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Three Remarks on Absolutely Solvable Groups

In fond memory of Edit Szabó (1958–2005)

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Abstract. First, we characterize finite groups that are not absolutely solvable, but every proper subgroup of them is absolutely solvable. Second, we show that every hereditary absolutely solvable group of odd order is an M-group. Third, we exhibit examples of groups which are not absolutely solvable, although they can be written as a product of two absolutely solvable normal subgroups.

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1. Introduction

The concept of absolutely solvable groups was invented by Gerhard Pazderski, and it was introduced in the paper [7] of his former student Edit Szabó.

Let G be a finite solvable group with a chief series

$$G = N_0 > N_1 > \cdots > N_{n-1} > N_n = 1\,,$$

that is, the N_i's are normal subgroups of G and this series is not refinable. Solvability of G yields that each chief factor N_{i-1}/N_i is an elementary abelian p_i-group

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for some prime p_i, so $|N_{i-1}/N_i| = p_i^{d_i}$ with some $d_i \geq 1$. Hence N_{i-1}/N_i can be considered as a vector space of dimension d_i over the p_i-element field. Every element $g \in G$ induces an automorphism of N_{i-1}/N_i by conjugation: $xN_i \mapsto gxg^{-1}N_i$ ($x \in N_{i-1}$). This way we obtain a linear representation $\Psi_i : G \to \mathrm{GL}(d_i, p_i)$. As each N_{i-1}/N_i is a chief factor, this representation is irreducible for every $i = 1, \ldots, n$.

As it is well known, a representation is called absolutely irreducible if it remains irreducible over any extension field. Absolutely irreducible representations play a significant role in representation theory. We now recall Pazderski's definition of absolutely solvable groups.

Definition 1. *A finite solvable group G is called absolutely solvable (abbreviated AS), if all representations induced on the chief factors are absolutely irreducible.*

Clearly, by the Jordan-Hölder theorem, the definition does not depend on the choice of the particular chief series.

Since subgroups of AS groups need not be AS, the following definition is more restrictive.

Definition 2. *A finite solvable groups G is called hereditary absolutely solvable (abbreviated HAS), if every subgroup of G is AS.*

In Section 2 we will summarize the main results of Edit Szabó from her papers [7] and [8]. The present work can be considered as a continuation of her investigations. The reader is advised to study the papers [7], [8] before reading this article.

In Section 3 we will determine the minimal non-AS groups, that is, those finite groups which are not AS themselves but every proper subgroup of them is AS.

The class of AS groups shows some similarities with the class of monomial groups (M-groups in short). In Section 4 we explore the relationship between the two classes. In particular, we show that every HAS group of odd order is an M-group.

Finally, in Section 5 we will give examples of groups which are the product of two AS normal subgroups, but they are not AS themselves.

2. Summary of Edit Szabó's results

We are going to give a short overview of Edit Szabó's two papers [7], [8] on AS groups, with emphasis on those results which we will apply in our proofs.

She showed that the class of AS groups is a formation, but not a saturated formation [7, Theorem 2.6]. However, the class of HAS groups is a saturated formation [7, Theorem 4.2], namely, the following local definition of HAS groups can be given using the notation from the Introduction.

Theorem 1. [7, Theorem 4.1] *A finite solvable group G is HAS iff for every $i = 1, \ldots, n$ the exponent of $G/\ker \Psi_i$ divides $p_i - 1$.*

In the examples and constructions she used the following easy observations (see [7, Lemmas 2.2, 2.4]):

(1) One-dimensional representations are absolutely irreducible.
(2) An irreducible representation of an abelian group is absolutely irreducible iff it is one-dimensional.
(3) If the dimension of an irreducible representation is a prime number and the image of the group is nonabelian, then the representation is absolutely irreducible.

Based on these elementary observations she was able to construct examples of groups what are AS but not HAS, and others that are HAS but not supersolvable [7, Proposition 4.4]. Note that, obviously, every supersolvable group is HAS.

We will also make use of the following result, which was actually proved under a weaker assumption that all normal subgroups of G are AS.

Theorem 2. [7, Theorem 3.2] *Every HAS group possesses an ordered Sylow tower. That is, if $|G| = p_1^{k_1} \cdots p_s^{k_s}$ with primes $p_1 < \cdots < p_s$, then there exists a series of normal subgroups $G = P_0 > P_1 > \cdots > P_{s-1} > P_s = 1$ such that $|P_{i-1}/P_i| = p_i^{k_i}$ for each $i = 1, \ldots, s$.*

In the other paper [8] embeddings into AS groups are investigated. Edit Szabó proved that every finite solvable group can be embedded as a subgroup into an AS group [8, Theorem 2.2]. However, she constructed a finite solvable group that cannot be embedded into an AS group as a subnormal subgroup [8, Theorem 4.1].

3. Minimal non-absolutely-solvable groups

It has been a recurrent topic in group theory to investigate groups that do not satisfy a certain property but all proper subgroups have that property. The best known example is the description of minimal non-nilpotent groups (see [1, Satz III.5.2] and [5]). For a survey of results of this type see [3]. For the class of AS groups we obtain the following.

Theorem 3. *Let G be a finite group which is not AS, but every proper subgroup of G is AS. Then $G = PQ$, where P is a normal Sylow p-subgroup, Q is a cyclic Sylow q-subgroup (p and q are distinct primes) and one of the following holds:*

(i) *q does not divide $p-1$ and G is a minimal non-nilpotent group;*
(ii) *q divides $p-1$, $|Q/\mathbf{C}_Q(P)| = q^{s+1}$ where q^s is the highest power of q dividing $p-1$, and either P is elementary abelian of order p^q or $q = 2$, P is a nonabelian group of order p^3 and exponent p.*

Furthermore, we have that every proper subgroup of G is supersolvable.

Proof. Let r be the smallest prime divisor of $|G|$. Every proper subgroup of G is HAS, hence it possesses an ordered Sylow tower (see Theorem 2). In particular, every proper subgroup of G is r-nilpotent.

If G itself is not r-nilpotent, then G is a minimal non-r-nilpotent group, and the theorem of Itô (see [1, Satz IV.5.4]) applies, so G is in fact a minimal non-nilpotent group and $G = PQ$ where P is a normal Sylow p-subgroup, Q is a cyclic

Sylow q-subgroup ($r = p \neq q$ primes). Since $p < q$, q does not divide $p-1$, so we are in case (i).

If G is r-nilpotent, then G itself has an ordered Sylow tower. Let p be the largest prime divisor of $|G|$ and let P be the normal Sylow p-subgroup of G. By the Schur-Zassenhaus Theorem there is a complement K to P. Since G is not HAS, Theorem 1 implies that there exists a chief factor N_{i-1}/N_i of G such that the exponent of $G/\mathbf{C}_G(N_{i-1}/N_i)$ does not divide $p_i - 1$ where $|N_{i-1}/N_i| = p_i^{d_i}$ for some prime p_i and $d_i \geq 1$. As $G/P \cong K$ is HAS, we must have $p_i = p$ and $N_{i-1} \leq P$. Since P is normal in G, it is contained in the Fitting subgroup of G, hence it acts trivially on each chief factor, and so K induces the same linear group on N_{i-1}/N_i what G does. Now $N_{i-1}K$ is not AS, hence we have $N_{i-1}K = G$, $N_{i-1} = P$. Since K acts completely reducibly on $P/\Phi(P)$, we get $N_i = \Phi(P)$.

As the exponent condition is not satisfied for the action of K on $P/\Phi(P)$, we can find a prime $q \neq p$ and an element $g \in K$ of q-power order such that the linear transformation induced by g on $P/\Phi(P)$ has order q^{s+1}, where q^s is the highest power of q dividing $p-1$ ($s \geq 0$). Then $P\langle g \rangle$ is not AS, hence $P\langle g \rangle = G$, so we have established that $G = PQ$ with normal Sylow p-subgroup P and cyclic Sylow q-subgroup $Q = \langle g \rangle$ in this case as well.

If q does not divide $p-1$ (i.e., $s = 0$), then g induces an automorphism of order q of P and acts irreducibly on $P/\Phi(P)$ and trivially on $\Phi(P)$. Hence in this case $G = PQ$ is a minimal non-nilpotent group, so we are again in case (i).

If q divides $p-1$ (i.e., $s \geq 1$), then the irreducible module $P/\Phi(P)$ of the cyclic group Q has dimension q. Then g^{q^s} acts by scalar multiplication of order q on $P/\Phi(P)$. The action of G on any chief factor below $\Phi(P)$ is absolutely irreducible, hence by the commutativity of Q, these chief factors have order p. Then the group of linear transformations induced on them by G has order dividing $p-1$, hence $Q_1 = \langle g^{q^s} \rangle$ acts trivially on every chief factor below $\Phi(P)$, and therefore also on $\Phi(P)$. The same holds obviously for every conjugate of Q_1. Since the normal closure of Q_1 in G is PQ_1, we obtain $\Phi(P) \leq \mathbf{Z}(PQ_1)$. In particular, P has nilpotence class at most 2. We can write $P/P' = \mathbf{C}_{P/P'}(Q_1) \times [P/P', Q_1] = \Phi(P)/P' \times [P/P', Q_1]$. Since we know that the only chief factor of G on which Q_1 acts nontrivially is $P/\Phi(P)$, we obtain $P' = \Phi(P)$.

If P is abelian, then it is elementary abelian of order p^q.

If P is nonabelian, then let $\Phi(P)/N$ be a chief factor of G. We know that $|\Phi(P)/N| = p$, hence we obtain that P/N is an extraspecial p-group. Since $|P/\Phi(P)| = p^q$, this is only possible if $q = 2$. Then $|P/\mathbf{Z}(P)| = p^2$ implies $|P'| = p$, so $|P| = p^3$. Since P cannot contain a characteristic subgroup of order p^2, we get that P has exponent p. This finishes the analysis of case (ii).

Now it is easy to check that the groups with the structure given in (i) or (ii) are not AS, but every proper subgroup of them is supersolvable, hence AS. □

Clearly, every group described in Theorem 3 can be generated by two elements. Hence we obtain the following.

Corollary 4. *If every 2-generated subgroup of G is AS, then G is HAS.*

4. Absolutely solvable groups and M-groups

There are some similarities between the class of AS groups and that of M-groups. By Taketa's theorem [9] every M-group is solvable. Subgroups of M-groups need not be M-groups, moreover, every finite solvable group can be embedded into an M-group (Dade, see [1, Satz V.18.11]). An analogous statement is valid for AS groups [8, Theorem 2.2]. Also, both concepts are related to representation theory. However, this similarity seems to be rather superficial, the classes of AS groups and M-groups are scarcely related.

It is quite easy to construct M-groups that are not AS groups. Namely, let p and q be distinct primes such that q does not divide $p-1$. Let $k>1$ be the multiplicative order of p modulo q, then the elementary abelian group of order p^k has an automorphism of order q. Forming the semidirect product we obtain a group G of order p^kq that is clearly not AS. However, it is an M-group, since it has an abelian normal subgroup with supersolvable quotient group (see [1, Satz V.18.4]).

For the reverse direction, let P be a noncommutative group of order 5^3 and exponent 5. Then a Sylow 2-subgroup Q of $\mathrm{SL}(2,5) < \mathrm{GL}(2,5) \cong \mathrm{Aut}(P/\Phi(P))$ can be lifted to act on P, and we can form the semidirect product $G = PQ$. Here Q is isomorphic to the quaternion group of order 8, and its exponent 4 divides (actually equals) $5-1$. Hence it is clear by Theorem 1 that G is HAS. However, since Q acts trivially on the center of P, any faithful irreducible character (of degree 5) of P is invariant under the action of Q, hence it can be extended to an irreducible character of degree 5 of G (see [2, Corollary 6.28]). It is easy to check that G does not have any subgroup of index 5, so this character is not monomial, and G is not an M-group. In fact, G is a minimal non-M-group, see [4, Lemma 2.2(c)].

The only positive result we were able to prove works for HAS groups of odd order.

Theorem 5. *Every HAS group of odd order is an M-group.*

Proof. Let G be a minimal counterexample. All proper subgroups and quotient groups of G are also HAS, hence by our minimal choice of G they are M-groups. Since G itself is not an M-group we can find a non-monomial irreducible character χ of G. Now, by the minimality of G, we see that χ is faithful and primitive. A minimal noncentral normal subgroup of a primitive solvable linear group is the central product of a central cyclic subgroup and an extraspecial p-group for some prime p (see [6]), hence G has a chief factor of even dimension.

On the other hand, we are going to show that the dimension of every chief factor of G is odd, and this will be a contradiction. Since G is HAS, the exponent condition in Theorem 1 holds. Therefore, G induces absolutely irreducible linear groups on chief facors that have order coprime to the characteristic of the given chief factor. In this case the dimension agrees with the dimension of an absolutely irreducible representation of G in charactersitic 0, hence it divides the order of G, so it is an odd number, indeed. □

In the proof we used the fact that G is hereditary absolutely solvable. Can we relax this condition?

Question. Is it true that every AS group of odd order is an M-group?

5. Products of absolutely solvable groups

In this section we answer a question Hermann Heineken asked from Edit Szabó at the Groups St Andrews 1993 conference in Galway. Our original example was just one particular group of order $2^4 5^4$. However, the referee suggested that this example can be generalized, and moreover, he produced examples of odd order as well.

Example 6. *Let* $Q = \langle a, b \mid a^8 = 1, b^2 = a^4, b^{-1}ab = a^{-1}\rangle$ *be a generalized quaternion group of order* 16. *Let* p *be a prime number congruent to* 3 *or* 5 *modulo* 8 *and take a faithful irreducible* Q*-module* P *over the* p*-element field. Let* $G = PQ$ *be the semidirect product, and* $N_1 = P\langle a^2, b\rangle$, $N_2 = P\langle a^2, ab\rangle$. *Then* N_1 *and* N_2 *are normal subgroups in* G, $G = N_1N_2$, *both* N_1 *and* N_2 *are AS, but* G *is not AS.*

Proof. Both $\langle a^2, b\rangle$ and $\langle a^2, ab\rangle$ are quaternion groups of order 8, hence both N_1 and N_2 have index 2 in G, so they are normal subgroups, and $N_1N_2 = G$ obviously holds. It is easy to check that every irreducible representation of the 8-element quaternion group is absolutely irreducible, hence both N_1 and N_2 are AS. (If $p \equiv 5 \pmod 8$, then the exponent condition from Theorem 1 holds, so N_1 and N_2 are even HAS in this case.)

Every faithful absolutely irreducible Q-module is 2-dimensional, since Q has an abelian subgroup of index 2. However, if $p \equiv 3$ or $5 \pmod 8$, then $\mathrm{GL}(2, p)$ does not contain any subgroup isomorphic to Q, hence our Q-module P cannot be 2-dimensional, so it is not an absolutely irreducible Q-module, thus G is not AS. $\square$

Now let q be an odd prime. We consider the wreath product of the cyclic group of order q^2 with the cyclic group of order q. The base group of the wreath product is $\{ (x_1, x_2, \ldots, x_q) \mid x_i \in \mathbb{Z}_{q^2}\ (i = 1, \ldots, q)\}$ and the cyclic shift of the coordinates will be denoted by b. We take the following subgroups of the base group:

$$A = \{ (x_1, x_2, \ldots, x_q) \mid x_i \in \mathbb{Z}_{q^2},\, x_1 \equiv x_2 \equiv \cdots \equiv x_q \pmod q,\, \textstyle\sum x_i = 0\}$$

and

$$A_0 = \{ (x_1, x_2, \ldots, x_q) \mid x_i \in \mathbb{Z}_{q^2},\, x_1 \equiv x_2 \equiv \cdots \equiv x_q \equiv 0 \pmod q,\, \textstyle\sum x_i = 0\}.$$

Then A_0 is elementary abelian of order q^{q-1}, A is abelian of order q^q and it has exponent q^2. Clearly, A_0 and A are invariant under b. Furthermore, let us choose an $a \in A \setminus A_0$, then $A = \langle A_0, a\rangle$ and a simple computation shows that ab has order q. Let us define $Q = A\langle b\rangle$.

Example 7. *Let q be an odd prime, and take the group Q defined above. Let p be a prime such that $p-1$ is divisible by q but not by q^2, and take a faithful irreducible Q-module P over the p-element field. Let $G = PQ$ be the semidirect product, and $N_1 = P\langle A_0, b\rangle$, $N_2 = P\langle A_0, ab\rangle$. Then N_1 and N_2 are normal subgroups in G, $G = N_1N_2$, both N_1 and N_2 are HAS, but G is not AS.*

Proof. Since the center of Q is cyclic (generated by a^q), Q indeed has a faithful irreducible representation (see [2, Theorem 2.32(b)]).

Since both $\langle A_0, b\rangle$ and $\langle A_0, ab\rangle$ are maximal subgroups of Q, we clearly have that N_1 and N_2 are normal subgroups of G, and $N_1N_2 = G$. Since the (isomorphic) groups $\langle A_0, b\rangle$ and $\langle A_0, ab\rangle$ have exponent q, it follows from Theorem 1 that N_1 and N_2 are HAS.

Any faithful absolutely irreducible representation of Q has dimension q, since A is an abelian normal subgroup of index q in Q. Our assumption on p implies that the Sylow q-subgroup of $\mathrm{GL}(q, p)$ is the wreath product of two cyclic groups of order q. Now Q and this Sylow q-subgroup have the same order, but they are not isomorphic, since the unique abelian maximal subgroup in the wreath product of two cyclic groups of order q is elementary abelian, while the abelian maximal subgroup $A < Q$ is not elementary abelian. This implies that Q does not have a faithful irreducible representation of dimension q over the p-element field, hence P is not an absolutely irreducible Q-module, thus G is not AS. □

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