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The Fundamental Group for the Complement of Cayley's Singularities*

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Abstract. Given a singular surface X, one can extract information on it by investigating the fundamental group $\pi_1(X - Sing_X)$. However, calculation of this group is non-trivial, but it can be simplified if a certain invariant of the branch curve of X – called the braid monodromy factorization – is known. This paper shows, taking the Cayley cubic as an example, how this fundamental group can be computed by using braid monodromy techniques ([18]) and their liftings. This is one of the first examples that uses these techniques to calculate this sort of fundamental group.

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1. On the Cayley cubic

The classification of singular cubic surfaces in $\mathbb{CP}^3$ was done in the 1860's, by Schläfli [22] and Cayley [7]. Surface XVI in Cayley's classification is now called Cayley cubic, and when embedded in $\mathbb{CP}^3$, it is defined by the following equation

$$4(X^3+Y^3+Z^3+W^3)-(X+Y+Z+W)^3=0,$$

and has four singularities, which are ordinary double points. Cayley noticed that this surface is the unique cubic surface having four ordinary double points, which is the maximum possible number of double points for a cubic surface (see, for example, Salmon's book [21]). In [12], one can find an important information about the Cayely cubic, such as the fact that it contains exactly nine lines, six of which connect the four nodes pairwise and the other three of which are coplanar. Note that the Cayley cubic is invariant under the symmetric group Sym_4.

Let us denote by C this surface and by $Sing_C = Sing$ the set of the four nodes. We are interested in the fundamental group of the complement of the set of singularities in the Cayley cubic $\pi_1(C-Sing)$. A direct computation of this group is elementary. Consider the smooth del-Pezzo surface S_6 of degree 6. The Cremona involution Cr is the regular automorphism of S_6 and has 4 fixed points, which we denote as $\mathrm{Fix}(Cr)$. The factor S_6/Cr is the Cayley cubic; singular points of the Cayley cubic are images of fixed points of Cr on S_6. Therefore, the universal cover of $(C-Sing)$ is $(S_6-\mathrm{Fix}(Cr))$, and hence the fundamental group $\pi_1(C-Sing)$ is $\mathbb{Z}/2\mathbb{Z}$.

However, for a general singular surface X in $\mathbb{CP}^3$ there is no general method for computing the fundamental group $\pi_1(X-Sing_X)$. We present here two other different approaches for this problem, demonstrating them on the Cayley cubic. We compute first the braid monodromy factorization of the branch curve of the Cayley cubic in $\mathbb{CP}^2$, based on the braid monodromy techniques of Moishezon-Teicher ([18], [19]). We then apply two methods in order to compute this fundamental group. The first method consists of lifting the factorization to a factorization in the mapping class group, from which we can find the desired group. The second method is based on [16] and finds the fundamental group using the Reidemeister-Schreier method ([15]).

The paper is divided as follows. In Section 2 we compute the braid monodromy factorization of the branch curve of the Cayley cubic C and the fundamental group of the complement of the branch curve, using a degeneration of the surface into a union of three planes. In Section 3 the fundamental group of the Cayley surface minus the singular points is computed using the results from Section 2.

2. The factorization Δ_6^2 and the fundamental group $\pi_1(\mathbb{CP}^2-\bar{S})$

In this section we give the braid monodromy factorization of the branch curve S in $\mathbb{C}^2$. We also present the fundamental groups $\pi_1(\mathbb{C}^2-S)$ and $\pi_1(\mathbb{CP}^2-\bar{S})$.

We begin with a few basic notations. Let $\bar{S}$ be a branch curve of the surface C in $\mathbb{CP}^2$, and let l_∞ be a line in $\mathbb{CP}^2$, transverse to $\bar{S}$. We introduce $S=\bar{S}-\bar{S}\cap l_\infty$.

Take a projection $\pi : \mathbb{CP}^2 \to \mathbb{CP}^1$. Let $\pi_{aff} : \mathbb{C}^2 \to \mathbb{C}$ be the affine generic projection. Then there is a finite subset of points $Z \subset \mathbb{C}$, which is the projection on $\mathbb{C}$ of the nodes and cusps of S and the branch points of $\pi_{aff}|_S$. Above each point of $Z \subset \mathbb{C}$ there is just one singular point of $\pi_{aff}|_S$. Let $\{\delta_i\}$ be a basis of non-intersecting loops in $\mathbb{C} - Z$ around each point of Z, starting from $x_0 \in \mathbb{C} - Z$.

Now we use the theorem of Zariski [24]: Let $z \in Z$ and δ be a loop in $\mathbb{C} - Z$ around z. Then there is a *braid monodromy* action $\varphi : \pi_1(\mathbb{C} - Z, x_0) \to B_n[\mathbb{C}_{x_0}, \mathbb{C}_{x_0} \cap S]$, s.t. $\mathbb{C}_{x_0}$ is the fiber of π_{aff} over x_0 and B_n is the braid group.

We define a halftwist Z_{ij} as the exchange of the positions of two points i and j, which occurs as follows: we take a tubular neighborhood of a path which connects i and j below the x-axis, then we let i and j rotate in a counterclockwise manner along the boundary of this neighborhood, until they exchange their position.

Notation 1. We denote by Z_{ij} (resp. $\bar{Z}_{ij}$) the counterclockwise halftwist of i and j below (resp. above) the axis. $Z^2_{i'j}$ is a fulltwist of i' around j. $Z^2_{i',jj'} = Z^2_{i'j'} Z^2_{i'j}$ is the fulltwist of i' around j and j'. In a similar way, we define also $Z^2_{ii',jj'} = Z^2_{i',jj'} Z^2_{i,jj'}$ and $Z^3_{i',jj'} = Z^3_{i'j} Z^3_{i'j'} (Z^3_{i'j})^{Z^2_{jj'}}$.

In the case when the singular point above z is a branch point of π_{aff}, a node, a cusp of S, or a point of tangency of two branches of the curve, then $\varphi(\delta) = H^\epsilon$, where H is a halftwist and $\epsilon = 1, 2, 3, 4$ (respectively).

When z can be given locally as an intersection on m lines, then $\varphi(\delta) = \Delta^2_m$, when Δ^2_m is a 360 degree rotation of the m points in the fiber.

More details for explicit computations and technical methods appear in [18] and [19].

Definition 2. *The braid monodromy w.r.t. S, π, u is the following factorization*

$$\Delta^2_S = \prod_i \varphi(\delta_i).$$

Remark 3. Let Δ^2 be the generator of the center of the braid group $B_n[D, K]$. Then, by a theorem of Artin (see [18]), $\Delta^2 = \prod \varphi(\delta_i)$. Note that Δ^2 is a 360 degree rotation of the disc D.

Denote the Cayley cubic as C and the set of the four nodes as $Sing$. We aim to compute the fundamental group $\pi_1(C - \mathrm{Sing})$. In general, the branch curve of a generic projection onto $\mathbb{CP}^2$ of a surface is singular and quite complicated. Therefore we wish to start from a more basic curve and do some "regeneration" in order to recover information on the branch curve.

Recall that another equation of the Cayley cubic can also be given by the following

$$XYZ + W(XY + XZ + YZ) = 0$$

(which is the original equation given in [7]). Looking at the following deformation

$$\mathbb{CP}^3 \times [0,1] \to [0,1], \quad \{XYZ + t \cdot W(XY + XZ + YZ) = 0\} \mapsto t.$$

We get a degeneration of the Cayley cubic into a union of three planes meeting at a point. The branch curve of the degenerated surface is an arrangement of three lines meeting at one point, one of which is set to be the "diagonal line".

The next step is to apply the regeneration process. The regeneration methods are actually, locally, the reverse process of the degeneration method. When regenerating a singular configuration consisting of lines and conics, the final stage in the regeneration process involves doubling each line, so that each point of K corresponding to a line labelled i is replaced by a pair of points, labelled i and i'. The purpose of the regeneration rules is to explain how the braid monodromy behaves when lines are doubled in this manner. We denote by $Z_{i,j} = H(z_{i,j})$ where $z_{i,j}$ is a path connecting points in K.

The rules are (see [20, pp. 336–337]):

1. **First regeneration rule**: The regeneration of a branch point of any conic – any branch point regenerates into two branch points:
   A factor of the braid monodromy of the form $Z_{i,j}$ is replaced in the regeneration by $Z_{i',j} \cdot \overset{(j)}{\underline{Z}}_{i,j'}$.
2. **Second regeneration rule**: The regeneration of a node – any node regenerates into two (or four) nodes:
   A factor of the form Z^2_{ij} is replaced by a factorized expression $Z^2_{ii',j} := Z^2_{i'j} \cdot Z^2_{ij}$, $Z^2_{i,jj'} := Z^2_{ij'} \cdot Z^2_{ij}$ or by $Z^2_{ii',jj'} := Z^2_{i'j'} \cdot Z^2_{ij'} Z^2_{i'j} \cdot Z^2_{ij}$.
3. **Third regeneration rule**: The regeneration of a tangent point – any tangent point regenerates into three cusps:
   A factor of the form Z^4_{ij} in the braid monodromy factorized expression is replaced by $Z^3_{i,jj'} := (Z^3_{ij})^{Z_{jj'}} \cdot (Z^3_{ij}) \cdot (Z^3_{ij})^{Z^{-1}_{jj'}}$.

The initial braid monodromy factorization of the degenerated surface is Δ^2_3. We first regenerate the "diagonal line" to a smooth conic which is tangent to the two other lines (see [20, Lemma 1]), as depicted in Figure 1. The braid monodromy factorization of this arrangement is

$$(Z^2_{1,3})^{Z^{-1}_{2',3}} \cdot Z_{2,2'}^{Z^{-2}_{1,2}\bar{Z}^2_{2,3}} \cdot (Z^4_{2,3})^{Z^2_{2,2'}} \cdot Z^4_{1,2'} \cdot Z_{2,2'}.$$

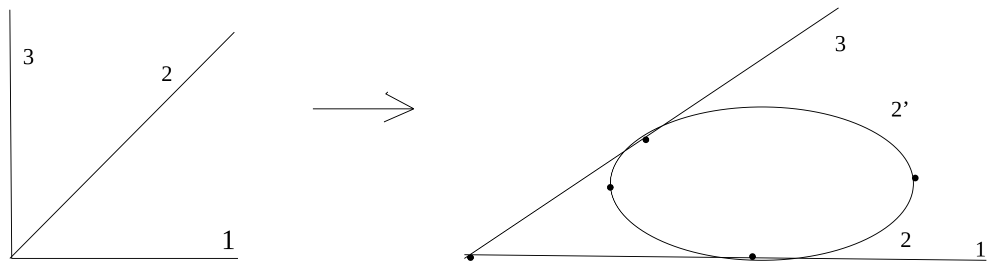

Figure 1.

We now regenerate the remaining two lines. By the second regeneration rule, the node is regenerated into four nodes, and each tangency point regenerates into three cusps. For example, the regeneration in a neighborhood of the node is depicted in Figure 2.

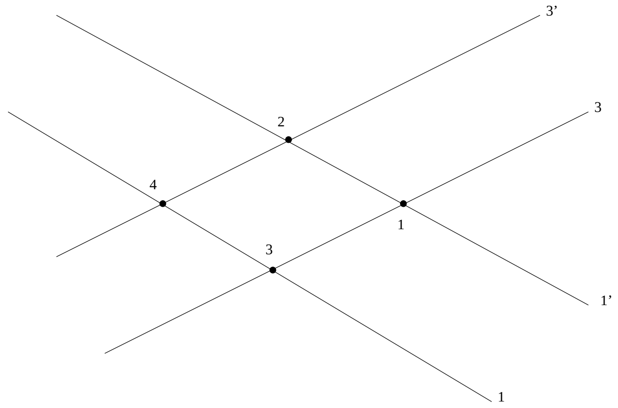

Figure 2.

We end up with a curve $\tilde{S}$ which has 6 cusps, 4 nodes and 2 branch points. The resulting factorization is

$$\begin{aligned}\widetilde{\Delta} = \ & (Z^2_{1'\,3})^{Z^{-2}_{2',3\,3'}} \cdot (Z^2_{1'\,3'})^{Z^2_{1'\,3} Z^{-2}_{2',3\,3'}} \cdot (Z^2_{1\,3})^{Z^{-2}_{2',3\,3'}} \cdot (Z^2_{1\,3'})^{Z^2_{1\,3} Z^{-2}_{2',3\,3'}} \\ & \cdot (Z_{2\,2'})^{Z^{-2}_{1\,1',2} \bar{Z}^2_{2,3\,3'}} \cdot (Z^3_{2,3\,3'})^{Z^2_{2\,2'}} \cdot Z^3_{1\,1',2'} \cdot Z_{2\,2'}.\end{aligned} \tag{1}$$

However, the resulting factorization $\widetilde{\Delta}$ is not a braid monodromy factorization of the branch curve S, due to the existence of extra branch points. Let D a disc in $\mathbb{C}_{x_0}$ containing all the points in the fiber. Define the forgetting homomorphisms:

$$1 \le i \le 3 \ f_i : B_6[D, \{1, 1', 2, 2', 3, 3'\}] \to B_2[D, \{i, i'\}].$$

It is clear that if $\widetilde{\Delta}$ was a BMF, then $\forall\, i, \deg(f_i(\widetilde{\Delta})) = 2$, by Remark 3. However, this is not the case in the current situation. It was proven in [13] (see also [14]), that if $\deg(f_i(\widetilde{\Delta})) = k < 2$, then there are $(2-k)$ extra branch points, and so there is a contribution of the factorization $\prod\limits_{m=1}^{2-k} Z_{i,i'}$ to $\widetilde{\Delta}$ (by contribution we mean that we multiply $\widetilde{\Delta}$ from the right by these $Z_{i,i'}$'s).

It is easy to see that $\deg(f_2(\widetilde{\Delta})) = 2$. However, we have the following

Lemma 4. $\deg(f_1(\widetilde{\Delta})) = \deg(f_3(\widetilde{\Delta})) = 1$.

Proof. We prove the lemma only for f_1; the proof for f_3 is identical. The braids coming from the nodes are sent by f_1 to Id, and also the braids $(Z_{2\,2'})^{Z^{-2}_{1\,1',2} \bar{Z}^2_{2,3\,3'}}$, $(Z^3_{2,3\,3'})^{Z^2_{2\,2'}}$, $Z_{2\,2'}$. By [20, Lemma 2, (i)], we see that $\deg(f_1(Z^3_{1\,1',2})) = 1$. □

Multiplying $\widetilde{\Delta}$ from the right by $Z_{1,1'} \cdot Z_{3,3'}$ we get a factorization of a curve with four nodes, six cusps and four branch points, which is the branch curve of the Cayley surface C.

Theorem 5. *The braid monodromy factorization of S is given in* (2) *and its factors are represented by paths in Figure* 3.

$$\begin{aligned} \Delta_S^2 = \ & (Z^2_{1'\,3})^{Z^{-2}_{2',3\,3'}} \cdot (Z^2_{1'\,3'})^{Z^2_{1'\,3} Z^{-2}_{2',3\,3'}} \cdot (Z^2_{1\,3})^{Z^{-2}_{2',3\,3'}} \cdot (Z^2_{1\,3'})^{Z^2_{1\,3} Z^{-2}_{2',3\,3'}} \\ & \cdot (Z_{2\,2'})^{Z^{-2}_{1\,1',2} \bar{Z}^2_{2,3\,3'}} \cdot (Z^3_{2,3\,3'})^{Z^2_{2\,2'}} \cdot Z^3_{1\,1',2'} \cdot Z_{2\,2'} \cdot Z_{1,1'} \cdot Z_{3,3'}. \end{aligned} \tag{2}$$

Note that the first, the fourth and the last two paths correspond to braids of branch points. The second and third ones correspond to braids of cusps and the rest correspond to braids of nodes.

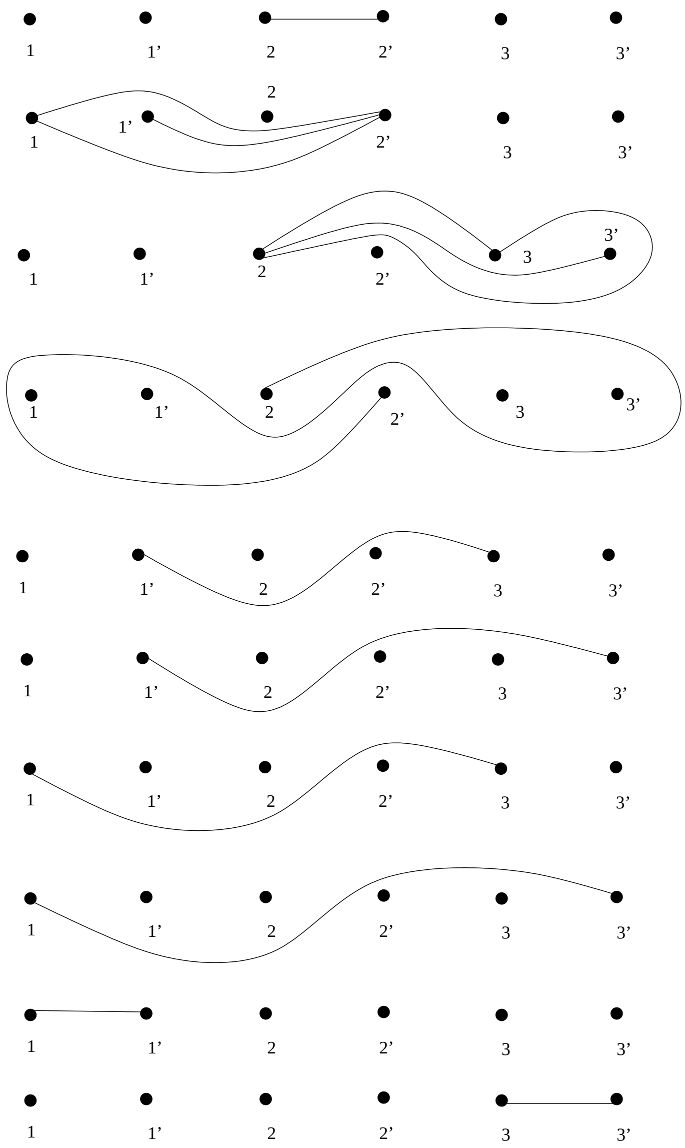

Figure 3.

2.1. The fundamental group $\pi_1(\mathbb{C}^2 - S)$

The van Kampen theorem [23] states that there is a "good" geometric base $\{\gamma_j\}$ of $\pi_1(\mathbb{C}_{x_0} - S \cap \mathbb{C}_{x_0}, *)$ (where $\mathbb{C}_{x_0}$ is the fiber of the projection $\pi|_{aff}$ above x_0), such that the group $\pi_1(\mathbb{C}^2 - S, *)$ is generated by the images of $\{\gamma_j\}$ in $\pi_1(\mathbb{C}^2 - S, *)$ with the following relations: $\varphi(\delta_i)\ \gamma_j = \gamma_j\ \forall i, j$. We recall that

$$\pi_1(\mathbb{CP}^2 - \bar{S}) \simeq \pi_1(\mathbb{C}^2 - S)/ < \prod \gamma_j > .$$

Recall that S has only branch points, nodes and cusps (when the cusp is locally defined by the equation $y^2 = x^3$). Denote by a and b the two branches of S at a neighborhood of a singular point, and γ_a, γ_b be two non-intersecting loops in $\pi_1(\mathbb{C}_{x_0} - S \cap \mathbb{C}_{x_0}, *)$ around the intersection points of the branches with the fiber $\mathbb{C}_{x_0}$ (constructed by cutting each of the paths and creating two loops, which proceed along the two parts and encircle a and b). Then by the van Kampen theorem, we have the relation $< \gamma_a, \gamma_b >= \gamma_a\gamma_b\gamma_a\gamma_b^{-1}\gamma_a^{-1}\gamma_b^{-1} = 1$ for cusp, the relation $[\gamma_a, \gamma_b] = \gamma_a\gamma_b\gamma_a^{-1}\gamma_b^{-1} = 1$ for node, the relation $\gamma_a = \gamma_b$ for branch point.

Theorem 6. *The fundamental group $\pi_1(\mathbb{C}^2 - S)$ is generated by $\gamma_1, \gamma_2, \gamma_3$ subject to the relations*

$$\begin{aligned} \langle \gamma_1, \gamma_2 \rangle &= e &(3)\\ \langle \gamma_2, \gamma_3 \rangle &= e &(4)\\ [\gamma_2, \gamma_1^2\gamma_3^2] &= e &(5)\\ [\gamma_1, \gamma_2^{-1}\gamma_3\gamma_2] &= e. &(6) \end{aligned}$$

The group $\pi_1(\mathbb{CP}^2 - \bar{S})$ has relations (3), (4), (6) *and an additional relation*

$$\gamma_3^2\gamma_2^2\gamma_1^2 = e. \qquad (7)$$

Proof. By the above explanation and Figure 3, we have the following relations

$$\begin{aligned} \gamma_2 &= \gamma_{2'} &(8)\\ \langle \gamma_1, \gamma_{2'} \rangle = \langle \gamma_{1'}, \gamma_{2'} \rangle = \langle \gamma_{1'}\gamma_1\gamma_{1'}^{-1}, \gamma_{2'} \rangle &= e &(9)\\ \langle \gamma_{2'}\gamma_2\gamma_{2'}^{-1}, \gamma_3 \rangle = \langle \gamma_{2'}\gamma_2\gamma_{2'}^{-1}, \gamma_{3'} \rangle = \langle \gamma_{2'}\gamma_2\gamma_{2'}^{-1}, \gamma_{3'}\gamma_3\gamma_{3'}^{-1} \rangle &= e &(10)\\ \gamma_1^{-1}\gamma_{1'}^{-1}\gamma_{2'}^{-1}\gamma_{3'}\gamma_3\gamma_{2'}\gamma_2\gamma_{2'}^{-1}\gamma_3^{-1}\gamma_{3'}^{-1}\gamma_{2'}\gamma_{1'}\gamma_1 &= \gamma_{2'} &(11)\\ [\gamma_1, \gamma_{2'}^{-1}\gamma_3\gamma_{2'}] = [\gamma_{1'}, \gamma_{2'}^{-1}\gamma_3^{-1}\gamma_{3'}\gamma_3\gamma_{2'}] &= e &(12)\\ [\gamma_1, \gamma_{2'}^{-1}\gamma_3\gamma_{2'}] = [\gamma_1, \gamma_{2'}^{-1}\gamma_3^{-1}\gamma_{3'}\gamma_3\gamma_{2'}] &= e &(13)\\ \gamma_1 &= \gamma_{1'} &(14)\\ \gamma_3 &= \gamma_{3'}. &(15) \end{aligned}$$

We want to simplify this presentation. By (8) and (14), relation (9) gets the form (3). By (8) and (15), relation (10) gets the form (4). Relation (11) is transformed (by (3), (8), (14) and (15)) to (5), and relations (12) and (13) are transformed (by (8), (14) and (15)) to (6).

In order to get the group $\pi_1(\mathbb{CP}^2 - \bar{S})$, we add the projective relation $\gamma_{3'}\gamma_3\gamma_{2'}\gamma_2\gamma_{1'}\gamma_1 = e$, which is transformed to $\gamma_3^2\gamma_2^2\gamma_1^2 = e$. Therefore relation (5) is omitted and $\pi_1(\mathbb{CP}^2 - \bar{S})$ is generated by $\gamma_1, \gamma_2, \gamma_3$ with relations (3), (4), (6) and (7).

We note that if we consider the relations which are derived from the complex conjugates of the braids of Figure 3, we gain no new relations, therefore the presentation is complete. □

Remark 7. Since we deal with a singular cubic surface, we note that a result of Zariski [24] for a smooth cubic surface in $\mathbb{CP}^3$ was generalized by Moishezon [17] for any degree. Let $\bar{S}_n$ (resp. S_n) be the branch curve of a smooth surface in $\mathbb{CP}^3$ in $\mathbb{CP}^2$ (resp. $\mathbb{C}^2$). Moishezon proved that

$$\pi_1(\mathbb{C}^2 - S_n) \cong B_n \quad \text{and} \quad \pi_1(\mathbb{CP}^2 - \bar{S}_n) \cong B_n/\mathrm{Center}(B_n)$$

3. Finding the fundamental group $\pi_1(C - Sing)$

In this section we give two different ways to find the fundamental group of the complement of the singular points in the Cayley surface.

3.1. Using a lifting to the mapping class group

The projection π defines a pencil of lines on $\mathbb{CP}^2$. Considering the preimages of these lines under the projection of C onto $\mathbb{CP}^2$, we obtain a pencil of elliptic curves on C, intersecting transversely at the base locus, namely three smooth points (the preimages in C of the pole of the projection π). This pencil has eight nodal fibers, of which four pass through the singular points of C and the four others pass through the preimages of the branch points of S with respect to the projection π. The monodromy of this fibration can be encoded by a factorization in a mapping class group, which can be obtained from the braid monodromy of S by a simple lifting algorithm. See Section 5.2 of [6] and Section 3.3 of [5] for details.

Among the various factors of (2), those corresponding to cusps of S (i.e. $Z^3_{1\,1',2'}$ and $(Z^3_{2,3\,3'})^{Z^2_{2\,2'}}$) lie in the kernel of the lifting homomorphisms and do not contribute to the monodromy of the elliptic pencil. This is because the preimage of the fiber of π through a cusp of S is actually a smooth elliptic curve.

To determine the lifts of the other factors, we view the fiber E of the elliptic pencil as a triple cover of a line in $\mathbb{CP}^2$ (the reference fiber of π on which the braid monodromy acts) branched at six points (the points where S intersects the considered line), which we label $1, 1', 2, 2', 3, 3'$ as before. Each of these branch points corresponds to a simple ramification, i.e. involving only two of the three sheets of the covering. Looking at the combinatorics of the degeneration of C to a union of three planes, we can label these sheets by elements of $\{1, 2, 3\}$ in such a way that the monodromy of the triple cover maps γ_1 and $\gamma_{1'}$ to the transposition (23), γ_2 and $\gamma_{2'}$ to (13), and γ_3 and $\gamma_{3'}$ to (12).

The lifting homomorphism (see [6, 5]) maps the halftwist $Z_{1\,1'}$ to a positive Dehn twist along the simple closed loop on E formed by the two lifts of the supporting arc of the halftwist in sheets 2 and 3 of the covering. Similarly for the other halftwists appearing in the braid monodromy factorization. Because the halftwists $Z_{i\,i'}$ ($1 \leq i \leq 3$) have disjoint supporting arcs, the corresponding Dehn twists $\tau_{i\,i'}$ also have disjoint supporting loops; moreover it is easy to check that these loops are homotopically non-trivial (see below). Since disjoint non-trivial simple closed loops on an elliptic curve are homotopic, the Dehn twists $\tau_{i\,i'}$ correspond to mutually homotopic vanishing cycles, and represent the same element in the mapping class group $\mathrm{Map}_1 = SL(2,\mathbb{Z})$; we call $\alpha \in \pi_1(E)$ the homotopy class of these vanishing cycles, and τ_α the corresponding Dehn twist. Next we observe that the support of the halftwist $t = (Z_{2\,2'})^{Z^{-2}_{1\,1',2}\bar{Z}^2_{2,3\,3'}}$ (Figure 3, fourth line) is also disjoint from those of $Z_{1\,1'}$ and $Z_{3\,3'}$, which indicates that the corresponding vanishing cycle again represents the homotopy class α in $\pi_1(E)$.

For a generic projection of a smooth surface, the nodes of the branch curve correspond to smooth fibers of the pencil, and the corresponding braid monodromies lie in the kernel of the lifting homomorphism. However, in our case the four nodes of the branch curve correspond to nodal fibers of the pencil; the corresponding braid monodromies are squares of liftable halftwists, which lift to the squares of the corresponding Dehn twists.

We first consider $\nu = (Z_{1'\,3})^{Z^{-2}_{2',3\,3'}}$ (Figure 3, fifth line): the supporting arc of this halftwist intersects that of $Z_{1\,1'}$ only once, at the common endpoint $1'$. Hence, the double lifts of the supporting arcs (which give the supporting loops of the corresponding Dehn twists) intersect transversely exactly once (at the branch point which lies above $1'$). Calling $\beta \in \pi_1(E)$ the homotopy class of the vanishing cycle corresponding to the lift of $(Z_{1'\,3})^{Z^{-2}_{2',3\,3'}}$, the intersection number $\alpha \cdot \beta = 1$ implies that α and β form a basis of $\pi_1(E) \simeq \mathbb{Z}^2$ (and confirms that the vanishing cycles are indeed not homotopically trivial as claimed above). The same argument could have been used considering $Z_{3\,3'}$ (whose support intersects that of ν once at the common end point 3), or $Z_{2\,2'}$ or t instead of $Z_{1\,1'}$ (in that case the supporting arcs intersect transversely once at interior points, but their double lifts each live in only two of the three sheets of the covering, and it is easy to check that the supporting loops of the corresponding Dehn twists intersect transversely once). In any case, we conclude that the braid monodromy factor ν^2 lifts to τ_β^2, the square of the positive Dehn twist about a loop in the homotopy class β.

Finally, the three other nodes of S ($(Z^2_{1'\,3'})^{Z^2_{1'\,3}Z^{-2}_{2',3\,3'}}$, $(Z^2_{1\,3})^{Z^{-2}_{2',3\,3'}}$ and $(Z^2_{1\,3'})^{Z^2_{1\,3}Z^{-2}_{2',3\,3'}}$) correspond to the conjugates of ν^2 by the braids $Z^{-1}_{3\,3'}$, $Z^{-1}_{1\,1'}$, and $Z^{-1}_{1\,1'}Z^{-1}_{3\,3'}$ respectively (Figure 3, sixth, seventh, eighth lines). Applying the lifting homomorphism, we obtain that the corresponding mapping class group elements are respectively $(\tau_\beta^2)^{\tau_\alpha^{-1}} = \tau^2_{\beta-\alpha}$, $(\tau_\beta^2)^{\tau_\alpha^{-1}} = \tau^2_{\beta-\alpha}$, and $(\tau_\beta^2)^{\tau_\alpha^{-2}} = \tau^2_{\beta-2\alpha}$. In other words, the vanishing cycles represent respectively the homotopy classes $\beta-\alpha$, $\beta-\alpha$, and $\beta-2\alpha$.

In conclusion, the mapping class group monodromy factorization of the elliptic pencil (in Map_1) is

$$\mathrm{Id} = \tau_\alpha \cdot \tau_\alpha \cdot \tau_\beta^2 \cdot \tau^2_{\beta-\alpha} \cdot \tau^2_{\beta-\alpha} \cdot \tau^2_{\beta-2\alpha} \cdot \tau_\alpha \cdot \tau_\alpha. \tag{16}$$

As a verification, one can consider the isomorphism $\mathrm{Map}_1 \simeq SL(2,\mathbb{Z})$ given by the action on $H_1(E,\mathbb{Z})$, working in the basis $\{\alpha,\beta\}$: then, recalling that the action of a Dehn twist on homology is given by $[\tau_\delta(\gamma)] = [\gamma] + ([\delta]\cdot[\gamma])[\delta]$, we have

$$\tau_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \tau_{\beta-\alpha} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_{\beta-2\alpha} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix},$$

and the identity (16) indeed holds (recalling that our products are written in the braid order, i.e. with composition from left to right, while the usual product of matrices is a composition from right to left).

Let us recall the quasi-projective Lefschetz Hyperplane Section Theorem: Let $X := Y - Z$, (of dimension d) where Y is an algebraic subset of the complex projective space $\mathbb{CP}^n$, $n \geq 2$, and where Z is an algebraic subset of Y. Let L be a projective hyperplane which is in generic position with respect to X. If X is nonsingular, then the natural maps

$$H_q(L \cap X) \to H_q(X) \quad \text{and} \quad \pi_q(L \cap X, *) \to \pi_q(X)$$

are bijective for $0 \leq q \leq d-2$ and surjective for $d-1$ (see [11] and [10]). In our case X is the Cayley cubic C minus the singular locus. The above generically chosen central projection $\pi : \mathbb{CP}^2 - P_0 \to \mathbb{CP}^1$ (where P_0 is the center of the projection) defines a pencil of lines in $\mathbb{CP}^2$ which lifts to a generic pencil of planes in $\mathbb{CP}^3$ whose axis is a line M. The above curve E is then the intersection of C with a generic member L of the pencil of planes. It follows that the natural map

$$\pi_1((C - Sing) \cap L) = \pi_1(E) \to \pi_1(C - Sing)$$

is a surjection. In particular, the fundamental group of C is abelian, so we can work with homology groups instead of fundamental groups. It follows that we are left to determine the kernel of the natural map

$$\varphi : H_1(E) \to \pi_1(C - Sing).$$

Let L_i be the exceptional hyperplanes of the above pencil (the planes for which $L_i \cap (C - Sing)$ are not isotopic to E). In [8], Cheniot defines homological variation operators

$$\mathrm{var}_{i,q} : H_q(X \cap L, M \cap X) \to H_q(X \cap L), \quad i \in I,$$

by patching each relative cycle on $X \cap L$ modulo $M \cap X$ with its transform by monodromy around the exceptional lines. It is then shown in [8], that

$$\mathrm{Kernel}(\varphi) = \sum_{i \in I} \mathrm{Im}(\mathrm{var}_{i,q}).$$

If we choose a basis $\{\alpha,\beta\}$ of $H_1(E) \simeq H_1(E, M \cap X)$ as above, then the image of $\mathrm{var}_{i,q}$ is nothing else as the image of the lifted braid, viewed as an element in the mapping class group Map_1. (This can be seen by unravelling the definitions of Cheniot and the above construction of the mapping class group factorization, see also [9], Section 2). It follows that in our case $\pi_1(X) = \pi_1(C - Sing)$ is isomorphic to the quotient of $\pi_1(E) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ by the relations $\gamma = \rho_*(\gamma)$ for every $\gamma \in \pi_1(E)$ and for every factor ρ in the mapping class group factorization (16). The relation $\tau_\alpha(\beta) = \beta$ implies that $[\alpha]$ is trivial in $\pi_1(C - Sing)$, while the relation $\tau_\beta^2(\alpha) = \alpha$ implies that $2[\beta]$ is trivial; hence, $\pi_1(C - Sing)$ is a quotient of $\mathbb{Z}/2$. It follows from the monodromy factorization given in formula (16) that for every element ρ of the monodromy subgroup, the image of $\rho - \mathrm{Id}$ is in the span of α and 2β. Therefore, we get no further relations, and we recover that $\pi_1(C - Sing) = \mathbb{Z}/2$.

3.2. Using the RMS method

We find $\pi_1(C - Sing)$ by a second method, using the Reidemeister-Schreier algorithm. We follow the method proposed in [16]. We recall this method briefly.

Denote by $Gr^*_{d,n}$ the set of the graphs with d labelled vertices and n labelled edges. Assuming we have a homomorphism $f : \pi_1(\mathbb{C}^2 - S) \to Sym_d$ and let $\gamma_1, \dots, \gamma_n$ be generators of $\pi_1(\mathbb{C}^2 - S)$. Denote by g the homomorphism from the free group with n generators $F_n = \langle \bar{\gamma}_1, \dots, \bar{\gamma}_n \rangle$ to $\pi_1(\mathbb{C}^2 - S)$ such that $g(\bar{\gamma}_i) = \gamma_i$, $\forall\, i = 1, \dots, n$.

Assume we have a homomorphism $\bar{f} : F_n \to Sym_d$ such that $\bar{f}(\bar{\gamma}_i)$ is a transposition $\forall\, i = 1, \dots, n$. So we can associate to it a graph $\Gamma_{\bar{f}} \in Gr^*_{d,n}$ in the following way. If $\bar{f}(\bar{\gamma}_i) = (h, k)$, then the edge i will have the vertices h and k. Given a monodromy map $f : \pi_1(\mathbb{C}^2 - S) \to Sym_d$, the monodromy graph associated to it is the graph $\Gamma = \Gamma_{\bar{f}}$, where $\bar{f}$ is the lifting of f to F_n under the map g.

In order to compute $\pi_1(C - Sing)$, let us consider the projection $\pi|_{C-D} : C - D \to \mathbb{C}^2 - S$, where $D = \pi^{-1}(S)$ and $D = 2R + F$ (R is the ramification locus of π). As this is an unramified cover, we can identify $\pi_1(C - D)$ with the subgroup of $\pi_1(\mathbb{C}^2 - S)$ given by those elements γ such that γ stabilizes a vertex of Γ (i.e. the γ's such that $f(\gamma)(j) = j$ for a fixed vertex j of Γ).

Explicitly, taking a base point in $C - D$ to be the preimage of the base point of $\mathbb{C}^2 - S$ lying in the sheet labelled 1, then a loop g in $\mathbb{C}^2 - S$ lifts to an arc in $C - D$, whose other end point is the preimage in the sheet $f(g)(1)$; hence we obtain a closed loop in $C - D$ if and only if $f(g)$ maps 1 to 1.

Let us fix a numeration on Γ, and let Γ' be a maximal subtree. By abuse of notation, let us denote by γ_i the edges of Γ.

Definition 8. *A sequence $c = (k_j)_{j=1,\dots,l}$ of distinct edges of Γ such that the edge k_i intersects the edge k_{i+1} only in a single vertex is called a chain of Γ (of length l). A p-chain is a chain with p as a starting vertex. A p,q-chain is a chain with p as a starting vertex and q as an ending vertex.*

If $c = (k_j)_{j=1,\dots,l}$ is a 1-chain in Γ', then set $\gamma_c = \gamma_{k_1} \cdots \gamma_{k_l}$, and if c is the trivial 1-chain, then set $\gamma_c = id$. The set of all $\sigma \in Sym_d$ such that $\sigma = f(\gamma_c)$ for c a 1-chain in Γ' is a complete set of representatives for left cosets of the stabilizer of the vertex 1 in Sym_d : if c is a $1, q$-chain in Γ' then $f(\gamma_c)(1) = q$.

So in order to calculate $\pi_1(C-D)$, we apply the Reidemeister-Schreier method ([15]) to the Schreier set $RS = \{\gamma_c | \, c \text{ is a 1-chain in } \Gamma'\}$ to get the following proposition [16, Proposition 6.1]:

Proposition 9. *$\pi_1(C-D)$ is generated by $\eta_{c,k} = \gamma_c\gamma_k(\overline{\gamma_c\gamma_k})^{-1}$, where c is a 1-chain in Γ', k an edge of Γ (such that $c \cup \{k\}$ is not a 1-chain in Γ'), and is defined by the relators $\gamma_c R \gamma_c^{-1}$ (written in terms of η's), where c is a 1-chain in Γ', R is a relator of $\pi_1(\mathbb{C}^2 - S)$.*

As the proof of this proposition describes, in order to obtain a presentation for $\pi_1(C - Sing)$, we must quotient by the normal subgroup generated by all loops around the components $D = 2R + F$. The loops around the components of R are those $\eta_{c,k}$ with k equal to the last edge of c and loops $\eta_{c,k}\eta_{c',k}$ in case k is an edge of Γ such that $c \cup \{k\}$ is not a 1-chain in Γ', and $\gamma_{c'} = \overline{\gamma_c\gamma_k} \in RS$; while the loops around the components of F are those $\eta_{c,k}$ with k an edge which does not pass through the ending vertex of c.

Using this proposition, we can find $\pi_1(C - Sing)$ in our case. Recall that the monodromy maps γ_1 to the transposition $(2,3)$, γ_2 to $(1,3)$ and γ_3 to $(1,2)$. In this way, we create the map $f : \pi_1(\mathbb{C}^2 - S) \to Sym_3$, and thus we can associate to it the graph $\Gamma = \Gamma_{\bar{f}}$ in Figure 4.

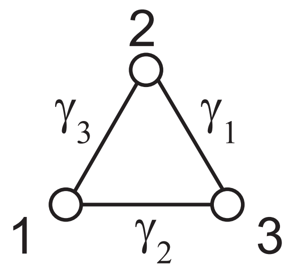

Figure 4. The graph Γ

Denote by Γ' the maximal subtree composed of the edges γ_1 and γ_3 (and the vertices $\{1,2,3\}$). Let RS $= \{Id, \gamma_3, \gamma_3\gamma_1\}$ be the Schreier set of the stabilizer of the vertex 1. Let us denote $\gamma_{c_1} = Id, \gamma_{c_2} = \gamma_3, \gamma_{c_3} = \gamma_3\gamma_1$, $k_i = i$, and let $\eta_{i,j} = \gamma_{c_i}\gamma_{k_j}(\overline{\gamma_{c_i}\gamma_{k_j}})^{-1}$ be the generators of $\pi_1(C - D)$.

We get the following generators:

$\eta_{1,1} = Id \cdot \gamma_1(\overline{Id \cdot \gamma_1})^{-1} = \gamma_1$
$\eta_{1,2} = Id \cdot \gamma_2(\overline{Id \cdot \gamma_2})^{-1} = \gamma_2\gamma_1^{-1}\gamma_3^{-1}$
$\eta_{1,3} = Id \cdot \gamma_3(\overline{Id \cdot \gamma_3})^{-1} = \gamma_3\gamma_3^{-1} = Id$
$\eta_{2,1} = \gamma_3 \cdot \gamma_1(\overline{\gamma_3 \cdot \gamma_1})^{-1} = \gamma_3\gamma_1\gamma_1^{-1}\gamma_3^{-1} = Id$
$\eta_{2,2} = \gamma_3 \cdot \gamma_2(\overline{\gamma_3 \cdot \gamma_2})^{-1} = \gamma_3\gamma_2\gamma_3^{-1}$
$\eta_{2,3} = \gamma_3 \cdot \gamma_3(\overline{\gamma_3 \cdot \gamma_3})^{-1} = \gamma_3^2$
$\eta_{3,1} = \gamma_3\gamma_1 \cdot \gamma_1(\overline{\gamma_3\gamma_1 \cdot \gamma_1})^{-1} = \gamma_3\gamma_1^2\gamma_3^{-1}$
$\eta_{3,2} = \gamma_3\gamma_1 \cdot \gamma_2(\overline{\gamma_3\gamma_1 \cdot \gamma_2})^{-1} = \gamma_3\gamma_1\gamma_2$
$\eta_{3,3} = \gamma_3\gamma_1 \cdot \gamma_3(\overline{\gamma_3\gamma_1 \cdot \gamma_3})^{-1} = \gamma_3\gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1}$.

Thus we have 7 generators; to find $\pi_1(C - Sing)$ we have to eliminate all loops around D. That is – we quotient by the generators $\eta_{1,1}, \eta_{2,2}, \eta_{2,3}, \eta_{3,1}, \eta_{3,3}$, leaving us with $\eta_{1,2}$ and $\eta_{3,2}$. However, $\eta_{1,2} \cdot \eta_{3,2} = \gamma_2^2$, which also represents a loop around D, and thus $\gamma_2^2 = Id$. Thus, $\pi_1(C - Sing)$ is generated by one elements – γ_2 – and isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

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