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Embeddings of Projective Klingenberg Planes in the Projective Space $\mathsf{PG}(5, \mathbb{K})$

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Abstract. In this paper embeddings of projective Klingenberg planes in a 5-dimensional projective space are classified. It is proved that if a PK-plane is fully embedded in $\mathsf{PG}(5, \mathbb{K})$, for some skewfield $\mathbb{K}$, then it is either isomorphic to the Desarguesian projective Klingenberg plane (projective Hjelmslev plane for bijective σ) $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}, \sigma))$ over a ring of ordinary or twisted dual numbers or it is a subgeometry of an ordinary projective plane. As a consequence we have in the finite case that, if a projective Klingenberg plane of order (qt, t) is embedded in PG(5,q), then it is a projective Hjelmslev plane $\mathsf{PH}(2, \mathbb{D}(q, \sigma))$ over a ring of ordinary or twisted dual numbers over the Galois field $\mathsf{GF}(q)$. The embeddings related to the twisted case are new.

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1. Introduction

Projective Klingenberg and Hjelmslev planes are natural generalizations of ordinary projective planes. After having been studied intensively in the eighties (see e.g. [7] and [15]) those ring geometries were somewhat forgotten, until they made their comeback in the theory of linear codes over finite rings (see [6] and [11]). This revival was one motivation for us to study embeddings of Klingenberg planes. Only one result, proved by Artmann [1], about the embedding of a class of Desarguesian Klingenberg planes was known. On the other hand embeddings of other point-line geometries such as generalized polygons were studied thoroughly (for a survey see [13] and [14]). Embeddings have helped in understanding the corresponding geometries, and have also directly and indirectly influenced the coding theory that emerged from the geometries. As an example we mention the codes arising from quadrics embedded in projective space, and the codes arising from geometric hyperplanes of hexagons (and the latter arise from embeddings!), see [4]. Hence, studying embeddings of Hjelmslev planes is certainly a worthwhile job.

Another motivating reason for writing this paper comes from a characterization theorem of Cronheim [3]. He proved that the only finite uniform Desarguesian projective Hjelmslev planes are the planes over rings of twisted dual numbers over a Galois field and the planes over Witt rings of length two over a Galois field. He also gives a characterization of both classes in terms of the automorphism group. We prove that the planes over the twisted dual numbers are the only ones that can be embedded in a 5-dimensional projective space, giving a new geometric characterization of this class of planes, and hence of the corresponding class of rings.

The paper is organized as follows. In Section 2 we give some basic definitions about Klingenberg and Hjelmslev planes needed in our main theorem. Section 3 gives an explicit description of the classical embedding of the plane $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}, \sigma))$ over the ring of twisted dual numbers over a skewfield. Our description is more explicit (using coordinates) and more general than the result of Artmann (the latter is only valid for the non-twisted case). New embeddings are contained in this description for proper PK-planes which are not PH-planes. Section 4 formulates the main theorem: the classification of all (full) embeddings of PK-planes in $\mathsf{PG}(5, \mathbb{K})$. As a consequence we obtain in the finite case a characterization of the PH-planes over the rings of twisted dual numbers over a Galois field. Finally, in Section 5, the main theorem is proved in a series of lemmas.

2. Definitions and preliminaries

Definition 2.1. *A projective Klingenberg plane (PK-plane) $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \in, \sim)$ is a point-line incidence structure with neighbor relation $\sim = (\sim_\mathcal{P}, \sim_\mathcal{L})$ satisfying the following three axioms :*

(PK1) *There exists an epimorphism ϕ from $\mathcal{S}$ onto a projective plane $\overline{\mathcal{S}}$ such that $\phi(p) = \phi(q)$ if and only if $p \sim_\mathcal{P} q$ for all $p, q \in \mathcal{P}$ and $\phi(L) = \phi(M)$*

if and only if $L \sim_{\mathcal{L}} M$ *for all* $L, M \in \mathcal{L}$.

(PK2) *Two non-neighboring points are incident with exactly one common line.*

(PK3) *Two non-neighboring lines are incident with exactly one common point.*

Any ordinary projective plane is a PK-plane with the epimorphism ϕ the identity map and with the neighbor relations $\sim_{\mathcal{P}}$ and $\sim_{\mathcal{L}}$ the trivial equality relations. A PK-plane is called *proper*, if it is not a projective plane.

Definition 2.2. *A projective Hjelmslev plane (PH-plane)* $\mathcal{H}$ *is a projective Klingenberg plane with two additional axioms concerning the behaviour of neighboring elements. More precisely* $\mathcal{H}$ *is a PH-plane if:*

(PH1) $\mathcal{H}$ *is a PK-plane.*

(PH2) *Two neighboring points are incident with at least two distinct common lines.*

(PH3) *Two neighboring lines are incident with at least two distinct common points.*

Projective Klingenberg and Hjelmslev planes were introduced by Wilhelm Klingenberg in [9] and [10].

Next we pay attention to finite PK- and PH-planes. Let $\mathcal{S}$ be a finite PK-plane. Then there exists a unique pair (s,t) of non-zero integers such that for any flag (p, L) of $\mathcal{S}$ there are exactly t points on L neighboring with p and exactly s points on L not neighboring with p. The pair (s,t) is called the *order* of $\mathcal{S}$.

In a finite PK-plane of order (s,t) the following holds : $|\mathcal{P}| = s^2 + st + t^2$, $|\mathcal{L}| = s^2 + st + t^2$, any line is incident with $s+t$ points, any point is incident with $s+t$ lines, any point has t^2 neighbors, any line has t^2 neighbors, $t|s$ and $r = \frac{s}{t}$ is the order of the projective plane $\overline{\mathcal{S}}$ and $s \leq t^2$ or $t = 1$ (see [5] and [8]). The PK-planes of order $(s, 1)$ are the ordinary projective planes of order s.

The first examples of projective Klingenberg and Hjelmslev planes given by Klingenberg in [9] are constructed in an algebraic manner and they are now called *Desarguesian* PK- (viz. PH-planes). We recall here briefly this construction.

For a local ring R with unique maximal ideal J the incidence structure $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is defined as follows.

The points are the triples $(x, y, z) \in R \times R \times R$ up to a right scalar which is a unit in R and with $(x, y, z) \notin J \times J \times J$. The lines are the triples $[u, v, w] \in R \times R \times R$ up to a left scalar which is a unit in R and with $[u, v, w] \notin J \times J \times J$. The point represented by (x, y, z) is incident with the line represented by $[u, v, w]$ if and only if $u \cdot x + v \cdot y + w \cdot z = 0$. Finally, two points, represented by (x, y, z) and (x', y', z') are neighbors if and only if $(x', y', z') - (x, y, z)\lambda \in J \times J \times J$ for some $\lambda \in R \setminus J$ and similarly for lines.

The projective ring plane $\mathcal{S}$ defined in this way is a PK-plane (with the epimorphism from $\mathcal{S}$ onto a projective plane $\overline{\mathcal{S}}$ induced by the natural mapping from the local ring R onto its residue skewfield $\overline{R} = R/J$) and $\mathcal{S}$ is denoted by $\mathsf{PK}(2,R)$. If R is finite, the plane $\mathsf{PK}(2,R)$ has order (s,t) with $s = |R|$ and $t = |J|$.

Two additional properties of the local ring make the plane $\mathsf{PK}(2, R)$ a projective Hjelmslev plane. Indeed, if R is a left and right chain ring and if every nonunit is a left and right zero divisor in R, then neighboring points (lines) are incident with at least two lines (points). A local ring which is a left and right chain ring and whose maximal ideal consists of two sided zero divisors is called a *Hjelmslev ring* or *H-ring*. In the finite case the maximal ideal always consists of two sided zero divisors. Hence, a finite chain ring is always a H-ring.

An important class of H-rings are the so-called *twisted dual numbers over a skewfield.*

Let $\mathbb{K}$ be a skewfield and σ an automorphism of $\mathbb{K}$. Then the ring of σ-dual numbers over $\mathbb{K}$ is defined as the set $\mathbb{K} \times \mathbb{K}$ with addition $(a + bt) + (c + dt) = (a + c) + (b + d)t$ and multiplication $(a + bt) \cdot (c + dt) = ac + (ad + bc^\sigma)t$.

It is easy to see that this is an H-ring with unique maximal ideal $J = \mathbb{K}t$ satisfying $J^2 = (0)$ and we use the notation $\mathbb{D}(\mathbb{K}, \sigma)$ for this ring of *twisted dual numbers.* If $\mathbb{K}$ is the finite field $\mathsf{GF}(q)$, the rings are denoted $\mathbb{D}(q, \sigma)$.

For σ the identity automorphism one obtains the well-known ring of dual numbers $\mathbb{D}(\mathbb{K})$ over $\mathbb{K}$.

If σ is in the above definition an endomorphism, but not an automorphism, then we obtain a left chain ring that is not a right chain ring. With abuse of notation, we will denote the corresponding projective Klingenberg plane by $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}, \sigma))$, although it is not a Hjelmslev plane.

3. The classical embedding of $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}, \sigma))$

In [1] B. Artmann shows that the PH-plane $\mathsf{PH}(2, \mathbb{K}[t]/t^n)$ over the ring of polynomials with coefficients in the field $\mathbb{K}$ modulo t^n can be embedded in the $(3n-1)$-dimensional projective space over $\mathbb{K}$. As a special case he obtains an embedding of the projective Hjelmslev plane $\mathsf{PH}(2, \mathbb{K}[t]/t^2) \cong \mathsf{PH}(2, \mathbb{D}(\mathbb{K}))$ over the ring of dual numbers over the field $\mathbb{K}$ in the projective space $\mathsf{PG}(5,\mathbb{K})$.

For an explicit description of this embedding, we make use of an adaptation of the embedding given by Thas in [12] for the more general case of projective planes over full matrix rings $M_n(\mathsf{GF}(q))$. From that embedding one easily derives an embedding for finite PH-planes over non-twisted dual numbers $\mathbb{D}(q) = \{a + bt \mid a, b \in \mathsf{GF}(q)\}$ as this ring can be identified with the subring

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathsf{GF}(q) \right\}$$

of the full matrix ring $M_2(\mathsf{GF}(q))$.

In fact, it is easy to see that the construction also works in the infinite case, and so we present it in full generality for general dual numbers $\mathbb{D}(\mathbb{K})$ over a field or even a skewfield $\mathbb{K}$.

The embedding α goes as follows. Any point of $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}))$ represented by $(x_0 + x_1t, y_0 + y_1t, z_0 + z_1t)$ is mapped by α to the line of $\mathsf{PG}(5, \mathbb{K})$ through the points represented by $(x_0, 0, y_0, 0, z_0, 0)$ and $(x_1, x_0, y_1, y_0, z_1, z_0)$. These two 6-tuples represent indeed two distinct points of $\mathsf{PG}(5, \mathbb{K})$ since $(x_0, y_0, z_0) \neq (0, 0, 0)$

and it is easy to see that the line $\alpha(p)$ is independent of the choice of the representative triple for p. The representatives of neighboring points can always be chosen as $(x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t)$ and $(x_0 + x'_1 t, y_0 + y'_1 t, z_0 + z'_1 t)$. Hence if p and q are neighboring, then the corresponding lines p^α and q^α have the point $(x_0, 0, y_0, 0, z_0, 0)$ in common.

Now we look at the lines. Any line of $\mathsf{PH}(2, \mathbb{K})$ represented by $[u_0 + u_1 t, v_0 + v_1 t, w_0 + w_1 t]$ is mapped to the 3-space of $\mathsf{PG}(5, \mathbb{K})$ which is the intersection of the two hyperplanes $[0, u_0, 0, v_0, 0, w_0]$ and $[u_0, u_1, v_0, v_1, w_0, w_1]$. Again the two 6-tuples represent two distinct hyperplanes of $\mathsf{PG}(5, \mathbb{K})$ since $(u_0, v_0, w_0) \neq (0, 0, 0)$ and the 3-space L^α is independent of the choice of the representative triple for L. The 3-spaces corresponding to neighboring lines (with chosen representatives $[u_0 + u_1 t, v_0 + v_1 t, w_0 + w_1 t]$ and $[u_0 + u'_1 t, v_0 + v'_1 t, w_0 + w'_1 t]$) are contained in the same hyperplane $[0, u_0, 0, v_0, 0, w_0]$.

The incidence relation in the PH-plane corresponds with the natural incidence in $\mathsf{PG}(5, \mathbb{K})$.

The embedding described above is called the *classical embedding* of $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}))$ in $\mathsf{PG}(5, \mathbb{K})$.

We can generalize this embedding to the case of twisted dual numbers $\mathbb{D}(\mathbb{K}, \sigma)$ by defining the image of a point represented by $(x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t)$ as the line through the points represented by $(x_0, 0, y_0, 0, z_0, 0)$ and $(x_1, x_0^\sigma, y_1, y_0^\sigma, z_1, z_0^\sigma)$ and the image of a line represented by $[u_0 + u_1 t, v_0 + v_1 t, w_0 + w_1 t]$ as the 3-space which is the intersection of the hyperplanes represented by $[0, u_0^\sigma, 0, v_0^\sigma, 0, w_0^\sigma]$ and $[u_0, u_1, v_0, v_1, w_0, w_1]$. Here, σ can be non-bijective.

We also call this embedding the *classical embedding* of $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}, \sigma))$ in $\mathsf{PG}(5, \mathbb{K})$.

We mention that the same kind of embedding for the projective line over $\mathbb{D}(\mathbb{K}, \sigma)$ can be found as Example 5.4 in [2].

4. Main result

In the preceding section we saw how the PH-plane over the (twisted) dual numbers over the skewfield $\mathbb{K}$ can be classically embedded in $\mathsf{PG}(5, \mathbb{K})$. In this section we characterize this embedding under some natural hypotheses.

Definition 4.1. *Consider a projective Klingenberg plane $\mathcal{S}$ with point set $\mathcal{P}$ and line set $\mathcal{L}$ possessing a proper epimorphism ϕ onto a projective plane $\overline{\mathcal{S}}$. Let $\mathbb{K}$ be a skewfield and let $\mathsf{PG}(5, \mathbb{K})$ be the 5-dimensional projective space over $\mathbb{K}$ with line set Π and set of solids Σ (a solid is a 3-dimensional subspace). Let α be a map from $\mathcal{P}$ to Π and from $\mathcal{L}$ to Σ satisfying the following properties.*

(PE1) *For $x \in \mathcal{P}$ and $L \in \mathcal{L}$, we have $x \in L$ if and only of $x^\alpha \in L^\alpha$.*

(PE2) *For $x, y \in \mathcal{P}$, we have that $x \sim y$ if and only if x^α meets y^α nontrivially; for $L, M \in \mathcal{L}$, we have that $L \sim M$ if and only if L^α and M^α are contained in a hyperplane.*

Then we call α an embedding.

By Lemma 5.5 below, there is a natural embedding of $\overline{\mathcal{S}}$ into a plane π of $\mathsf{PG}(5,\mathbb{K})$. If this embedding is *full*, i.e. if all points and lines of π are images of points and lines of $\overline{\mathcal{S}}$, then we call α *full*. In the next section, we will prove the following theorem.

Theorem 4.2. *If $\mathcal{S}$ is a projective Klingenberg plane with a natural proper epimorphism onto a projective plane $\overline{\mathcal{S}}$ and if $\mathcal{S}$ is fully embedded in $\mathsf{PG}(5,\mathbb{K})$, for some skewfield $\mathbb{K}$, then either*

(i) *$\mathcal{S}$ is a projective Hjelmslev (or Klingenberg for non-bijective σ) plane $\mathsf{PH}(2,\mathbb{D}(\mathbb{K},\sigma))$ over a ring of twisted dual numbers and the embedding is the classical one, or*

(ii) *$\mathcal{S}$ is a subgeometry of a projective plane (and in this case two distinct lines always meet in at most one point, and, dually, every two distinct points are joined by at most one line).*

This has the following consequence in the finite case.

Corollary 4.3. *If $\mathcal{S}$ is a finite projective Klingenberg plane of order (qt,t), for some natural numbers q,t, and if $\mathcal{S}$ is embedded in $\mathsf{PG}(5,q)$, then $\mathcal{S}$ is a projective Hjelmslev plane $\mathsf{PH}(2,\mathbb{D}(q,\sigma))$ over a ring of twisted dual numbers and the embedding is the classical one.*

5. Proof of the main result

We prove the theorem in a series of lemmas. Throughout we assume that $\mathcal{S}$ is a projective Klingenberg plane with point set $\mathcal{P}$ and line set $\mathcal{L}$, and with a natural proper epimorphism ϕ onto a projective plane $\overline{\mathcal{S}}$. We assume that $\mathcal{S}$ is embedded in $\mathsf{PG}(5,\mathbb{K})$, for some skewfield $\mathbb{K}$.

Lemma 5.1. *Let x_1,x_2,x_3 be three distinct points of $\mathcal{S}$ with $x_1 \sim x_2 \sim x_3$. Then x_1^α, x_2^α and x_3^α all contain the same point z.*

Proof. Since there are at least 3 lines through x_1^ϕ in $\overline{\mathcal{S}}$, and since the lines of $\mathcal{S}$ through both x_1 and x_2, and through both x_1 and x_3, determine at most two line neighborhood classes, we can select a line $L \in \mathcal{L}$ incident with x_1 and not incident with either x_2 or x_3. Let $x_4 \in \mathcal{P}$ be such that $x_1 \sim x_4$ and x_4 is incident with L. Let π be the plane determined by x_1^α and x_4^α. Note that $\pi \subseteq L^\alpha$. Since neither x_2 nor x_3 are incident with L, none of x_2^α, x_3^α is contained in π. But since both x_2^α, x_3^α must meet both x_1^α and x_4^α, we deduce that $x_1^\alpha, x_2^\alpha, x_3^\alpha, x_4^\alpha$ all meet in the same point z. $\square$

Lemma 5.1 easily implies that all lines of $\mathsf{PG}(5,\mathbb{K})$ that are the image under α of points of the same point neighborhood N, meet in a unique point, that we may denote as N^α. Dually, every line of a fixed line neighborhood M maps under α to a solid contained in a fixed hyperplane M^α.

Lemma 5.2. *If N_1, N_2, N_3 are three distinct point neighborhoods of $\mathcal{S}$, then N_1^ϕ, N_2^ϕ, N_3^ϕ are collinear if and only if $N_1^\alpha, N_2^\alpha, N_3^\alpha$ are collinear. Hence all images under α of the point neighborhoods are contained in a same plane $\overline{\pi}$ and there is a natural monomorphism $\varphi_p : \overline{\mathcal{S}} \to \overline{\pi}$.*

Proof. Suppose first that $N_1^\phi, N_2^\phi, N_3^\phi$ are collinear, and assume, by way of contradiction, that $N_1^\alpha, N_2^\alpha, N_3^\alpha$ are not collinear, say they span the plane π. Then for every line L of $\mathcal{S}$ contained in the neighborhood M of lines that meet N_1, N_2, N_3 nontrivially, we see that $\pi \subseteq L^\alpha \subseteq M^\alpha$. This implies that, whenever a point $x \in N_1$ is incident with at least two elements of M (and every such point is!), then it is incident with all members of M, a contradiction.

In order to show the converse, we first claim that it suffices to prove that not all N^α, with N running through the set of point neighborhoods, are incident with a common line of $\mathsf{PG}(5, \mathbb{K})$. Indeed, suppose this is true, and suppose that N_1, N_2, N_3 are three point neighborhoods with $N_1^\alpha, N_2^\alpha, N_3^\alpha$ on a common line K of $\mathsf{PG}(5, \mathbb{K})$, but with $N_1^\phi, N_2^\phi, N_3^\phi$ not collinear. Let N be an arbitrary point neighborhood. Then choose any point neighborhood $N' \neq N$ and the line joining N^ϕ and N'^ϕ meets the union of the three lines determined by $N_1^\phi, N_2^\phi, N_3^\phi$ in at least two different points $N^{*\phi}$ and $N^{**\phi}$. By the previous paragraph, both $N^{*\alpha}$ and $N^{**\alpha}$ are incident with K, and hence, again by the previous paragraph, so is N^α, which shows our claim. So suppose by way of contradiction that all N^α are incident with a common line K. Then clearly, for every line $L \in \mathcal{L}$, the image L^α contains K. Let $x \in \mathcal{P}$ be arbitrary and let $L_1, L_2 \in \mathcal{L}$ both be incident with x and such that $L_1 \not\sim L_2$. Then L_1^α and L_2^α span $\mathsf{PG}(5, \mathbb{K})$ and hence their intersection is precisely K. This means that $x^\alpha = K$, contradicting injectivity of α on $\mathcal{P}$. $\square$

We now prove two easy properties of the plane $\overline{\pi}$.

Lemma 5.3. *Let $\overline{\pi}$ be the plane defined in Lemma 5.2. Then for every point $x \in \mathcal{P}$, the line x^α meets $\overline{\pi}$ in a unique point.*

Proof. Let x be any point of $\mathcal{S}$, and suppose that it belongs to the point neighborhood class N. Then N^α belongs to the intersection of $\overline{\pi}$ and x^α. Hence we only must show that x^α is not contained in $\overline{\pi}$. Suppose, by way of contradiction, it is. Choose two points y, z of $\mathcal{S}$ not in N such that the lines xy and xz are not neighboring. Since y and z do not belong to N, the lines y^α and z^α do not meet x^α. Hence these lines meet $\overline{\pi}$ in unique points and it follows that both $(xy)^\alpha$ and $(xz)^\alpha$ contain the plane $\overline{\pi}$, contradicting the fact that xy and xz are not neighboring. $\square$

Lemma 5.4. *Let $\overline{\pi}$ be the plane defined in Lemma 5.2. Then for every line $L \in \mathcal{L}$, the solid L^α meets $\overline{\pi}$ in a line.*

Proof. Clearly L^α contains a line of $\overline{\pi}$. Suppose, by way of contradiction, that $\overline{\pi}$ is contained in L^α. Select a point $x \in \mathcal{P}$ on L, and a line $L' \in \mathcal{L}$ not neighboring L, but incident with x. The solid L'^α contains a line of $\overline{\pi}$, and it contains x^α (and these two lines are distinct by Lemma 5.3). Hence L^α and L'^α share a plane, a contradiction. $\square$

Let π be a plane of $\mathsf{PG}(5,\mathbb{K})$. With the *dual* of π, we mean the projective plane obtained from π by considering as points all solids through π, and as lines all hyperplanes through π.

Dually to Lemma 5.2 one proves that the hyperplanes M^α, with M running through all line neighborhood classes, are lines of a dual plane $\overline{\pi}'$ of $\mathsf{PG}(5,\mathbb{K})$, and we denote the corresponding natural monomorphism by φ_l. Note also that, for any line neighborhood class M, the hyperplane M^α is generated by all lines x^α, with x running through the points of $\mathcal{S}$ incident with a member of M.

So with every point neighborhood class N of $\mathcal{S}$ corresponds a point N^α in the plane $\overline{\pi}$, and also a solid through the plane $\overline{\pi}'$ (which is the intersection of all hyperplanes M^α, with M running through the set of line neighborhood classes such that M^ϕ is incident with N^ϕ). We denote that solid by $N^{\alpha *}$. Likewise, for a line neighborhood class M, there is a line $M^{\alpha *}$ of $\mathsf{PG}(5,\mathbb{K})$ spanned by the points N^α, with N running through the point neighborhood classes such that N^ϕ is incident with M^ϕ.

Note that $M^{\alpha *} \subseteq M^\alpha$, for M a line neighborhood, and $N^\alpha \subseteq N^{\alpha *}$, for N any point neighborhood.

The condition on the embedding of being "full" now precisely means that the monomorphism φ_p is an isomorphism. However, for the time being, we do not assume this extra condition yet.

Next we prove that both planes $\overline{\pi}$ and $\overline{\pi}'$ either are disjoint, or coincide.

A *digon* is a pair of distinct lines, each one incident with a pair of distinct points.

Lemma 5.5. *The planes $\overline{\pi}$ and $\overline{\pi}'$ either are disjoint, or coincide. Also, as soon as $\mathcal{S}$ contains a digon, the planes are disjoint.*

Proof. We first show that, if $\mathcal{S}$ contains two lines L, L' that meet in at least two points x, x', then $\overline{\pi}$ and $\overline{\pi}'$ are not disjoint. Suppose, by way of contradiction, that they are disjoint.

Let R be the intersection in $\mathsf{PG}(5,\mathbb{K})$ of L^α and $\overline{\pi}$. By Lemma 5.4, R is a line, which is clearly also contained in L'^α (as L and L' are neighboring). Since x, x' are incident with L, L' in $\mathcal{S}$, we also have that x^α and x'^α are contained in L^α and L'^α; hence the plane $L^\alpha \cap L'^\alpha$ contains the line R and the points $z := x^\alpha \cap \overline{\pi}'$ and $z' := x'\alpha \cap \overline{\pi}'$ (the latter are indeed points by the dual of Lemma 5.4). If $z \neq z'$, then the line R, contained in $\overline{\pi}$, meets the line zz' in a point of $\overline{\pi}'$, contradicting our hypothesis. Hence $z = z'$. But since x^α and x'^α also share a point in $\overline{\pi}$, this easily implies $x^\alpha = x'^\alpha$, and hence $x = x'$.

Now we assume that $\overline{\pi}$ and $\overline{\pi}'$ are different, but not disjoint. Notice that $\overline{\pi}'$ is the intersection of any three hyperplanes M_i^α, $i = 1,2,3$, where M_1, M_2, M_3 are three line neighborhood classes with non-concurrent epimorphic images under ϕ. But we can choose M_i, $i = 1,2,3$, in such a way that none of $M_1^{\alpha *}, M_2^{\alpha *}, M_3^{\alpha *}$ contains the intersection $\overline{\pi} \cap \overline{\pi}'$. Since M_i^α contains both $\overline{\pi}'$ and $M_i^{\alpha *}$, it then follows that M_i^α contains $\overline{\pi}$ and the proof is complete. $\square$

We now first treat the case $\overline{\pi} \neq \overline{\pi}'$.

Lemma 5.6. *If $\overline{\pi} \neq \overline{\pi}'$, then $\mathcal{S}$ is a subgeometry of* $\mathsf{PG}(2,\mathbb{K})$.

Proof. By Lemma 5.5, we know that $\overline{\pi}$ and $\overline{\pi}'$ are disjoint. Hence, for any point neighborhood N, the solid $N^{\alpha *}$ is generated by N^α and $\overline{\pi}'$. It follows easily that, for each point $x \in \mathcal{P}$, the line x^α meets $\overline{\pi}'$ in a unique point, which we denote by x^β, and for each line $L \in \mathcal{L}$, the solid L^α meets $\overline{\pi}'$ in a unique line, which we denote by L^β. Also the mapping β thus defined is clearly injective and preserves incidence and non-incidence. Hence β is a monomorphism and the result follows. $\square$

From now on, we assume that the embedding is full, and we classify the case $\overline{\pi} = \overline{\pi}'$.

Lemma 5.7. *Suppose $\overline{\pi} = \overline{\pi}'$. If α is full, and if N is a point neighborhood class of $\mathcal{S}$, then the set $\{x^\alpha : x \in N\}$ runs through all lines of* $\mathsf{PG}(5,\mathbb{K})$ *that are incident with N^α and are contained in $N^{\alpha *}$, except for the lines in $\overline{\pi}$.*

Proof. Let N be a point neighborhood class. Let $x \in N$ be arbitrary. The solid $N^{\alpha *}$ contains $\overline{\pi}$ and x^α. Let $K \neq x^\alpha$ be an arbitrary line through N^α in $N^{\alpha *}$. Let M be the unique line neighborhood class of $\mathcal{S}$ with the property that $M^{\alpha *}$ is contained in the plane generated by K and x^α. Let $L \in M$ be such that x is incident with L, and let $y \in N$ be such that y is not incident with L. The plane generated by y^α and K meets $\overline{\pi}$ in a unique line $M'^{\alpha *}$. Let $L' \in M'$ be such that y is incident with L'. Note that $M \neq M'$ since both L^α and L'^α contain the line K, and since, if $M' = M$, they would also both contain $M^{\alpha *}$, they would both meet the solid $x^{\alpha *}$ in the same set, implying that y would belong to both lines L, L', a contradiction. Hence L and L' meet in a unique point z, and so L^α and L'^α meet in the unique line z^α. But they both contain K, so $K = z^\alpha$. $\square$

Lemma 5.8. *If $\overline{\pi} = \overline{\pi}'$, then the projective Klingenberg plane $\mathcal{S}$ is isomorphic to* $\mathsf{PH}(2,\mathbb{D}(\mathbb{K},\sigma))$, *for some endomorphism σ and the embedding α is classical.*

Proof. We introduce coordinates $X_0, X_1, X_2, \ldots, X_5$ in $\mathsf{PG}(5,\mathbb{K})$. We choose for $\overline{\pi}$ the plane with equation $X_1 = X_3 = X_5 = 0$. Consider the plane π' with equation $X_0 = X_2 = X_4 = 0$. Let N be any point neighborhood class and let $N^{\alpha *}$ be the corresponding solid. Since it contains $\overline{\pi}$ as a consequence of Lemma 5.5, it meets π' in a unique point u_N outside $\overline{\pi}$ and by Lemma 5.7, the line of $\mathsf{PG}(5,\mathbb{K})$ through u_N and N^α is the image x^α of a point $x \in N$. Since solids of the form L^α, $L \in \mathcal{L}$, meet π' in a line, it is now easy to see that the mapping $\beta : \overline{\pi} \to \pi' : N^\alpha \mapsto u_N$ is an injective collineation. We can choose the coordinates such that β maps $(1,0,0,0,0,0)$, $(0,0,1,0,0,0)$, $(0,0,0,0,1,0)$ and $(1,0,1,0,1,0)$, respectively, to $(0,1,0,0,0,0)$, $(0,0,0,1,0,0)$, $(0,0,0,0,0,1)$ and $(0,1,0,1,0,1)$, respectively. It follows with elementary linear algebra (a version of the Fundamental Theorem of Projective Geometry) that there exists an endomorphism σ of $\mathbb{K}$ such that β maps $(x_0,0,y_0,0,z_0,0)$ to $(0,x_0^\sigma,0,y_0^\sigma,0,z_0^\sigma)$. It now follows easily that the points of $\mathcal{S}$ are mapped under α onto lines generated by points $(x_0,0,y_0,0,z_0,0)$ and $(x_1,x_0^\sigma,y_1,y_0^\sigma,z_1,z_0^\sigma)$. All these lines are precisely the lines of the standard embedding of $\mathsf{PH}(2, \mathbb{D}(\mathbb{K}, \sigma))$. Since the solids of $\mathsf{PG}(5, \mathbb{K})$ corresponding to lines of $\mathcal{S}$ are determined by the lines of $\mathsf{PG}(5, \mathbb{K})$ corresponding to points of $\mathcal{S}$, the embedding is completely determined and standard. The lemma is proved. $\square$

In the finite case, the embedding is automatically full under the conditions of Corollary 4.3. Moreover, it is clear that finite proper Klingenberg planes always contain digons. Indeed, if not, then consider a line L of a finite Klingenberg plane of order (qt, t) without digons. Through each point of L, there are t lines neighboring L. Since there are no digons, all these lines are different and we obtain a set of $(q+1)t(t-1)+1$ lines of the same line neighborhood class. This contradicts $(q+1)t(t-1)+1 > t^2$. Consequently Corollary 4.3 follows.

Remark 5.9. By Lemma 5.7 it follows that all point neighborhoods are affine planes. A PH-plane with this property is called *uniform.* Cronheim [3] has proved that the only finite uniform Desarguesian projective Hjelmslev planes are either planes over a ring of twisted dual numbers over a Galois field or planes over a Witt ring of length two over a Galois field. Corollary 4.3 now gives a characterization of the first class. These are the only uniform planes that are embeddable in $\mathsf{PG}(5, q)$.

Remark 5.10. The problem in the non-full case is considerably more involved, and it is even not clear what the examples are. Possibly there might be non-Desarguesian PK-planes non-fully embedded, as some preliminary work seems to indicate.

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