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On Armendariz Rings

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Abstract. In this work, we construct a class of Armendariz rings and a class of non-Armendariz rings. For this we study the transfer of the Armendariz property to trivial ring extension and direct product. The article includes a brief discussion of the scope and precision of our results.

Keywords: Armendariz ring, Gaussian ring, trivial ring extension, direct product

1. Introduction

Throughout this paper all rings are assumed to be commutative with identity elements and all modules are unital.

Let R be a commutative ring. The content $C(f)$ of a polynomial $f \in R[X]$ is the ideal of R generated by all coefficients of f. One of its properties is that $C(.)$ is semi-multiplicative, that is $C(fg) \subseteq C(f)C(g)$; and a polynomial $f \in R[X]$ is said to be Gaussian over R if $C(fg) = C(f)C(g)$, for every polynomial $g \in R[X]$. A polynomial $f \in R[X]$ is Gaussian provided $C(f)$ is locally principal by [8, Remark 1.1]. A ring R is said a Gaussian ring if $C(fg) = C(f)C(g)$ for any polynomials f, g with coefficients in R. A domain is Gaussian if and only if it is a Prüfer domain. See for instance [1], [3], [6], [8].

A ring R is called an Armendariz ring if whenever polynomials $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{i=0}^{n} b_i X^i \in R[X]$ satisfy $fg = 0$, we have $C(f)C(g) = 0$ (that is $a_i b_j = 0$ for every i and j). It is easy to see that subrings of Armendariz rings are also Armendariz. E. Armendariz ([2, Lemma 1]) noted that any reduced ring (i.e., ring without non-zero nilpotent elements) is an Armendariz ring. Also, D. D. Anderson and V. Camillo ([1]) show that a ring R is Gaussian if and only if every homomorphic image of R is Armendariz. See for instance [1], [2], [11], [12].

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Considerable work has been concerned with trivial ring extensions. Part of it has been summarized in Glaz's book [5] and Huckaba's book (where R is called the idealization of E by A) [9]. See for instance [5], [9], [10].

The goal of this work is to exhibit a class of Armendariz rings and a class of non-Armendariz rings. For this purpose, we study the transfer of the Armendariz property to trivial ring extension and direct product.

2. Main results

This section develops a result of the transfer of the Armendariz property for a particular context of trivial ring extensions. And so, we will construct a new class of Armendariz rings (with zero-divisors).

First, we examine the context of trivial ring extension of a local ring (A, M) by an A-module E such that $ME = 0$. Remark that this ring is a total ring by the proof of [10, Theorem 2.6 (1)].

Theorem 2.1. *Let (A, M) be a local ring, E an A-module such that $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of A by E. Then, R is an Armendariz ring if and only if A is it too.*

Proof. If R is an Armendariz ring, then so is A since A is a subring of R. Conversely, assume that A is an Armendariz ring. Let $f = \sum_{i=0}^{n} (a_i, e_i)X^i$ and $g = \sum_{i=0}^{m} (b_i, f_i)X^i$ be two polynomials in $R[X]$ such that $fg = 0$, where n and m are positive integers. Two cases are then possible.

Case 1. $a_i \notin M$ for some $i = 0, \dots, n$. In this case, a_i is invertible in A and then (a_i, e_i) is invertible in R. Hence, $C_R(f) = R$ and f is a Gaussian polynomial (by [8, Remark 1.1]) and so $C_R(f)C_R(g) = C_R(fg) = 0$ as desired.

Case 2. $a_i \in M$ for each $i = 0, \dots, n$. Two cases are then possible:

$*$ If there exists $b_j \notin M$ for some $j = 0, \dots, m$, then by Case 1, g is a Gaussian polynomial and then $C_R(f)C_R(g) = C_R(fg) = 0$ as desired.

$*$ If $b_j \in M$ for each $j = 0, \dots, m$, we set the two polynomials of $A[X]$: $f_A = \sum_{i=0}^{n} a_iX^i$ and $g_A = \sum_{i=0}^{m} b_iX^i$. We have $f_Ag_A = 0$ since $fg = 0$. Hence, $C_A(f_A)C_A(g_A) = 0$ since A is an Armendariz ring. But $C(f)C(g) = (C_A(f_A)C_A(g_A), 0)$

since $a_i, b_j \in M$ for each $i = 0, \ldots, n$ and for each $j = 0, \ldots, m$ and $ME = 0$. Therefore, $C(f)C(g) = 0$ and this completes the proof of Theorem 2.1.

By Theorem 2.1 and since each domain is Armendariz, we have:

Corollary 2.2. *Let (A, M) be a local domain, E an A-module such that $ME = 0$. Then the trivial ring extension $R := A \propto E$ of A by E is an Armendariz ring.*

Next, we explore the Armendariz property to the trivial ring extension of the form $R := A \propto B$, where $A \subseteq B$ is an extension of domains.

Theorem 2.3. *Let $A \subseteq B$ be two domains. Then the trivial ring extension $R := A \propto B$ of A by B is an Armendariz ring.*

Proof. Let $f = \sum_{i=0}^{n}(a_i, e_i)X^i$ and $g = \sum_{i=0}^{m}(b_i, f_i)X^i$ be two non-zero polynomials in $R[X]$ such that $fg = 0$, where n and m are positive integers. Set $f_A = \sum_{i=0}^{n} a_iX^i$ and $g_A = \sum_{i=0}^{m} b_iX^i$. We have $f_Ag_A = 0$ (since $fg = 0$) and so $f_A = 0$ or $g_A = 0$ (since A is a domain). We can assume that $f_A = 0$, that is $a_i = 0$ for each $i = 0, \ldots, n$ (the case $g_A = 0$ is similar).

Set $f_B = \sum_{i=0}^{m} e_iX^i \in B[X]$. Notice that $f_B \neq 0$ since $f \neq 0$ and $f_A = 0$. We have $f_Bg_A = 0 \in B[X]$ (since $fg = 0$) and so $g_A = 0$ (since B is a domain and $f_B \neq 0$). Therefore, $C_R(f) = \sum_{i=0}^{n} R(0, e_i)X^i$ and $C_R(g) = \sum_{i=0}^{m} R(0, f_i)X^i$ and so $C_R(f)C_R(f) = 0$ as desired.

The next two examples prove that the condition A and B are domains in Theorem 2.3 is necessary even if A is Armendariz and $B = A$.

Example 2.4. Let K be a field, $A = K \propto K$ be the trivial ring extension of K by K, and let $R = A \propto A$ be the trivial ring extension of A by A. Then:

1) A is an Armendariz ring.
2) R is not an Armendariz ring.

Proof. 1) The ring A is Gaussian by [3, Example 2.3 (1.b)]. In particular, A is an Armendariz ring.

2) Our aim is to show that R is not Armendariz. Let $f = ((0,1),(0,0)) + ((0,0),(1,0))X$ and $g = ((0,1),(0,0)) + ((0,0),(-1,0))X$ be two polynomials in $R[X]$. We easily check that $fg = 0$ and $C(f)C(g) = [R((0,1),(0,0)) + R((0,0),(1,0))][R((0,1),(0,0)) + R((0,0),(-1,0))] = R((0,0),(0,1)) \neq 0$, as desired.

Example 2.5. Let $R := \mathbb{Z}/8\mathbb{Z} \propto \mathbb{Z}/8\mathbb{Z}$ be the trivial ring extension of $\mathbb{Z}/8\mathbb{Z}$ by $\mathbb{Z}/8\mathbb{Z}$. Then:

1) $\mathbb{Z}/8\mathbb{Z}$ is an Armendariz ring by [12, Theorem 2.2].
2) $R := \mathbb{Z}/8\mathbb{Z} \propto \mathbb{Z}/8\mathbb{Z}$ is not an Armendariz ring by [12, Example 3.2].

Now, we will construct a wide class of rings satisfying the Armendariz property. For this, we study the transfer of this property to direct product.

Theorem 2.6. *Let $(R_i)_{i=1,\dots,n}$ be a family of rings. Then $\prod_{i=1}^{n} R_i$ is an Armendariz ring if and only if so is R_i for each $i = 1, \dots, n$.*

Proof. We will prove the result for $i = 1, 2$, and the theorem will be established by induction on n.

Assume that $R_1 \times R_2$ is an Armendariz ring. We show that R_1 is an Armendariz ring (it is the same for R_2).

Let $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{i=0}^{m} b_i X^i$ be two polynomials in $R_1[X]$ such that $fg = 0$, where n and m are positive integers. Set $f_1 = \sum_{i=0}^{n} (a_i, 0) X^i$ and $g_1 = \sum_{i=0}^{m} (b_i, 0) X^i$ ($\in (R_1 \times R_2)[X]$). We have $f_1 g_1 = (fg, 0) = (0, 0)$. Hence, $C_{R_1\times R_2}(f_1)$ $C_{R_1\times R_2}(g_1) = 0$ since $R_1 \times R_2$ is an Armendariz ring.

But $C_{R_1\times R_2}(f_1) C_{R_1\times R_2}(g_1) = (C_{R_1}(f) C_{R_1}(g), 0)$. Therefore, $C_{R_1}(f) C_{R_1}(g) = 0$ and this shows that R_1 is an Armendariz ring.

Conversely, assume that R_1 and R_2 are Armendariz rings. Let $f = \sum_{i=0}^{n} (a_i, e_i) X^i$ and $g = \sum_{i=0}^{m} (b_i, f_i) X^i$ be two polynomials in $(R_1 \times R_2)[X]$ such that $fg = 0$, where n and m are positive integers. Set $f_1 = \sum_{i=0}^{n} a_i X^i \in R_1[X]$, $f_2 = \sum_{i=0}^{n} e_i X^i \in R_2[X]$, $g_1 = \sum_{i=0}^{m} b_i X^i \in R_1[X]$ and $g_2 = \sum_{i=0}^{m} f_i X^i \in R_2[X]$. We have $0 = fg = (f_1 g_1, f_2 g_2)$ which implies that $f_1 g_1 = 0$ and $f_2 g_2 = 0$. Hence $C_{R_1}(f_1) C_{R_1}(g_1) = 0$ and $C_{R_2}(f_2) C_{R_2}(g_2) = 0$ since R_1 and R_2 are Armendariz rings. But $C_{R_1\times R_2}(f) C_{R_1\times R_2}(g) = (C_{R_1}(f_1) C_{R_1}(g_1), C_{R_2}(f_2) C_{R_2}(g_2))$. Therefore, $C_{R_1\times R_2}(f) C_{R_1\times R_2}(g) = 0$ and this completes the proof of Theorem 2.4.

By Theorem 2.6 and since each domain is Armendariz, we have:

Corollary 2.7. *Let $(R_i)_{i=1,\dots,n}$ be a family of domains. Then $\prod_{i=1}^{n} R_i$ is an Armendariz ring.*

Now, we study the localization of Armendariz ring.

Theorem 2.8. *Let R be a ring. Then:*

1) *Assume that R is an Armendariz ring and S is a multiplicative subset of R. Then $S^{-1}R$ is an Armendariz ring.*
2) *A ring R is Armendariz if and only if R_M is Armendariz for each maximal ideal M of R.*

Proof. 1) Without loss of generality, we may consider the polynomials of the form $S^{-1}f$ and $S^{-1}g$ where $f = \sum_{i=0}^{n} a_iX^i$ and $g = \sum_{i=0}^{m} b_iX^i \in R[X]$, such that $S^{-1}(f)S^{-1}(g) = 0$. Hence, there exists $t \in S$ such that $tfg = 0$ and so $tC_R(f)C_R(g) = C_R(tf)C_R(g) = 0$ since R is Armendariz. Then we have:

$$\begin{aligned} C_{S^{-1}R}(S^{-1}f)C_{S^{-1}R}(S^{-1}g) &= S^{-1}(C_R(f))S^{-1}(C_R(g)) \\ &= S^{-1}[C_R(f)C_R(g)] \\ &= S^{-1}[tC_R(f)C_R(g)] \\ &= 0. \end{aligned}$$

Therefore, $S^{-1}R$ is an Armendariz ring.
2) If R is Armendariz, then so is R_M for each maximal ideal M of R by 1).
Conversely, assume that R_M is Armendariz for each maximal ideal M and let $f, g \in R[X]$ such that $fg = 0$. Then $C(fg)_M = 0$ and so $[C(f)C(g)]_M (= C(f)_M C(g)_M) = 0$ for each maximal ideal M since R_M is Armendariz. Therefore, $C(f)C(g) = 0$ as desired.

By Theorem 2.8 and since each domain is Armendariz, we have:

Corollary 2.9. *A locally domain is an Armendariz ring.*

Remark 2.10. Let R be a non-Prüfer domain. Then R is an Armendariz ring which is not Gaussian. Hence, there exists an ideal I of R such that R/I is not an Armendariz ring by [1]. This shows that the homomorphic image of an Armendariz ring is not necessarily an Armendariz ring.

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