p_2p_3 and p_1p_4 coincide with those of $x_1\hat{x}_0$ and $x_0\hat{x}_1$, respectively.

The edges of the triangles $T_i, \hat{T}_i, T'_i, \hat{T}'_i$ dissect P into finitely many polygons that appear in k pairs symmetric with respect to c. Each pair consists of one member contained in $\bigcup_{i\in\mathbb{Z}} T_i$ and one member covered by $\bigcup_{i\in\mathbb{Z}} \hat{T}_i$. Images of the k first members under suitable integer powers of τ form a dissection of T (see Figure 3). Similarly, we find one element in every pair of symmetric pieces of P

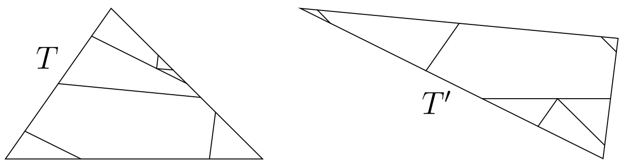

Figure 3. Dissections of T and T'

such that images of these k elements under suitable integer powers of τ' constitute a dissection of T'. This shows that $\deg_{\mathrm{Isom}^+}(T, T') \le k$.

It remains to establish an upper bound for k. We prepare this by proving

$$\alpha + \beta' + \delta \le \pi. \tag{3}$$

Among all triangles $\triangle x_0x_1y$ with $y \in \hat{l}$ and of diameter $d(x_0, x_1) = d$ the isosceles one with $d(x_1, y) = d(x_0, x_1) = d$ and $|\angle x_1x_0y| = |\angle x_0yx_1| = \alpha_0 \ge \frac{\pi}{3}$ maximizes the size of the inner angle at x_0, in particular $\alpha \le \alpha_0$. Its area coincides with that of T and can be computed by $\frac{1}{2}d^2 \sin(|\angle x_0x_1y|) = \frac{1}{2}d^2 \sin(\pi - 2\alpha_0) = \frac{1}{2}d^2 \sin 2\alpha_0$. So

$$\alpha \le \alpha_0, \quad \text{where} \quad \tfrac{\pi}{3} \le \alpha_0 < \tfrac{\pi}{2} \quad \text{and} \quad d^2 \sin 2\alpha_0 = 2\lambda(T) = \lambda(P) = dd' \sin\delta.$$

Similarly,

$$\beta' \le \beta'_0, \quad \text{where} \quad \tfrac{\pi}{3} \le \beta'_0 < \tfrac{\pi}{2} \quad \text{and} \quad d'^2 \sin 2\beta'_0 = dd' \sin\delta.$$

These admit the estimate

$$\begin{aligned} dd' \sin(\pi - \delta) &= \tfrac{1}{2} 2dd' \sin\delta \\ &\le \tfrac{1}{2}\Big(\tfrac{d\cos\alpha_0}{d'\cos\beta'_0} + \tfrac{d'\cos\beta'_0}{d\cos\alpha_0}\Big) dd' \sin\delta \\ &= \tfrac{1}{2}\Big(\tfrac{d\cos\alpha_0}{d'\cos\beta'_0}\, d'^2 \sin 2\beta'_0 + \tfrac{d'\cos\beta'_0}{d\cos\alpha_0}\, d^2 \sin 2\alpha_0\Big) \\ &= dd' \left(\cos\alpha_0 \sin\beta'_0 + \cos\beta'_0 \sin\alpha_0\right) \\ &= dd' \sin(\alpha_0 + \beta'_0). \end{aligned}$$

Hence $\sin(\alpha_0+\beta'_0) \ge \sin(\pi-\delta)$ and therefore $\alpha_0+\beta'_0 \le \pi-\delta$, because $\alpha_0+\beta'_0, \pi-\delta \in \left[\frac{\pi}{2}, \pi\right]$. This implies (3), namely $\alpha + \beta' + \delta \le \alpha_0 + \beta'_0 + \delta \le \pi$.

Since $\delta < \frac{\pi}{2}$ and $d(x_0, c_4) = \frac{d(x_0,\hat{x}_1)}{2} \le \frac{d}{2} \le \frac{d'}{2} = d(p_1, c_4)$ in the triangle $\triangle x_0p_1c_4$, we have $p_1 \in \bigcup_{i=0}^{\infty} x_ix_{i+1}$. Thus $x_0\hat{x}_0$ meets p_1p_4 as well as p_2p_3 and splits P into two polygons symmetric with respect to c. We denote the one that contains p_1p_2 by P_1. Now k is the number of pieces of P contained in P_1.

The dissection of P_1 is induced by those segments $x_i\hat{x}_{1-i}$, $x_i\hat{x}_{-i}$, $x'_i\hat{x}'_{1-i}$, $x'_i\hat{x}'_{-i}$ that intersect $\mathrm{int}(P_1)$. Among the edges of the triangles from Σ these are exactly the $x_i\hat{x}_{1-i}$, $x_i\hat{x}_{-i}$ satisfying $x_i \in x_0p_2 \setminus \{x_0, p_2\}$, that is, $1 \leq i \leq i_0 = \lceil \frac{d(x_0,p_2)}{d} \rceil - 1$.

Next we shall see that $x'_0\hat{x}'_0$ and $x'_1\hat{x}'_0$ are the only edges of triangles from Σ' that can intersect $\mathrm{int}(P_1)$. For this it suffices to show that $\hat{x}'_0 \in p_2p_3 \cap P_1$, because then all $\hat{x}'_ix'_{1-i}$, $\hat{x}'_ix'_{-i}$, $i \geq 1$, are separated from $\mathrm{int}(P_1)$ by $x_0\hat{x}_0$ and all $\hat{x}'_ix'_{-i}$, $\hat{x}'_ix'_{1-i}$, $i \leq -1$, are separated from $\mathrm{int}(P_1)$ by p_1p_2. We have

$$d(c, \hat{x}'_0) = \tfrac{d(x'_0,\hat{x}'_0)}{2} \leq \tfrac{d'}{2} = d(c, c_1) < d(c, p_2).$$

Hence $\hat{x}'_0$ belongs to the open half-line $\{p_2 + \mu(p_3 - p_2) : \mu > 0\}$. On the other hand, the slope $\tan\alpha$ of $x_0\hat{x}_0$ relative to the "horizontal" straight line l is larger than the slope $\tan(\alpha' - \delta)$ of $x'_0\hat{x}'_0$, since $\alpha' \leq \alpha$. This yields $\hat{x}'_0 \in p_2p_3 \cap P_1$. So the dissection of P_1 induced by the triangles from Σ' is realized by the two segments $c\hat{x}'_0$, $\hat{x}'_0c_1$ and consists of the three polygons $P_1 \cap T'$, $P_1 \cap \hat{T}'_0$, $P_1 \cap \hat{T}'_{-1}$.

Now the full dissection of P_1 is completed by the segments $x_i\hat{x}_{1-i}$, $x_i\hat{x}_{-i}$ with $1 \leq i \leq i_0 = \lceil \frac{d(x_0,p_2)}{d} \rceil - 1$. First let $1 \leq i \leq i_0 - 1$, that is, $x_i \in x_0p_1 \setminus \{x_0, p_1\}$. Then each $x_i\hat{x}_{1-i}$ or $x_i\hat{x}_{-i}$ passes through $\mathrm{int}(P_1 \cap T')$ and intersects at most one of $\mathrm{int}(P_1 \cap \hat{T}'_0)$ and $\mathrm{int}(P_1 \cap \hat{T}'_{-1})$. Hence subdividing the previous dissection by such a segment increases the number of pieces by at most 2. After having used all these $2(i_0 - 1)$ segments we have at most $3 + 4(i_0 - 1) = 4i_0 - 1$ pieces.

We show now that the two remaining segments $x_{i_0}\hat{x}_{1-i_0}$, $x_{i_0}\hat{x}_{-i_0}$ together increase the total number of pieces of P_1 by at most 4. We have $x_{i_0} \in p_1p_2 \setminus \{p_2\}$. If $x_{i_0} \in p_1c_1 \setminus \{c_1\}$ we can argue as above. In the case $x_{i_0} = c_1$ each of $x_{i_0}\hat{x}_{1-i_0}$ and $x_{i_0}\hat{x}_{-i_0}$ either intersects both $\mathrm{int}(P \cap T')$ and $\mathrm{int}(P_1 \cap \hat{T}'_0)$, but misses $\mathrm{int}(P_1 \cap \hat{T}'_{-1})$, or is collinear with $c_1\hat{x}'_0$, or intersects only $\mathrm{int}(P_1 \cap \hat{T}'_{-1})$. This yields the claim. In the final case $x_{i_0} \in c_1p_2 \setminus \{c_1, p_2\}$ (which is displayed in Figure 2) we compute

$$|\angle p_1c_1\hat{x}'_0| + |\angle p_2x_{i_0}\hat{x}_{-i_0}| = (\beta' + \delta) + \alpha \leq \pi$$

by (3). Thus $x_{i_0}\hat{x}_{-i_0}$ meets only $P_1 \cap \hat{T}'_{-1}$, but misses $\mathrm{int}(P \cap T')$ and $\mathrm{int}(P_1 \cap \hat{T}'_0)$. The segment $x_{i_0}\hat{x}_{1-i_0}$ may intersect all three open polygons $\mathrm{int}(P \cap T')$, $\mathrm{int}(P_1 \cap \hat{T}'_0)$, and $\mathrm{int}(P_1 \cap \hat{T}'_{-1})$. However, dissecting by both $x_{i_0}\hat{x}_{-i_0}$ and $x_{i_0}\hat{x}_{1-i_0}$ enlarges the number of pieces of P_1 by at most 4.

The last observation shows that the total number k of pieces in P_1 is bounded by $k \leq (4i_0 - 1) + 4 = 4i_0 + 3$. Now the proof of Lemma 3 is completed by

$$\begin{aligned}
\deg_{\mathrm{Isom}^+}(T, T') &\leq k \leq 4i_0 + 3 = 4\left(\left\lceil \tfrac{d(x_0,p_2)}{d} \right\rceil - 1\right) + 3 \\
&\leq 4\left(\left\lceil \tfrac{1}{d}(d(x_0, c) + d(c, c_1) + d(c_1, p_2)) \right\rceil - 1\right) + 3 \\
&\leq 4\left(\left\lceil \tfrac{1}{d}\left(\tfrac{d}{2} + \tfrac{d'}{2} + \tfrac{d}{2}\right) \right\rceil - 1\right) + 3 = 4\left\lceil \tfrac{d'}{2d} \right\rceil + 3 \\
&= 4\left\lceil \tfrac{1}{2} \max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\} \right\rceil + 3. \qquad \square
\end{aligned}$$

2.3. An improved estimate in terms of vertex numbers and diameters

Theorem 2. *Let P_m and P'_n be convex polygons of the same area whose numbers of vertices are m and n and whose diameters are d and d', respectively. Then*

$$\deg_{\mathrm{Isom}^+}(P_m, P'_n) \leq (m+n-5)\left(4\left\lceil \max\left\{\tfrac{\lfloor\frac{n}{2}\rfloor d}{d'}, \tfrac{\lfloor\frac{m}{2}\rfloor d'}{d}\right\}\right\rceil + 3\right).$$

In particular

$$\deg_{\mathrm{Isom}^+}(P_m, P'_n) < 2(m+n)^2 \max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}.$$

The proof is prepared by two more lemmas.

Lemma 4. *Let the area $\lambda(T)$ of a triangle T be represented as a sum $\lambda(T) = \lambda_1 + \cdots + \lambda_k$ of k positive real numbers. Then one can dissect T into k triangles $T_1, \ldots, T_k$ such that*

$$\lambda(T_i) = \lambda_i \quad \text{and} \quad \mathrm{diam}(T_i) > \tfrac{\mathrm{diam}(T)}{2} \quad \text{for } 1 \leq i \leq k.$$

Proof. Let $T = \triangle x_1x_2x_3$ and suppose that $d(x_1,x_2) \geq d(x_2,x_3) \geq d(x_1,x_3)$. We fix points $y_1, \ldots, y_{k-1}$ of the edge x_1x_3 such that $d(x_1,y_1) = \frac{\lambda_1}{\lambda(T)}\, d(x_1,x_3)$, $d(y_{i-1},y_i) = \frac{\lambda_i}{\lambda(T)}\, d(x_1,x_3)$ for $2 \leq i \leq k-1$, and $d(y_{k-1},x_3) = \frac{\lambda_k}{\lambda(T)}\, d(x_1,x_3)$. Then T splits into $T_1 = \triangle x_1x_2y_1$, $T_i = \triangle y_{i-1}x_2y_i$ for $2 \leq i \leq k-1$, and $T_k = \triangle y_{k-1}x_2x_3$. The areas of these triangles are proportional to the lengths of their edges contained in x_1x_3. Hence $\lambda(T_i) = \lambda_i$, $1 \leq i \leq k$. An estimate of their diameters can be obtained by the aid of the orthogonal projection π onto the long edge x_1x_2, namely

$$\mathrm{diam}(T_i) \geq d(y_{i-1},x_2) > d(\pi(y_{i-1}),x_2) > d(\pi(x_3),x_2) \geq \tfrac{d(x_1,x_2)}{2} = \tfrac{\mathrm{diam}(T)}{2}$$

for $2 \leq i \leq k$. For T_1 we even have $\mathrm{diam}(T_1) = d(x_1,x_2) = \mathrm{diam}(T)$. □

Lemma 5. *Let P_m and P'_n be as in Theorem 2. Then there exist dissections of P_m into $m+n-5$ triangles $T_1, \ldots, T_{m+n-5}$ and of P'_n into $m+n-5$ triangles $T'_1, \ldots, T'_{m+n-5}$ such that, for $1 \leq i \leq m+n-5$,*

$$\lambda(T_i) = \lambda(T'_i), \quad \tfrac{d}{2\lfloor\frac{m}{2}\rfloor} < \mathrm{diam}(T_i) \leq d, \quad \text{and} \quad \tfrac{d'}{2\lfloor\frac{n}{2}\rfloor} < \mathrm{diam}(T'_i) \leq d'.$$

Proof. By Lemma 1, there exist dissections of P_m into triangles $S_1, \ldots, S_{m-2}$ with $\mathrm{diam}(S_i) \geq \frac{d}{\lfloor\frac{m}{2}\rfloor}$ and of P'_n into triangles $S'_1, \ldots, S'_{n-2}$ with $\mathrm{diam}(S'_j) \geq \frac{d'}{\lfloor\frac{n}{2}\rfloor}$. Let $\mu_i = \sum_{l=1}^{i} \lambda(S_l)$, $0 \leq i \leq m-2$. Then the intervals $I_i = [\mu_{i-1}, \mu_i]$, $1 \leq i \leq m-2$, have the lengths $\lambda(S_i)$ and together constitute a dissection of $[0, \lambda(P_m)]$. Similarly, let $\nu_j = \sum_{l=1}^{j} \lambda(S'_l)$, $0 \leq j \leq n-2$. The intervals $J_j = [\nu_{j-1}, \nu_j]$, $1 \leq j \leq n-2$, have the lengths $\lambda(S'_j)$ and form a dissection of $[0, \lambda(P'_n)] = [0, \lambda(P_m)]$. The numbers $\mu_1, \ldots, \mu_{m-3}, \nu_1, \ldots, \nu_{n-3}$ cut $[0, \lambda(P_m)]$ into at most $m+n-5$ subintervals each being completely covered by some I_i and by some J_j. Hence there exists a dissection of $[0, \lambda(P_m)]$ into closed intervals $K_1, \ldots, K_{m+n-5}$ of positive lengths which refines both subdivisions $\{I_1, \ldots, I_{m-2}\}$ and $\{J_1, \ldots, J_{n-2}\}$ simultaneously.

The interval I_i of length $\lambda(S_i)$ splits into suitable $K_{i_1},\dots,K_{i_l}$ of the lengths $\lambda_{i_1},\dots,\lambda_{i_l}$. By Lemma 4, S_i can be decomposed into triangles $T_{i_1},\dots,T_{i_l}$ such that

$$\lambda(T_{i_r}) = \lambda_{i_r} \quad \text{and} \quad \operatorname{diam}(T_{i_r}) > \tfrac{\operatorname{diam}(S_i)}{2} \geq \tfrac{d}{2\lfloor\frac{m}{2}\rfloor} \quad \text{for } 1 \leq r \leq l.$$

This gives the dissection of P_m into $T_1,\dots,T_{m+n-5}$. In the same way we can dissect P'_n into triangles $T'_1,\dots,T'_{m+n-5}$ with

$$\lambda(T'_i) = \lambda_i \quad \text{and} \quad \operatorname{diam}(T'_i) > \tfrac{d'}{2\lfloor\frac{n}{2}\rfloor} \quad \text{for } 1 \leq i \leq m+n-5$$

by partitioning $S'_1,\dots,S'_{n-2}$. In particular $\lambda(T_i) = \lambda_i = \lambda(T'_i)$. This completes the proof. □

Proof of Theorem 2. We dissect P_m and P'_n by Lemma 5. Then, for every $i \in \{1,\dots,m+n-5\}$, we apply Lemma 3 to T_i and T'_i. This gives dissections of P_m and P'_n proving the first estimate. The second one can be shown as follows.

$$\begin{aligned}
\deg_{\mathrm{Isom}^+}(P_m,P'_n) &< (m+n-5)\left(4\left(\max\left\{\tfrac{\lfloor\frac{n}{2}\rfloor d}{d'},\tfrac{\lfloor\frac{m}{2}\rfloor d'}{d}\right\}+1\right)+3\right)\\
&\leq (m+n-5)\left(4\max\left\{\tfrac{\frac{n}{2}d}{d'},\tfrac{\frac{m}{2}d'}{d}\right\}+7\right)\\
&= (m+n-5)\big(2\max\left\{\tfrac{nd}{d'},\tfrac{md'}{d}\right\}+7\big)\\
&\leq (m+n-5)\big(2(m+n-3)\max\left\{\tfrac{d}{d'},\tfrac{d'}{d}\right\}+7\big)\\
&\leq (m+n-5)\big(2(m+n-3)+7\big)\max\left\{\tfrac{d}{d'},\tfrac{d'}{d}\right\}\\
&= 2(m+n-5)\big(m+n+\tfrac{1}{2}\big)\max\left\{\tfrac{d}{d'},\tfrac{d'}{d}\right\}\\
&< 2(m+n)^2\max\left\{\tfrac{d}{d'},\tfrac{d'}{d}\right\}.
\end{aligned}$$ □

2.4. An estimate in terms of vertex numbers, diameters, and radii of inscribed circles

Theorem 3. *Let P_m be a convex m-gon of diameter d containing a circle of radius r and let P'_n be a convex n-gon of the same area having the diameter d' and containing a circle of radius r'. Then*

$$\deg_{\mathrm{Isom}^+}(P_m,P'_n) \leq (m+n-1)\left(4\left\lceil\tfrac{1}{2}\max\left\{\tfrac{d}{r'},\tfrac{d'}{r}\right\}\right\rceil+3\right).$$

In particular

$$\deg_{\mathrm{Isom}^+}(P_m,P'_n) < (m+n)\big(2\max\left\{\tfrac{d}{r'},\tfrac{d'}{r}\right\}+7\big).$$

Again the problem is reduced to piecewise congruences of triangles.

Lemma 6. *Let P_m and P'_n be as in Theorem* 3. *Then there exist dissections of P_m into $k \leq m+n-1$ triangles $T_1,\dots,T_k$ and of P'_n into k triangles $T'_1,\dots,T'_k$ such that, for $1 \leq i \leq k$,*

$$\lambda(T_i) = \lambda(T'_i), \qquad r < \operatorname{diam}(T_i) \leq d, \qquad \textit{and} \qquad r' < \operatorname{diam}(T'_i) \leq d'.$$

Proof. Let $x_0, \dots, x_{m-1}$ and $x'_0, \dots, x'_{n-1}$ be the vertices of P_m and P'_n, respectively, ordered counterclockwise along the boundaries. Let c and c' be the midpoints of the inscribed circles of the respective polygons. We define a bijection p of the half-open interval $[0, \lambda(P_m))$ onto $\mathrm{bd}(P_m)$ such that the counterclockwise arc from x_0 to $p(\lambda)$ along $\mathrm{bd}(P_m)$ together with the segments cx_0 and $cp(\lambda)$ bounds a polygon of area λ. Similarly, we introduce $p' : [0, \lambda(P_m)) = [0, \lambda(P'_n)) \to \mathrm{bd}(P'_n)$.

Now let $\{0 = p^{-1}(x_0), \dots, p^{-1}(x_{m-1})\} \cup \{0 = (p')^{-1}(x'_0), \dots, (p')^{-1}(x'_{n-1})\} = \{\lambda_0, \dots, \lambda_{k-1}\}$ be ordered such that $0 = \lambda_0 < \cdots < \lambda_{k-1}$. Of course, $k \le m+n-1$. We define

$$T_i = \begin{cases} \mathrm{conv}\{c, p(\lambda_{i-1}), p(\lambda_i)\}, & 1 \le i \le k-1, \\ \mathrm{conv}\{c, p(\lambda_{k-1}), x_0\}, & i = k, \end{cases}$$

and

$$T'_i = \begin{cases} \mathrm{conv}\{c', p'(\lambda_{i-1}), p'(\lambda_i)\}, & 1 \le i \le k-1, \\ \mathrm{conv}\{c', p'(\lambda_{k-1}), x'_0\}, & i = k, \end{cases}$$

this way obtaining dissections of P_m into $T_1, \dots, T_k$ and of P'_n into $T'_1, \dots, T'_k$. Then $\lambda(T_i) = \lambda_i - \lambda_{i-1} = \lambda(T'_i)$ for $1 \le i \le k-1$ and $\lambda(T_k) = \lambda(P_m) - \lambda_{k-1} = \lambda(T'_k)$. Since every T_i contains the centre c as well as at least one vertex outside the inscribed circle of radius r, the lower estimate is obvious. The upper one is trivial. The triangles T'_i behave analogously. $\square$

Now Theorem 3 can be inferred from Lemma 6 as Theorem 2 has been proved by Lemma 5.

2.5. An estimate for regular polygons

Theorem 4. *Let P^r_m and P^r_n be regular polygons of the same area having m and n vertices, respectively. Then*

$$\deg_{\mathrm{Isom}^+}(P^r_m, P^r_n) \le 7(m+n-1).$$

Theorem 4 is an immediate consequence of the following claim and of Lemma 3.

Lemma 7. *Let P^r_m and P^r_n be regular polygons of area 1 having m and n vertices, respectively. Then there exist dissections of P^r_m into $k \le m+n-1$ triangles $T_1, \dots, T_k$ and of P^r_n into k triangles $T'_1, \dots, T'_k$ such that, for $1 \le i \le k$,*

$$\lambda(T_i) = \lambda(T'_i) \quad \text{and} \quad \tfrac{1}{2} < \tfrac{\mathrm{diam}(T_i)}{\mathrm{diam}(T'_i)} < 2.$$

Proof. Simple trigonometric calculations show that the radius r_m of the largest inscribed circle, the radius R_m of the smallest circumscribed circle, and the edge length e_m of P^r_m are

$$r_m = \frac{1}{\sqrt{m \tan \frac{\pi}{m}}}, \qquad R_m = \sqrt{\frac{1+\tan^2 \frac{\pi}{m}}{m \tan \frac{\pi}{m}}}, \qquad e_m = 2\sqrt{\frac{1}{m} \tan \frac{\pi}{m}}.$$

We assume $3 \le m < n$ without loss of generality.

Case 1. $n = 4$. Then $m = 3$. We cut P_3^r along an axis of symmetry into T_1, T_2 and P_4^r along a diagonal into T_1', T_2'. Then $\lambda(T_i) = \lambda(T_i') = \frac{1}{2}$ and $\frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_i')} = \frac{e_3}{2R_4} = 2^{\frac{1}{2}}3^{-\frac{1}{4}} = 1.07...$ for $i = 1, 2$.

Case 2. $n \geq 5$. Now we define dissections of P_m^r into $T_1, \ldots, T_k$, $k \leq m+n-1$, and of P_n^r into $T_1', \ldots, T_k'$ as we did in the proof of Lemma 6. We obtain in particular

$$\lambda(T_i) = \lambda(T_i'), \quad r_m < \mathrm{diam}(T_i), \quad \text{and} \quad r_n < \mathrm{diam}(T_i')$$

for $1 \leq i \leq k$. For the remaining estimate of $\frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_i')}$ let i be fixed.

Case 2.1. $m = 3$. The triangle T_i' is of the form $\triangle c'y_1'y_2'$, c' being the centre of P_n^r and y_1', y_2' lying on a common edge of P_n^r. We can estimate $\mathrm{diam}(T_i') = \max\{|c'y_1'|, |c'y_2'|, |y_1'y_2'|\}$ by

$$r_5 \leq r_n < \mathrm{diam}(T_i') \leq \max\{R_n, R_n, e_n\} \leq \max\{R_5, e_5\} = e_5. \tag{4}$$

Similarly, we obtain $T_i = \triangle c y_1 y_2$, where c is the centre of P_3^r and y_1, y_2 lie on a common edge of P_3^r. The respective triangle T_i', which has the same area as T_i, is contained in one of the n pairwise congruent triangles defined as the convex hulls of c' and an edge of P_n^r. Hence $\lambda(T_i) = \lambda(T_i') \leq \frac{1}{n} \leq \frac{1}{5}$. Since r_3 is the height of T_i over the edge y_1y_2, we obtain $\frac{|y_1y_2|\, r_3}{2} = \lambda(T_i) \leq \frac{1}{5}$ and hence $|y_1y_2| \leq \frac{2}{5r_3}$. Thus $\mathrm{diam}(T_i)$ satisfies the estimate

$$r_3 < \mathrm{diam}(T_i) = \max\{|cy_1|, |cy_2|, |y_1y_2|\} \leq \max\left\{R_3, R_3, \tfrac{2}{5r_3}\right\} = \tfrac{2}{5r_3}.$$

Combining this with (4) we obtain the required inequalities, namely

$$0.57... = \frac{r_3}{e_5} < \frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_i')} < \frac{\frac{2}{5r_3}}{r_5} = 1.73...$$

Case 2.2. $m \geq 4$. Arguments similar to those for showing (4) give

$$\begin{aligned} r_4 \leq r_m < \mathrm{diam}(T_i) &\leq \max\{R_m, R_m, e_m\} \leq \max\{R_4, e_4\} = e_4, \\ r_4 < r_n < \mathrm{diam}(T_i') &\leq \max\{R_n, R_n, e_n\} < \max\{R_4, e_4\} = e_4. \end{aligned}$$

Consequently, $\frac{1}{2} = \frac{r_4}{e_4} < \frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_i')} < \frac{e_4}{r_4} = 2$. This completes the proof. □

3. Remarks concerning similarities and translations

3.1. Congruence by dissection with respect to similarities

Any two polygons P, P' are congruent by dissection with respect to the group Sim of similarities, since P and the similar image $\sqrt{\frac{\lambda(P)}{\lambda(P')}}P'$ of P' have the same area and hence are congruent by dissection with respect to Isom by the Wallace-Bolyai-Gerwien theorem. It is shown in [7] that any convex m-gon P_m and any convex n-gon P_n' satisfy $\deg_{\mathrm{Sim}}(P_m, P_n') \leq 3(\max\{m, n\} - 2)$. This motivates the definition

$$\deg_{\mathrm{Sim}}(m, n) = \max\{\deg_{\mathrm{Sim}}(P_m, P_n') : P_m \text{ a convex } m\text{-gon}, P_n' \text{ a convex } n\text{-gon}\}$$

for $m, n \geq 3$. We introduce $\deg_{\mathrm{Sim}^+}(m,n)$ analogously, Sim^+ denoting the subgroup of proper similarities. Of course,

$$\deg_{\mathrm{Sim}}(m,n) = \deg_{\mathrm{Sim}}(n,m) \leq \deg_{\mathrm{Sim}^+}(m,n) = \deg_{\mathrm{Sim}^+}(n,m).$$

The above mentioned estimate from [7] now reads as

$$\deg_{\mathrm{Sim}}(m,n) \leq 3(n-2) \quad \text{for all} \quad 3 \leq m \leq n. \tag{5}$$

We could show the following.

Theorem 5. ([8], Theorems 5 and 6) *For arbitrary* $3 \leq m \leq n$,

$$\deg_{\mathrm{Sim}^+}(m,n) \leq \begin{cases} \left\lfloor \frac{5m-9}{2} \right\rfloor = \left\lfloor \frac{7m+8n-27}{6} \right\rfloor & \text{if } n = m, \\ \left\lfloor \frac{7m+8n-24}{6} \right\rfloor & \text{if } n > m \end{cases} \tag{6}$$

and

$$\deg_{\mathrm{Sim}}(m,n) \leq \begin{cases} n+3 & \text{if } m = 3, \\ m+n+\left\lfloor \frac{m}{3} \right\rfloor & \text{if } 4 \leq m \leq 11, \\ m+n+4 & \text{if } m \geq 12. \end{cases} \tag{7}$$

The estimate (6) can be realized by dissections into convex pieces, whereas our construction for (7) uses up to two non-convex simple polygons in the corresponding dissections. The bound (7) is stronger than (6) if n is sufficiently large, but can be improved by (6) for certain small m, n.

The estimate (6) is better than (5) apart from the case $m = n = 3$, where both bounds attain the value 3. In the latter situation the estimates are sharp. This is a consequence of the following observation.

Lemma 8. ([8], Corollary 2) *Two triangles T, T' satisfy $\deg_{\mathrm{Sim}}(T,T') \leq 2$ if and only if T has an angle of size α and T' has an angle of size α' such that $\alpha = \alpha'$ or $\alpha + \alpha' = \pi$.*

3.2. Congruence by dissection with respect to translations

Two polygons of the same area are not necessarily equidissectable with respect to the group Trans of translations. Since directions are invariant under translations, congruence by dissection with respect to Trans depends on the directional behaviour of the corresponding polygons, too. We recall a necessary and sufficient condition from [5].

Let $\mathbb{S}^1$ be the set of all vectors of length 1 in $\mathbb{R}^2$. Given a (not necessarily convex) polygon P and a direction $x \in \mathbb{S}^1$, $l(P,x)$ is to denote the sum of the lengths of all edges of P having the outer normal vector x. If no such edges exist, we put $l(P,x) = 0$. Now let $L_x(P) = l(P,x) - l(P,-x)$. It is shown in [5] that two polygons P, P' are congruent by dissection with respect to Trans if and only if they have the same area and $L_x(P) = L_x(P')$ for all $x \in \mathbb{S}^1$.

Suppose from now on that P, P' are arbitrary convex polygons congruent by dissection under translations. In contrast with Theorem 2, a general upper estimate for $\deg_{\mathrm{Trans}}(P, P')$ cannot be given in terms of the vertex numbers and the diameters of P and P'. For example, the two rectangles $R(m) = [0, m] \times [0, 1]$ and $R'(m) = [0, 1] \times [0, m]$ satisfy $\deg_{\mathrm{Trans}}(R(m), R'(m)) = m$ for every $m \in \{1, 2, \ldots\}$, even though their diameters agree. The last observation reveals a more general principle.

Given a bounded non-empty set $A \subseteq \mathbb{R}^2$ and a vector $x \in \mathbb{S}^1$, the *width* of A in direction x is given by

$$w(A, x) = \sup\{\langle a_1 - a_2, x\rangle : a_1, a_2 \in A\},$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product. Of course, this functional is monotone and invariant under translations, that is, $w(A, x) \leq w(B, x)$ if $A \subseteq B \subseteq \mathbb{R}^2$ and $w(A, x) = w(A + t, x)$ for all $t \in \mathbb{R}^2$.

Since P can be dissected into $\deg_{\mathrm{Trans}}(P, P')$ translates Q_i of pieces of P', there is a convex set of width $w(P, x)$ that can be covered by $\deg_{\mathrm{Trans}}(P, P')$ sets of width $w(Q_i, x) \leq w(P', x)$. This yields $\deg_{\mathrm{Trans}}(P, P') \geq \left\lceil \frac{w(P,x)}{w(P',x)} \right\rceil$ and, by symmetry,

$$\deg_{\mathrm{Trans}}(P, P') \geq \sup\left\{ \left\lceil \max\left\{ \tfrac{w(P,x)}{w(P',x)}, \tfrac{w(P',x)}{w(P,x)} \right\} \right\rceil : x \in \mathbb{S}^1 \right\}$$

as a general lower estimate for $\deg_{\mathrm{Trans}}(P, P')$.

We denote the right hand side of the last formula by $W(P, P')$. Can one find a general upper bound for $\deg_{\mathrm{Trans}}(P, P')$ only in terms of $W(P, P')$ and the vertex numbers of P and P'?

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