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Integral Formulas Related to Ovals

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Abstract. Using the notion of isoptics introduced and investigated in [1] and [2], we derive some new integral Cauchy-Crofton type formulas related to ovals in the plane.

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1. Introduction

In what follows we give a preview of certain facts concerning isoptics as the fundamental tool in our constructions (cf. [1] and [2]).

Definition 1.1. *A plane, closed, simple, positively oriented C^2-curve C of positive curvature is called an oval.*

We take a coordinate system with origin O in the interior of C. Let $p(t)$, $t \in [0, 2\pi]$, be the distance from O to the support line $l(t)$ of C perpendicular to the vector $e^{it} = \cos t + i \sin t$. It is well-known that the parametrization of C in terms of $p(t)$ is given by the formula

$$z(t) = p(t)e^{it} + p'(t)ie^{it},$$

where $ie^{it} = -\sin t + i\cos t$. Note that the support function p can be extended to a periodic function on $\mathbb{R}$ with the period 2π.

Now we want to define the notion of isoptics. To avoid confusion, we note that this notion is sometimes also used with different meanings in other fields and concepts, such as in classical illumination geometry (see, e.g., [7]) or in the theory of the light field (cf. [4]), for example describing so called isophotic families of area elements.

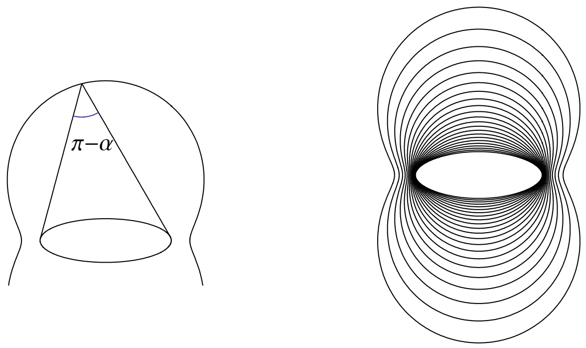

Figure 1. Construction of an isoptic to C and a few isoptics of an ellipse

Definition 1.2. *Let C_α be a locus of apices of a fixed angle $\pi-\alpha$, where $\alpha \in (0,\pi)$, formed by two support lines of the oval C. The curve C_α will be called an α-isoptic of C.*

It is convenient to parametrize the α-isoptic C_α by the same angle t so that the equation of C_α takes the form

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha)\right)ie^{it}.$$

Note that

$$z_\alpha(t) = z(t) + \lambda(t,\alpha)ie^{it} = z(t+\alpha) + \mu(t,\alpha)ie^{i(t+\alpha)}$$

and

$$z'_\alpha(t) = -\lambda(t,\alpha)e^{it} + \varrho(t,\alpha)ie^{it}$$

for suitable functions λ, μ and ϱ. Since $z_\alpha(t) = z(t+\alpha) + \mu(t,\alpha)ie^{i(t+\alpha)}$, then $\mu(t,\alpha)$ is negative for all values t and α. Moreover, we have

$$\lambda(t,\alpha) = \frac{p(t+\alpha) - p'(t)\sin\alpha - p(t)\cos\alpha}{\sin\alpha},$$
$$\varrho(t,\alpha) = \frac{p(t)\sin\alpha + p'(t+\alpha) - p'(t)\cos\alpha}{\sin\alpha},$$
$$\mu(t,\alpha) = \lambda(t,\alpha)\cos\alpha - \varrho(t,\alpha)\sin\alpha.$$

If we introduce the notation

$$q(t,\alpha) = z(t) - z(t+\alpha),$$

then there exists a smooth function $\varphi(t,\alpha)$ such that

$$\frac{q}{\|q\|}(t,\alpha) = e^{i\varphi(t,\alpha)}.$$

After some calculations we get

$$\frac{\partial \varphi}{\partial t}(t,\alpha) = \frac{[q,q']}{\|q\|^2}(t,\alpha) = \frac{\lambda(t,\alpha)R_\alpha(t) - \mu(t,\alpha)R(t)}{(\lambda^2(t,\alpha)+\varrho^2(t,\alpha))\sin\alpha} > 0,$$

where $[\ ,\ ]$ denotes the determinant of the arguments and $R(t)$ is the radius of curvature of C at t, and then again $R_\alpha(t) = R(t+\alpha)$. This formula states that for each α we have a smooth function $t = \varphi^{-1}(w,\alpha)$, and thus we can reparametrize the isoptics by using the mapping $(t,\alpha) = (\varphi^{-1}(w,\alpha),\alpha)$. In what follows we take

$$\tilde{z}_\alpha(w) = z_\alpha(\varphi^{-1}(w,\alpha)) = z(\varphi^{-1}(w,\alpha)) + \lambda(\varphi^{-1}(w,\alpha))ie^{i\varphi^{-1}(w,\alpha)}.$$

We will make use of this formula in the next section.

2. Cauchy-Crofton type formula

Let Ω be the exterior of an oval C. We define a mapping

$$F\colon (\varphi(0),\varphi(2\pi)) \times (0,\pi) \to \Omega \setminus \{\text{a certain support half-line}\},$$
$$F(w,\alpha) = \tilde{z}_\alpha(w).$$

The partial derivatives of F at (w,α) are given by

$$\frac{\partial F}{\partial w}(w,\alpha) = \left(\varrho(\varphi^{-1}(w,\alpha),\alpha)ie^{i\varphi^{-1}(w,\alpha)} - \lambda(\varphi^{-1}(w,\alpha),\alpha)e^{i\varphi^{-1}(w,\alpha)}\right)\frac{\partial \varphi^{-1}}{\partial w}(w,\alpha),$$
$$\frac{\partial F}{\partial \alpha}(w,\alpha) = -\lambda(\varphi^{-1}(w,\alpha),\alpha)\frac{\partial \varphi^{-1}}{\partial \alpha}(w,\alpha)e^{i\varphi^{-1}(w,\alpha)} - \frac{\mu(\varphi^{-1}(w,\alpha),\alpha)}{\sin\alpha}ie^{i\varphi^{-1}(w,\alpha)} +$$
$$+\varrho(\varphi^{-1}(w,\alpha),\alpha)\frac{\partial \varphi^{-1}}{\partial \alpha}(w,\alpha)ie^{i\varphi^{-1}(w,\alpha)}.$$

Hence the jacobian $\frac{\partial(\tilde{z}_\alpha(w))}{\partial(w,\alpha)}$ of F at (w,α) is equal to

$$\frac{\partial(\tilde{z}_\alpha(w))}{\partial(w,\alpha)} = \left[\frac{\partial F}{\partial w},\frac{\partial F}{\partial \alpha}\right](w,\alpha) = \frac{\mu(\varphi^{-1}(w,\alpha),\alpha)\lambda(\varphi^{-1}(w,\alpha),\alpha)}{\sin\alpha}\cdot\frac{\partial \varphi^{-1}}{\partial w}(w,\alpha) < 0.$$

But taking into consideration that

$$\frac{\partial \varphi^{-1}}{\partial w}(w,\alpha) = \frac{1}{\frac{\partial \varphi}{\partial t}(\varphi^{-1}(w,\alpha),\alpha)} = \frac{\|q\|^2}{[q,q']}(\varphi^{-1}(w,\alpha),\alpha),$$

we obtain

$$\frac{\partial(\tilde{z}_\alpha(w))}{\partial(w,\alpha)} = \frac{\mu\lambda}{\sin^2\alpha}\cdot\frac{\|q\|^2}{\lambda R_\alpha - \mu R}. \tag{2.1}$$

Theorem 2.1. *Under the above assumptions and with notation introduced in Figure 2 we have*

$$\iint\limits_{\Omega} \frac{\sin^2\omega}{h^2}\left(\frac{R_1}{t_1}+\frac{R_2}{t_2}\right)dxdy = 2\pi^2,$$

where R_1 and R_2 denote the curvature radii at the tangency points t_1 and t_2, respectively.

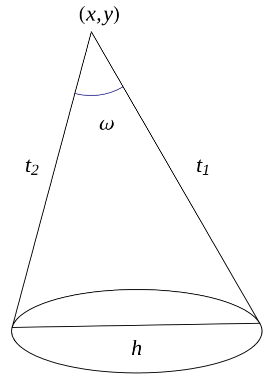

Figure 2. Notation to be used in Theorem 2.1

Proof. Using the formula for change of variables in multiple integrals for the mapping

$$(x, y) = F(w, \alpha)$$

and the formula (2.1) we get $t_1 = \lambda(\varphi^{-1}(w, \alpha), \alpha)$, $t_2 = -\mu(\varphi^{-1}(w, \alpha), \alpha)$, $\omega = \pi - \alpha$, $\|q(\varphi^{-1}(w, \alpha), \alpha)\| = h$, $R_1 = R$, $R_2 = R_\alpha$ and

$$\iint\limits_{\Omega} \frac{\sin^2 \omega}{h^2} \left(\frac{R_1}{t_1} + \frac{R_2}{t_2} \right) dxdy =$$

$$= \int\limits_0^\pi \int\limits_{\varphi(0)}^{\varphi(2\pi)} \frac{\sin^2 \alpha}{h^2} \left(\frac{R}{\lambda(\varphi^{-1}(w, \alpha), \alpha)} - \frac{R_\alpha}{\mu(\varphi^{-1}(w, \alpha), \alpha)} \right) \left| \frac{\partial(\tilde{z}_\alpha(w))}{\partial(w, \alpha)} \right| dwd\alpha =$$

$$= \int\limits_0^\pi \int\limits_{\varphi(0)}^{\varphi(2\pi)} dwd\alpha = \pi \left(\varphi(2\pi) - \varphi(0) \right) = 2\pi^2. \qquad \square$$

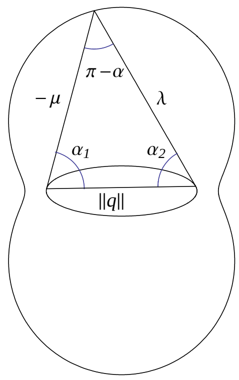

Figure 3. The sine theorem

At the end of this chapter we give an interesting form of the above formula. For this purpose we will need the sine theorem proved in [2].

Theorem 2.2. *With the notation introduced in Figure* 3*, the following identities hold:*

$$\frac{\|q\|}{\sin\alpha} = \frac{\lambda}{\sin\alpha_1} = \frac{-\mu}{\sin\alpha_2}.$$

Having suitably manipulated this formula, we get the following corollary.

Corollary 2.1. *Under the above assumptions and with the notation from Fig.* 2 *we have*

$$\iint_\Omega \left(\frac{R_1\sin^2\alpha_1}{t_1^3} + \frac{R_2\sin^2\alpha_2}{t_2^3}\right) dxdy = 2\pi^2,$$

where R_1 *and* R_2 *denote the curvature radii at the tangency points of* t_1 *and* t_2*, respectively.*

3. Some by-product formulas

Let us canonically associate a regular surface $\widetilde{C}$ in $\mathbb{R}^3$ with the oval C. This surface is given by a single parametrization $r(t,\alpha) = (q(t,\alpha),\alpha)$, for $t \in]0,2\pi[$, $\alpha \in]0,\pi[$. Note that the first fundamental form of $\widetilde{C}$ is nonzero in its domain, since

$$EG - F^2 = R_\alpha(t)^2 R^2(t)\sin^2\alpha + R^2(t) + R_\alpha^2(t) - 2R(t)R_\alpha(t)\cos\alpha > 0$$

for $t \in]0,2\pi[$, $\alpha \in]0,\pi[$.

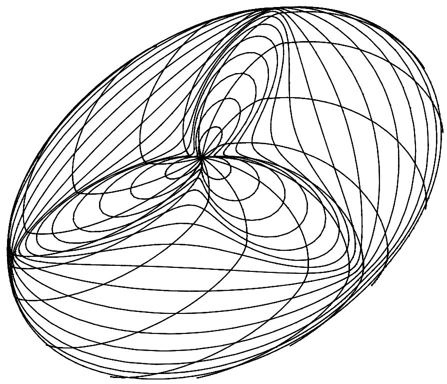

Figure 4. Surface $\widetilde{C}$ associated with a curve given by the support function $p(t) = 10 + \cos 3t$

Note that for a fixed oval C we have two natural quantities – the area $A(\widetilde{C})$ of $\widetilde{C}$ and its volume $V(\widetilde{C})$, which will be called the extended area and extended volume of the oval C.

Theorem 3.1. *Under the above assumptions and with the notation from Figure* 2 *we have*

$$\iint_\Omega \sin^2\omega \left(\frac{R_1}{t_1} + \frac{R_2}{t_2}\right) dxdy = 2V(\widetilde{C}),$$

where R_1 and R_2 denote the curvature radii at the tangency points of t_1 and t_2, respectively.

Proof. Let q_α denote the curve given by the equation

$$q_\alpha \colon]0, 2\pi[\to \mathbb{R}^2, \quad q_\alpha(t) = q(t, \alpha),$$

and let $A(q_\alpha)$ denote its area. Using two times the formula for change of variables in multiple integrals, we get

$$\iint\limits_{\Omega} \sin^2 \omega \left(\frac{R_1}{t_1} + \frac{R_2}{t_2} \right) dxdy = \int\limits_0^\pi \int\limits_{\varphi(0)}^{\varphi(2\pi)} \| q(\varphi^{-1}(x, \alpha), \alpha) \|^2 dw d\alpha =$$

$$= \int\limits_0^\pi \int\limits_0^{2\pi} [q, q'](t, \alpha) dt d\alpha = 2V(\widetilde{C}),$$

since

$$\int\limits_0^{2\pi} [q, q'](t, \alpha) dt = 2A(q_\alpha),$$

and from the elementary calculus it follows that

$$\int\limits_0^\pi A(q_\alpha) d\alpha = V(\widetilde{C}). \qquad \square$$

Finally we prove an integral formula involving the extended area of $\widetilde{C}$.

Theorem 3.2. *Under the above assumptions and the notation from Figure 2 we have*

$$\iint\limits_{\Omega} \frac{\sin \omega}{t_1 t_2} \sqrt{R_2^2 R_1^2 \sin^2 \alpha + R_1^2 + R_2^2(t) - 2R_1 R_2(t) \cos \alpha}\, dxdy = A(\widetilde{C}),$$

where R_1 and R_2 denote the curvature radii at the tangency points of t_1 and t_2, respectively.

Proof. This time we use the following substitution for change of variables in multiple integrals:

$$(x, y) = z_\alpha(t),$$

where by the formula (3.3) in [2] we have

$$\frac{\partial(z_\alpha(t))}{\partial(t, \alpha)} = \frac{\mu\lambda}{\sin \alpha} < 0.$$

Thus, using the formula for the area of a surface, we get

$$\iint_{\Omega} \frac{\sin\omega}{t_1 t_2}\sqrt{R_2^2 R_1^2 \sin^2\alpha + R_1^2 + R_2^2 - 2R_1 R_2 \cos\alpha}\, dxdy =$$

$$= \int_0^{\pi}\int_0^{2\pi} \sqrt{R_\alpha(t)^2 R^2(t)\sin^2\alpha + R^2(t) + R_\alpha^2(t) - 2R(t)R_\alpha(t)\cos\alpha}\, dtd\alpha =$$

$$= \int_0^{\pi}\int_0^{2\pi} \sqrt{EG - F^2} dtd\alpha = A(\widetilde{C}). \qquad \square$$

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